Branching Walks in Nonhomogeneous and Random Media

Elena Yarovaya

Moscow State University
Faculty of Mechanics and Mathematics
Dept. of Probability Theory

yarovaya@mech.math.msu.su

International Symposium "VISIONS in STOCHASTICS"

November 1-3, 2010, Moscow

Outline

Motivation

Models describable in terms of the branching processes with walking particles find numerous applications in various fields of the natural sciences. These evolutionary processes are essentially dependent on **the structure of the environment** where the walk runs. This explains the interest in branching random walks (BRW) on $\mathbf{Z}^{\mathbf{d}}$, $d \geq 1$, either non-random or random media and its applications (see, for example, Cranston and Molchanov, 2007).

The structure of a medium

The birth and the death of particles occur at

- every lattice point: the environment is **homogeneous**
- a few lattice points: the environment is **nonhomogeneous**

Motivation

Models describable in terms of the branching processes with walking particles find numerous applications in various fields of the natural sciences. These evolutionary processes are essentially dependent on **the structure of the environment** where the walk runs. This explains the interest in branching random walks (BRW) on $\mathbf{Z}^{\mathbf{d}}$, $d \geq 1$, either non-random or random media and its applications (see, for example, Cranston and Molchanov, 2007).

The structure of a medium

The birth and the death of particles occur at

- every lattice point: the environment is **homogeneous**
- a few lattice points: the environment is **nonhomogeneous**

Motivation

Models describable in terms of the branching processes with walking particles find numerous applications in various fields of the natural sciences. These evolutionary processes are essentially dependent on **the structure of the environment** where the walk runs. This explains the interest in branching random walks (BRW) on $\mathbf{Z}^{\mathbf{d}}$, $d \geq 1$, either non-random or random media and its applications (see, for example, Cranston and Molchanov, 2007).

The structure of a medium

The birth and the death of particles occur at

- every lattice point: the environment is **homogeneous**
- a few lattice points: the environment is **nonhomogeneous**

BRW in Nonhomogeneous Non-Random Media

Informal Description of BRW on $\mathbf{Z^d}$

The population of individuals is initiated at time $t = 0$ by a single particle at a point $x \in \mathbf{Z^d}$, $d \geq 1$.

Being outside of sources the particle performs a continuous time random walk on $\mathbf{Z^d}$ until reaching a source.

At the source it spends an exponentially distributed time and then either jumps to a point $y \in \mathbf{Z^d}$ (distinct from the source) or dies producing just before the death a random number of offsprings.

The newborn particles behave independently and stochastically in the same way as the parent individual.

Features of BRW in Nonhomogeneous Environments

The models of **BRW** on multidimensional lattices with a reproduction of particles at **a few lattice points** is of interest mainly in connection with the following circumstances:

- the branching medium, i.e., the set of branching characteristics at points of the phase space, is **nonhomogeneous**,
- the phase space in which the walk occurs is **unbounded**.

Features of BRW in Nonhomogeneous Environments

The models of **BRW** on multidimensional lattices with a reproduction of particles at **a few lattice points** is of interest mainly in connection with the following circumstances:

- the branching medium, i.e., the set of branching characteristics at points of the phase space, is **nonhomogeneous**,
- the phase space in which the walk occurs is **unbounded**.

Features of BRW in Nonhomogeneous Environments

The models of **BRW** on multidimensional lattices with a reproduction of particles at **a few lattice points** is of interest mainly in connection with the following circumstances:

- the branching medium, i.e., the set of branching characteristics at points of the phase space, is **nonhomogeneous**,
- the phase space in which the walk occurs is **unbounded**.

Objects of Study in BRW

Evolution of particles on $\mathbf{Z^d}$ is described by

- **the number of particles** $\mu_t(y)$ **at a point** $y \in \mathbf{Z^d}$ at time t,
- **the population size** $\mu_t := \sum_{y\in\mathbf{Z^d}} \mu_t(y)$ **on** $\mathbf{Z^d}$
- their **moments**

$$m_n(t,x,y) := E_x\mu_t^n(y), \quad m_n(t,x) := E_x\mu_t^n, \quad n \in \mathbf{N},$$

where E_x denotes the mathematical expectation under the condition $\mu_0(\cdot) = \delta_y(\cdot)$.

The Main Problem for BRW

The study of limit behavior of the processes $\mu_t(y)$ and μ_t is one of the main problems for BRW.

