On short-time asymptotics of one-dimensional Harris flows

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Outline

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The Definition

Definition

The point motion of a Harris flow is a two-parametric process $(X(u,t), u \in \mathbb{R}, t \geq 0)$, such that

- For each u the process $X(u,\cdot)$ is a continuous martingale adapted to a common filtration; $X(u,0) = u$.
- Their infinitesimal covariation is given by

$$d\langle X(u_1), X(u_2)\rangle_t = \varphi(X(u_1,t) - X(u_2,t))\,dt,$$

  where φ is an aperiodic real positive definite function.
- For each t the process $X(\cdot,t)$ is monotone.

We assume that $\varphi(0) = 1$ for convenience.

Examples

Examples

- The flow of solutions to an SDE

$$dX(u,t) = \int_{\mathbb{R}} \psi(p - X(u,t))\, W(dp, dt),$$

  where W is a Gaussian white noise in $\mathbb{R} \times \mathbb{R}_+$. $\varphi = \psi * \psi$, provided that ψ is symmetric.

- The Arratia flow, given by

$$\varphi(x) = \begin{cases} 1, & x = 0 \\ 0, & x \neq 0 \end{cases}$$

The Existence Theorem

Theorem

The process $(X(u,t))$ exists and is unique in distribution, provided that φ is strictly positive definite and locally Lipshitz outside zero.

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The Problem

We consider the deviation of the Brownian motions from their starting point in sup-norm

$$\sup_{u\in[0,1]} |X(u,t) - u|$$

as t tends to zero.

Examples

Examples

- If $\varphi \in C^2$ then there is a law of iterated logarithm:

$$\limsup_{t\to 0} \sup_{u\in[0,1]} \frac{|X(u,t)-u|}{\sqrt{2t\ln\ln t^{-1}}} = 1.$$

- The Arratia flow behaves in a different way:

$$\sup_{u\in[0,1]} |X(u,t)-u| = \sqrt{t\ln t^{-1}} + O\left(\sqrt{t\ln\ln t^{-1}}\right).$$

- For every possible Harris flow there is an inequality:

$$\limsup_{t\to 0} \sup_{u\in[0,1]} \frac{|X(u,t)-u|}{\sqrt{t\ln t^{-1}}} \le 1.$$

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The Tangent Process

There exists a Gaussian process $(Y(u,t))$ with the following properties:

- For each u $Y(u,\cdot)$ is a Brownian motion starting at u, adapted to a common filtration;
- The covariaton is given by

$$d\langle Y(u_1), Y(u_2)\rangle_t = \varphi(u_1 - u_2)\,dt.$$

It can be coupled with X in such a way that

$$d\langle X(u_1), Y(u_2)\rangle_t = \varphi(X(u_1,t) - u_2)\,dt.$$

Examples

Examples

- For the flow of solutions to an SDE, Y can be given by

$$Y(u,t) = u + \int_0^t \int_{\mathbb{R}} \psi(u-p)\, W(dp, dt).$$

- For the Arratia flow, $Y(u,\cdot)$ are independent Brownian motions.

Our Approach

- The asymptotics of $\sup_u |X(u,t) - u|$ can be described by that of Y up to the $O\left(\sqrt{t \ln\ln t^{-1}}\right)$ scale.
- In case Y has a continuous modification, we estimate $|X - Y|$ for the coupled X and Y.
- If Y has no continuous modification, the tail probabilities of X are compared to those of Y.

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The Result in Continuous Case

Theorem

Suppose that Y has a continuous modification. Then

$$\sup_{u\in[0,1]} |X(u,t) - Y(u,t)| \ll \sqrt{t \ln\ln t^{-1}}.$$

Corollary

In this case X satisfies

$$\limsup_{t\to 0} \sup_{u\in[0,1]} \frac{|X(u,t) - u|}{\sqrt{2t \ln\ln t^{-1}}} = 1.$$

Sketch of Proof

- Take a slowly increasing number of points u_{nk} at time $t_n = q^n$, $0 < q < 1$.
- Estimate the difference $|X(u_{nk}, t_n) - Y(u_{nk}, t)|$ uniformly in k.
- Use continuity of Y to show that

$$\sup_k \left| Y\left(u_{n,k+1}, t_n\right) - Y\left(u_{nk}, t_n\right) \right| \ll \sqrt{t_n \ln \ln t_n^{-1}},$$

$$t_n \to 0, \sup_k |\Delta u_{nk}| \to 0.$$

- Use monotonicity of X to handle the points other than u_{nk}.

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The Result in General Case

Theorem

Suppose that φ *is monotone on* $[0,\delta]$ *for some* $\delta > 0$. *Then*

$$\sup_{u\in[0,1]} |X(u,t) - u| = E(t) + O\left(\sqrt{t \ln\ln t^{-1}}\right),$$

where

$$E(t) = \mathrm{E} \sup_{\substack{0\le k<t^{-1/2} \\ k\in\mathbb{Z}}} \left| Y\left(kt^{1/2}, t\right) - kt^{1/2}\right|.$$

$E(t)$ may be estimated by using techniques of the theory of Gaussian processes.

Sketch of Proof

- Take the $\lfloor t^{-1/2} \rfloor$ points $u_k := kt^{1/2}$ and $\tilde{u}_k := kt^{1/2} \ln t^{-1}$.
- Estimate the tails of $\sup_k |Y(\tilde{u}_k, t) - \tilde{u}_k|$ by means of concentration-of-measure arguments.
- Estimate the upper tail of $\sup_k |X(u_k, t) - u_k|$ by that of $\sup_k |Y(\tilde{u}_k, t) - \tilde{u}_k|$.
- Exchange u_k and $\tilde{u}_k$ and estimate the lower tail in the same way.
- Use monotonicity of X to handle the points other than u_k.

Slepian Comparison for Martingales

Theorem

Suppose that M and N are continuous $\mathbb{R}^d$-valued martingales and that N is Gaussian. Let $f : \mathbb{R}^d \to \mathbb{R}$ be a function with second derivatives of at most exponential growth. Further, suppose that

$$d\left\langle M^i, M^i \right\rangle_t = d\left\langle N^i, N^i \right\rangle_t,$$

$$d\left\langle M^i, M^j \right\rangle_t \geq d\left\langle N^i, N^j \right\rangle_t,$$

$$\partial_{ij} f \leq 0, i \neq j.$$

Then

$$\mathrm{E}f\left(M(1)\right) \leq \mathrm{E}f\left(N(1)\right).$$