

PRICING BY HEDGING AND NO-ARBITRAGE BY HEDGING

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This talk is based on the papers

- 1 Bender, C., Sottinen, T. and Valkeila, E. (2008) Pricing by hedging and no-arbitrage beyond semimartingales. *Finance and Stochastics* **12**, 441-468.
- 2 Bender, C., Sottinen, T., and Valkeila, E. (2010) Fractional processes as models in stochastic finance. To appear in *Advanced Mathematical Methods for Finance* (Eds. G. Di Nunno and B. Oksendal) Series in Mathematical Finance, Springer.

OUTLINE

- 1 QUADRATIC VARIATION MARKET MODELS WITH
CONDITIONAL SMALL-BALL PROPERTY
- 2 NO-ARBITRAGE WITH ALLOWED STRATEGIES
- 3 LOCALLY CONTINUOUS STOPPING TIMES
- 4 NO-ARBITRAGE WITH STOPPING-ALLOWED
STRATEGIES
- 5 NO-ARBITRAGE WITH SIMPLE STRATEGIES

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QUADRATIC VARIATION MARKET MODELS WITH CONDITIONAL SMALL-BALL PROPERTY

We assume that the stock-price process $S = (S_t)_{t \in [0, T]}$ is (almost surely) continuous, strictly positive, starts from s_0 , and the information used in trading is generated by it.

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So, we work in the canonical space $\Omega = \mathcal{C}_{s_0, \sigma}[0, T]$ with $S_t(\eta) = \eta(t)$ and

$$\mathcal{F}_t = \sigma \{ \eta(s); s \leq t \},$$

$\mathcal{F} = \mathcal{F}_T$. (The index $\sigma > 0$ will be explained in the next slide.)

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We assume that the following **CONDITIONAL SMALL-BALL PROPERTY** is satisfied:

$$\mathbf{P} \left[\sup_{t \in [\tau, T]} |S_t - \eta(t)| < \varepsilon \middle| \mathcal{F}_\tau \right] > 0$$

P-a.s. for all paths η , positive ε , and stopping times τ .

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So, we have a collection of models **P** on the canonical filtered space $\mathcal{C}_{s_0, \sigma}[0, T]$, where **P** is restricted only by the assumptions of quadratic variation and conditional small-ball property.

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The no-arbitrage result followed basically from the fact that we can write the value $V_t(\Phi)(\eta)$ of an allowed strategy (almost surely) by using a value functional $v(t, \eta; \varphi)$:

$$V_t(\Phi)(\eta) = V_0(\Phi)(\eta) + v(t, \eta; \varphi) \quad \text{for } \mathbf{P}\text{-a.a. } \eta,$$

and $v(t, \eta; \varphi)$ is continuous in η uniformly in t . Here φ is the strategy functional associated to Φ :

$$\Phi_t(\eta) = \varphi\left(t, \eta(t), g_1(t, \eta), \dots, g_m(t, \eta)\right),$$

where φ is smooth and g_1, \dots, g_m are **HINDSIGHT FACTORS**.

NO-ARBITRAGE WITH ALLOWED STRATEGIES

The allowed strategies are natural from the hedging point of view: Hedging strategies of typical options are of this type. However, from the no-arbitrage point of view the allowed strategies are not so natural: They do not include stopping times.

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To include stopping times we need some technical assumptions. The key concept is **LOCAL CONTINUITY**.

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To include stopping times we need some technical assumptions. The key concept is **LOCAL CONTINUITY**.

While stopping times are rarely continuous, the author is not aware of any (reasonable) stopping time that is not locally continuous.

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LOCALLY CONTINUOUS STOPPING TIMES

DEFINITION (LOCAL CONTINUITY)

Let \mathcal{X} and \mathcal{Y} be metric spaces. A function $f : \mathcal{X} \rightarrow \mathcal{Y}$ is **LOCALLY CONTINUOUS** if for all $x \in \mathcal{X}$ there exists an open $U_x \subset \mathcal{X}$ such that $x \in \bar{U}_x$ and $f(x_n) \rightarrow f(x)$ whenever $x_n \rightarrow x$ in U_x .