The Aim of the Study

BRW models with one source of branching studied by

- Cranston and Molchanov (2007),
  [Simple Symmetric BRW: Simple SBRW]
- Albeverio et al. (1998), Bogachev and Yarovaya (1998), Yarovaya (2005, 2007),
  [Symmetric BRW: SBRW]
- Vatutin et al. (2003), Vatutin and Topchii (2004), Yarovaya (2009), Bulinskaya (2010)
  [Catalytic BRW: CBRW]

The present work aims to generalize BRW models with one source of branching by introducing **a few sources of three types**.

The Underlying Random Walk in BRW Models

The Symmetric Random Walk (SRW)

Let

$$A = \|a(x,y)\|_{x,y\in\mathbf{Z^d}}$$

be an infinitesimal transition matrix. The random walk (RW) is assumed to possess the following properties:

- symmetry: $a(x,y) \equiv a(y,x)$,
- homogeneity: $a(x,y) \equiv a(y-x)$,
- irreducibility: every point $y \in Z^d$ is reachable,
- regular: $a(x) \geq 0$, $x \neq 0$, $a(0) < 0$, $\sum_{x\in Z^d} a(x) = 0$,
- a finite variance of jumps: $\sum_{x\in Z^d} |x|^2 a(x) < \infty$, where $|x|$ is the Euclidean norm of a vector x on $\mathbf{Z^d}$

The Symmetric Random Walk (SRW)

Let

$$A = \|a(x,y)\|_{x,y \in \mathbf{Z^d}}$$

be an infinitesimal transition matrix. The random walk (RW) is assumed to possess the following properties:

- symmetry: $a(x,y) \equiv a(y,x)$,
- homogeneity: $a(x,y) \equiv a(y-x)$,
- irreducibility: every point $y \in Z^d$ is reachable,
- regular: $a(x) \geq 0$, $x \neq 0$, $a(0) < 0$, $\sum\limits_{x \in Z^d} a(x) = 0$,
- a finite variance of jumps: $\sum\limits_{x \in Z^d} |x|^2 a(x) < \infty$, where $|x|$ is the Euclidean norm of a vector x on $\mathbf{Z^d}$

The Symmetric Random Walk (SRW)

Let

$$A = \|a(x,y)\|_{x,y\in\mathbf{Z^d}}$$

be an infinitesimal transition matrix. The random walk (RW) is assumed to possess the following properties:

- symmetry: $a(x,y) \equiv a(y,x)$,
- homogeneity: $a(x,y) \equiv a(y-x)$,
- irreducibility: every point $y \in Z^d$ is reachable,
- regular: $a(x) \geq 0$, $x \neq 0$, $a(0) < 0$, $\sum_{x\in Z^d} a(x) = 0$,
- a finite variance of jumps: $\sum_{x\in Z^d} |x|^2 a(x) < \infty$, where $|x|$ is the Euclidean norm of a vector x on $\mathbf{Z^d}$

The Symmetric Random Walk (SRW)

Let

$$A = \|a(x,y)\|_{x,y\in\mathbf{Z^d}}$$

be an infinitesimal transition matrix. The random walk (RW) is assumed to possess the following properties:

- symmetry: $a(x,y) \equiv a(y,x)$,
- homogeneity: $a(x,y) \equiv a(y-x)$,
- irreducibility: every point $y \in Z^d$ is reachable,
- regular: $a(x) \geq 0$, $x \neq 0$, $a(0) < 0$, $\sum\limits_{x\in Z^d} a(x) = 0$,
- a finite variance of jumps: $\sum\limits_{x\in Z^d} |x|^2 a(x) < \infty$, where $|x|$ is the Euclidean norm of a vector x on $\mathbf{Z^d}$

The Symmetric Random Walk (SRW)

Let

$$A = \|a(x,y)\|_{x,y\in \mathbf{Z^d}}$$

be an infinitesimal transition matrix. The random walk (RW) is assumed to possess the following properties:

- symmetry: $a(x,y) \equiv a(y,x)$,
- homogeneity: $a(x,y) \equiv a(y-x)$,
- irreducibility: every point $y \in Z^d$ is reachable,
- regular: $a(x) \geq 0$, $x \neq 0$, $a(0) < 0$, $\sum_{x\in Z^d} a(x) = 0$,
- a finite variance of jumps: $\sum_{x\in Z^d} |x|^2 a(x) < \infty$, where $|x|$ is the Euclidean norm of a vector x on $\mathbf{Z^d}$

The Symmetric Random Walk (SRW)

Let

$$A = \|a(x,y)\|_{x,y\in \mathbf{Z^d}}$$

be an infinitesimal transition matrix. The random walk (RW) is assumed to possess the following properties:

- symmetry: $a(x,y) \equiv a(y,x)$,
- homogeneity: $a(x,y) \equiv a(y-x)$,
- irreducibility: every point $y \in Z^d$ is reachable,
- regular: $a(x) \geq 0$, $x \neq 0$, $a(0) < 0$, $\sum\limits_{x\in Z^d} a(x) = 0$,
- a finite variance of jumps: $\sum\limits_{x\in Z^d} |x|^2 a(x) < \infty$, where $|x|$ is the Euclidean norm of a vector x on $\mathbf{Z^d}$

The Transition Probability

The random walk transition probability $p(t,x,y)$ satisfies the system of differential equations (Kolmogorov's backward equations)

$$\partial_t p(t,x,y) = Ap(t,x,y), \quad p(0,x,y) = \delta_y(x),$$

where the (linear) operator A (the generator of the random walk) acts with respect to variable x, that is,

$$Ap(t,x,y) := \sum_{x'} a(x,x')p(t,x',y).$$

Remark about a Simple SRW.

In particular, a symmetric RW includes **a simple SRW** defined by $a(x,y) = -a(0)/2d$, for $|y-x| = 1$, $a(x,x) = a(0)$, and $a(x,y) = 0$ otherwise.
If $\varkappa = -a(0)$, then the SRW transition probability $p(t,x,y)$ satisfies the system of differential equations (Kolmogorov's backward equations)

$$\partial_t p(t,x,y) = \kappa \Delta p(t,x,y), \quad p(0,x,y) = \delta_y(x),$$

where the Laplace operator Δ (the generator of the simple random walk) acts with respect to variable x, that is,

$$\Delta p(t,x,y) := \frac{1}{2d} \sum_{x':|x'-x|=1} (p(t,x',y) - p(t,x,y)).$$

The Laplace Transformation

Under above conditions, the random walk transition probability has the asymptotics

$$p(t,x,y) \sim \gamma_d \cdot t^{-d/2}, \quad t \to \infty.$$

Denote the Laplace Transformation of the random walk by

$$G_\lambda(x,y) := \int_0^\infty e^{-\lambda t} p(t,x,y)\, dt.$$

Put

$$\beta_c := 1/G_0(0,0),$$

then

$$\beta_c = 0 \text{ for } d = 1,2 \qquad \text{and} \qquad \beta_c > 0 \text{ for } d \geq 3.$$

Branching Mechanism at the Source

The branching mechanism at the source is governed by the infinitesimal generating function

$$f(u) := \sum_{n=0}^{\infty} b_n u^n, \quad 0 \le u \le 1,$$

where

$$b_1 < 0, \quad b_n \ge 0, \quad n \ne 1, \quad \sum_n b_n = 0.$$

It will be supposed that

$$\beta_r := f^{(r)}(1) < \infty, \quad r \in \mathbb{N}, \qquad \beta := \beta_1.$$

We assume without restriction of generality: if there is only one branching source on $\mathbf{Z^d}$, then it is situated at the origin.

SBRW

The long-time behavior of $m_1(t,x,y)$ and $m_1(t)$ in this system is determined by the structure of the spectrum of the linear operator

$$H = A + \beta \mathit{\Delta}_0,$$

where the random walk generator A is a bounded self-adjoint operator in $l^2(\mathbf{Z^d})$, and the parameter β characterizes source intensity. Here, $\mathit{\Delta}_0 = \delta_0 \delta_0^T$ and $\delta_0 = \delta_0(\cdot)$ as usual denotes the column-vector on the lattice $\mathbf{Z^d}$ taking the value 1 at the origin and 0 at other points.

Remark. *To avoid confusion with the standard notation of the Laplace operator Δ, the above operator is denoted as "slanted delta": $\mathit{\Delta}$.*

Violation of Symmetry

Violation of Symmetry. CBRW

A modification of the BRW with a single source on **Z** was introduced by Vatutin et al. (2003) and named the catalytic branching random walks (CBRW).
The characteristic feature of this CBRW is an additional parameter $0 < \alpha < 1$ controlling the behavior of the process at the source.