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EXAMPLE

An indicator $\mathbf{1}_A : \mathbb{R} \rightarrow \mathbb{R}$

- 1 is locally continuous if $A = \bar{G}$, G is open,
- 2 is not locally continuous if A has an isolated point.

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In the case (3) above we say that τ is **FAT**. All the stopping times in the example above are fat.

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The functionals in the example above are locally continuous even if they were not stopping times.

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A trading strategy Φ is **STOPPING-ALLOWED** if it is of the form

$$\Phi_t = \sum_{k=1}^n \Phi_t^{(k)} \mathbf{1}_{(\tau_k, \tau_{k+1}]}(t),$$

where the $\Phi^{(k)}$'s are allowed and τ_k 's are locally continuous.

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The definition above is understood in the conditional sense, i.e. $\Phi^{(k)}$ may depend on \mathcal{F}_{τ_k} and $\tau_{k+1} \geq \tau_k$ is locally continuous in the conditioned, or quotient, space $\mathcal{C}_{S_{\tau_k}, \sigma}[\tau_k, T]$.

NO-ARBITRAGE WITH STOPPING-ALLOWED STRATEGIES

THEOREM (NO-ARBITRAGE WITH STOPPING-ALLOWED STRATEGIES)

Let Φ be a stopping-allowed strategy. Then Φ is not an arbitrage opportunity.

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Theorem (No-Arbitrage with Stopping-Allowed Strategies) follows by applying the conditional small-ball property n times with the following lemma:

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Theorem (No-Arbitrage with Stopping-Allowed Strategies) follows by applying the conditional small-ball property n times with the following lemma:

LEMMA (NO-ARBITRAGE WITH TAKE-THE-MONEY-AND-RUN STRATEGIES)

Let Φ be allowed strategy and let τ be a locally continuous stopping time. Then $\Phi \mathbf{1}_{[0,\tau]}$ is not an arbitrage opportunity.

NO-ARBITRAGE WITH STOPPING-ALLOWED STRATEGIES

PROOF OF LEMMA (NO-ARBITRAGE WITH
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Let $\Phi \mathbf{1}_{[0,\tau]}$ be a candidate for an arbitrage opportunity:
 $V_0(\Phi \mathbf{1}_{[0,\tau]}) = 0$ and $V_T(\Phi \mathbf{1}_{[0,\tau]}) \geq 0$ \mathbf{P} -a.s., or

$$v(\tau(\eta), \eta; \varphi) \geq 0 \quad \text{for } \mathbf{P}\text{-a.a. } \eta.$$

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NO-ARBITRAGE WITH STOPPING-ALLOWED STRATEGIES

PROOF OF LEMMA (NO-ARBITRAGE WITH TAKE-THE-MONEY-AND-RUN STRATEGIES).

Since $v(\tau(\eta), \eta; \varphi) \geq 0$ for all η we have in particular that $V_T(\Phi \mathbf{1}_{[0, \tau]}) \geq 0$ $\tilde{\mathbf{P}}$ -a.s. ($\tilde{\mathbf{P}}$ stands for the Black-Scholes reference model).

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NO-ARBITRAGE WITH STOPPING-ALLOWED STRATEGIES

PROOF OF THEOREM (NO-ARBITRAGE WITH STOPPING-ALLOWED STRATEGIES).

By using the conditional small-ball property instead of an unconditional one we see that Lemma (No-Arbitrage with Take-the-Money-and-Run Strategies) can be strengthened to:

$$\Phi^{(k)} \mathbf{1}_{(\tau_k, \tau_{k+1}]}$$

is not an arbitrage opportunity. Here the allowed strategy $\Phi^{(k)}$ may depend additionally on \mathcal{F}_{τ_k} , and τ_{k+1} is locally continuous on the quotient, or conditioned, space $\mathcal{C}_{S_{\tau_k}, \sigma}[\tau_k, T]$.

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But this means that the stopping-allowed strategy Φ does not generate arbitrage on any of the stochastic intervals $(\tau_k, \tau_{k+1}]$. Hence, it cannot generate arbitrage on the interval $[0, T]$. \square

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If we assume that the trading strategy is constant between the switching stopping times we can weaken the local continuity assumption.