As a side effect, introducing of α leads to violation of symmetry of the transition intensity matrix A.

The RW with Violation of Symmetry at the origin

Let $\bar{A} = \|\bar{a}(x,y)\|_{x,y\in Z^d}$ be an infinitesimal transition matrix. In contrast to SRW the RW is assumed to be:

nonsymmetric:

$$\begin{array}{lll} \bar{a}(x,y) = \bar{a}(y,x) & \text{for} \quad x \neq 0 \quad \text{and} \quad \text{all} \quad y \in \mathbf{Z^d}, \\ \bar{a}(0,y) = \frac{\alpha-1}{a(0)}\bar{a}(y,0) & \text{for} \quad y \neq 0 \\ \bar{a}(0,0) = \alpha - 1, & \end{array}$$

The RW with Violation of Symmetry at the origin

Let $\bar{A} = \|\bar{a}(x,y)\|_{x,y\in Z^d}$ be an infinitesimal transition matrix. In contrast to SRW the RW is assumed to be:

nonsymmetric:

$$\begin{array}{ll} \bar{a}(x,y) = \bar{a}(y,x) & \text{for} \quad x \neq 0 \quad \text{and} \quad \text{all} \quad y \in \mathbf{Z^d}, \\ \bar{a}(0,y) = \frac{\alpha-1}{\bar{a}(0)}\bar{a}(y,0) & \text{for} \quad y \neq 0 \\ \bar{a}(0,0) = \alpha - 1, & \end{array}$$

The RW with Violation of Symmetry at the origin

Let $\bar{A} = \|\bar{a}(x,y)\|_{x,y\in Z^d}$ be an infinitesimal transition matrix. In contrast to SRW the RW is assumed to be:

nonhomogeneity:

$$\begin{array}{ll} \bar{a}(x,y) = \bar{a}(y-x) & \text{for} \quad x \neq 0 \quad \text{and} \quad \text{all} \quad y \in \mathbf{Z^d}, \\ \bar{a}(0,y) = \frac{\alpha-1}{\bar{a}(0)}\bar{a}(y) & \text{for} \quad y \neq 0, \end{array}$$

The RW with Violation of Symmetry at the origin

Let $\bar{A} = \|\bar{a}(x,y)\|_{x,y\in Z^d}$ be an infinitesimal transition matrix. In contrast to SRW the RW is assumed to be:

regular:

$$\sum_{y\in \mathbf{Z^d}} \bar{a}(x,y) = 0 \qquad \text{for} \quad \text{all} \quad y \in \mathbf{Z^d},$$
$$\sum_{x\in \mathbf{Z^d}} \bar{a}(x,y) = (\alpha - 1 - \bar{a}(0))\, \frac{\bar{a}(y)}{\bar{a}(0)}.$$

Symmetrization of the operator $\bar{H}$

As was shown by Yarovaya (2009) for CBRW, the evolutionary operator $\bar{H} = \bar{A} + \beta\Delta_0$ for the first moments $m_1(t,x,y)$ and $m_1(t)$ has the following form

$$\bar{H} = A + \zeta\Delta_0 A + \beta\Delta_0, \quad \zeta = \left(\frac{\alpha-1}{\bar{a}(0)} - 1\right).$$

where the random walk generator $A + \zeta\Delta_0 A$ with $\zeta > 0$ is a bounded but in general non-self-adjoint operator in $l^2(\mathbf{Z^d})$.

However, the operator $\bar{H}$ can be symmetrized by the change of variables

$$D = I + (\sqrt{1+\zeta} - 1)\Delta_0,$$

that is the operator $D^{-1}\bar{H}D$ is a bounded self-adjoint operator in $l^2(\mathbf{Z^d})$.

Development BRW models with single source

Simple SBRW: $M = \varkappa\Delta + \beta\Delta_0$

$$\Downarrow$$

SBRW: $H = A + \beta\Delta_0$

$$\Downarrow$$

CBRW: $\bar{H} = A + \zeta\Delta_0 A + \beta\Delta_0$

The Critical Point. SBRW

The operator A in $\ell^2(\mathbf{Z}^\mathbf{d})$ has only the essential spectrum

$$\sigma(A) = [\min_\theta \phi(\theta), 0], \quad \phi(\theta) := \sum_x a(x) e^{i(x,\theta)}, \quad \theta \in [-\pi, \pi]^d,$$

which coincides with the essential spectrum of the operator

$$H = A + \beta \Delta_0.$$

Furthermore, for $\beta > \beta_c$ the operator H has the unique eigenvalue $\lambda_0 > 0$, which is a unique root of the equation

$$\beta G_\lambda(0,0) = 1.$$

Criticality

The point β_c is critical since the asymptotic behavior of the process is essentially different for

- $\beta > \beta_c$,
- $\beta = \beta_c$,
- $\beta < \beta_c$.

Criticality

The point β_c is critical since the asymptotic behavior of the process is essentially different for

- $\beta > \beta_c$,
- $\beta = \beta_c$,
- $\beta < \beta_c$.

Criticality

The point β_{c} is critical since the asymptotic behavior of the process is essentially different for

- $\beta > \beta_c$,
- $\beta = \beta_c$,
- $\beta < \beta_c$.

Criticality

The point β_{c} is critical since the asymptotic behavior of the process is essentially different for

- $\beta > \beta_c$,
- $\beta = \beta_c$,
- $\beta < \beta_c$.

Theorem (Asymptotics of the moments)

As $t \to \infty$, the moments m_n have the asymptotics

$$m_n(t,x,y) \sim C_n^d(x,y)u_n(t), \qquad m_n(t,x) \sim C_n^d(x)v_n(t), \qquad t \to \infty,$$

where $C_n^d(x,y), C_n^d(x)$ — the positive constants, depending on the dimention d are defined recursively in n and the functions u_n, v_n are of the form:

Supercritacal case

If $\beta > \beta_c$ then for all d

$$u_n = v_n = e^{n\lambda_0 t}.$$

Theorem (Asymptotics of the moments)

As $t \to \infty$, the moments m_n have the asymptotics

$$m_n(t,x,y) \sim C_n^d(x,y)u_n(t), \qquad m_n(t,x) \sim C_n^d(x)v_n(t), \qquad t \to \infty,$$

where $C_n^d(x,y), C_n^d(x)$ — the positive constants, depending on the dimention d are defined recursively in n and the functions u_n, v_n are of the form:

Supercritacal case

If $\beta > \beta_c$ then for all d

$$u_n = v_n = e^{n\lambda_0 t}.$$

Theorem (Asymptotics of the moments)

As $t \to \infty$, the moments m_n have the asymptotics

$$m_n(t,x,y) \sim C_n^d(x,y)u_n(t), \qquad m_n(t,x) \sim C_n^d(x)v_n(t), \qquad t \to \infty,$$

where $C_n^d(x,y), C_n^d(x)$ — the positive constants, depending on the dimention d are defined recursively in n and the functions u_n, v_n are of the form:

Subcritacal case

If $\beta < \beta_c$ then

$$\begin{array}{lll} u_n(t) = t^{-3/2}, & v_n(t) = t^{-1/2} & \textit{for} \quad d = 1; \\ u_n(t) = (t\ln^2 t)^{-1}, & v_n(t) = (\ln t)^{-1} & \textit{for} \quad d = 2; \\ u_n(t) = t^{-d/2}, & v_n(t) \equiv 1 & \textit{for} \quad d \geq 3. \end{array}$$

Theorem (Asymptotics of the moments)

As $t \to \infty$, the moments m_n have the asymptotics

$$m_n(t,x,y) \sim C_n^d(x,y)u_n(t), \qquad m_n(t,x) \sim C_n^d(x)v_n(t), \qquad t \to \infty,$$

where $C_n^d(x,y), C_n^d(x)$ — the positive constants, depending on the dimention d are defined recursively in n and the functions u_n, v_n are of the form:

Critacal case

If $\beta = \beta_c$ then

$$\begin{array}{lll}
u_n(t) = t^{-1/2}(\ln t)^{n-1}, & v_n(t) = t^{(n-1)/2} & \text{for } d = 1; \\
u_n(t) = t^{-1}, & v_n(t) = (\ln t)^{n-1} & \text{for } d = 2; \\
u_n(t) = t^{-1/2}(\ln t)^{n-1}, & v_n = t^{n-1/2} & \text{for } d = 3; \\
u_n(t) = t^{n-1}(\ln t)^{1-2n}, & v_n = t^{2n-1}(\ln t)^{1-2n} & \text{for } d = 4; \\
u_n(t) = t^{n-1}, & v_n(t) = t^{2n-1} & \text{for } d \geq 5.
\end{array}$$

Supercritacal case

If $\beta > \beta_c$ then for all d

$$u_n = v_n = e^{n\lambda_0 t}.$$

In the sense of convergence of moments under the normalization $e^{-\lambda_0 t}$ (where exponent λ_0 is determined from the equation $\beta G_\lambda(0,0) = 1$), the random variables $\mu_t(y)$ and μ_t have a limit distribution as $t \to \infty$.