One way to weaken the assumption is to ask only local lower semi-continuity instead of local “full” continuity.

NO-ARBITRAGE WITH SIMPLE STRATEGIES

DEFINITION (LOCAL LOWER SEMI-CONTINUITY)

Let \mathcal{X} be a metric space and let \mathcal{Y} be an ordered complete metric space. A function $f : \mathcal{X} \rightarrow \mathcal{Y}$ is **LOCALLY LOWER SEMI-CONTINUOUS** if for all $x \in \mathcal{X}$ there exists an open $U_x \subset \mathcal{X}$ such that $x \in \bar{U}_x$ and $\liminf f(x_n) \geq f(x)$ whenever $x_n \rightarrow x$ in U_x .

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EXAMPLE

An indicator $\mathbf{1}_A : \mathcal{X} \rightarrow \mathbb{R}$ is locally lower semi-continuous if for all $x \in A$ and $\varepsilon > 0$ there exists a ball $B \subset A$ such that $\text{dist}(x, B) < \varepsilon$.

NO-ARBITRAGE WITH SIMPLE STRATEGIES

DEFINITION (SIMPLE STRATEGY)

A trading Φ strategy is **SIMPLE** if it is of the form

$$\Phi_t = \sum_{k=1}^n \xi_k \mathbf{1}_{(\tau_k, \tau_{k+1}]},$$

where τ_k is locally lower semi-continuous stopping times (relative to τ_{k-1}) and ξ_k 's are \mathcal{F}_{τ_k} measurable.

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THEOREM (NO-ARBITRAGE WITH SIMPLE STRATEGIES)

Let Φ be a simple strategy. Then Φ is not an arbitrage opportunity.

NO-ARBITRAGE WITH SIMPLE STRATEGIES

Before going to the proof of Theorem (No-Arbitrage with Simple Strategies) let us note that the theorem is true even without the assumption on the quadratic variation. In contrast, Theorem (No-Arbitrage with Stopping-Allowed Strategies) fails to be true if the quadratic variation vanishes.

NO-ARBITRAGE WITH SIMPLE STRATEGIES

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Because of the “time-linearity” of the arbitrage and conditional small-ball property it is enough to show the following:

LEMMA (UP’N’DOWN)

Let τ be locally lower semi-continuous stopping time. Then

$$\mathbf{P}[S_\tau > s_0] > 0 \quad \text{and} \quad \mathbf{P}[S_\tau < s_0] > 0.$$

NO-ARBITRAGE WITH SIMPLE STRATEGIES

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We show that $\mathbf{P}[S_\tau > s_0] > 0$; the case $\mathbf{P}[S_\tau < s_0] > 0$ is symmetric.

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We show that the set $\{S_\tau > s_0\} = \{\eta; \eta(\tau(\eta)) > s_0\}$ contains a ball. Then the claim will follow from the small-ball property.

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Fix an increasing and concave path η_0 with $\eta_0(0) = s_0$ and a local lower semi-continuity set U_{η_0} of τ at η_0 .

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Since τ is lower semi-continuous on U_{η_0} we can find such an $\varepsilon < 1/2 (\eta_0(\tau(\eta_0)) - s_0)$ that $\tau(\eta) \geq 1/2 \tau(\eta_0)$ whenever $\eta \in B$, where B is some ball contained in $B_{\eta_0}(\varepsilon) \cap U_{\eta_0}$.

NO-ARBITRAGE WITH SIMPLE STRATEGIES

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Since η_0 is increasing and concave

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So, the ball B is contained in the set $\{S_\tau > s_0\}$, which implies that $\mathbf{P}[S_\tau > s_0] > 0$. \square

NO-ARBITRAGE WITH SIMPLE STRATEGIES

REMARK (ε -DELAY)

The Lemma (Up'n'Down) is true with local lower semi-continuity replaced by a weaker assumption of ε -DELAY:

For all η_0 there are positive $\varepsilon = \varepsilon(\eta_0)$ and $\delta = \delta(\eta_0)$ such that

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- The End -