Classification of BRW depending on intensity of a source

BP	**RW**	**BRW**	$u(t)$	$v(t)$
$\beta > 0$	$d = 1, 2,\ \beta_c = 0$	Sp. $\beta > \beta_c$	$e^{\lambda t}$	$e^{\lambda t}$
$\beta > 0$	$d \geq 3,\ \beta_c > 0$	Sp. $\beta > \beta_c$	$e^{\lambda t}$	$e^{\lambda t}$
$\beta > 0$	$d = 3,\ \beta_c > 0$	Cr. $\beta = \beta_c$	$1/\sqrt{t}$	$\sqrt{t}$
$\beta > 0$	$d = 4,\ \beta_c > 0$	Cr. $\beta = \beta_c$	$1/\ln t$	$t/\ln t$
$\beta > 0$	$d \geq 5,\ \beta_c > 0$	Cr. $\beta = \beta_c$	1	t
$\beta > 0$	$d \geq 3,\ \beta_c > 0$	Sb. $\beta_c > \beta > 0$	$t^{-d/2}$	1
$\beta = 0$	$d = 1,\ \beta_c = 0$	Cr. $\beta = \beta_c$	$1/\sqrt{t}$	1
$\beta = 0$	$d = 2,\ \beta_c = 0$	Cr. $\beta = \beta_c$	$1/t$	1
$\beta = 0$	$d \geq 3,\ \beta_c > 0$	Sb. $\beta_c > \beta = 0$	$t^{-d/2}$	1
$\beta < 0$	$d = 1,\ \beta_c = 0$	Sb. $\beta < \beta_c$	$t^{-3/2}$	$1/\sqrt{t}$
$\beta < 0$	$d = 2,\ \beta_c = 0$	Sb. $\beta < \beta_c$	$(t \ln^2 t)^{-1}$	$1/\ln t$
$\beta < 0$	$d \geq 3,\ \beta_c > 0$	Sb. $\beta < \beta_c$	$t^{-d/2}$	1

Let

$$\bar{G}_\lambda(x,y) := \int_0^\infty e^{-\lambda t}\bar{p}(t,x,y)\,dt, \qquad \bar{\beta}_c := 1/\bar{G}_0(0,0).$$

Theorem. CBRW (2010)

The spectrum of the operator $\bar{H}$ lies on the real line , and all its positive points (provided that such points exist) are isolated eigenvalues.
In addition, the operator $\bar{H}$ has one positive eigenvalue $\bar{\lambda}$, where the eigenvalue $\bar{\lambda}$ is determined from the equation $\beta\,\bar{G}_\lambda(0,0) = 1$, if and only if the condition $\beta > \bar{\beta}_c$ holds.

Theorem. Asymptotics of local moments for CBRW (2010)

If $\bar\beta > \bar\beta_c$ and $s = 1 - \bar a(0)(\alpha - 1)^{-1}$, then

$$m_n(t;x,y) \sim \bar C_n(x,y)e^{n\bar\lambda t}, \quad t \to \infty,$$

where

$$\bar C_1(x,0) = \frac{(1-s)\bar G_{\bar\lambda}(x,0)\,\bar G_{\bar\lambda}}{\|\bar G_{\bar\lambda}(\cdot,0)\|^2 - s\bar G_{\bar\lambda}^2}, \qquad \bar C_1(x,y) = \frac{\bar G_{\bar\lambda}(x,0)\,\bar G_{\bar\lambda}(y,0)}{\|\bar G_{\bar\lambda}(\cdot,0)\|^2 - s\bar G_{\bar\lambda}^2}, \quad y \neq 0,$$

$$\bar C_n(x,y) = g_n(\bar C_1(0,y), \bar C_2(0,y), \dots, \bar C_{n-1}(0,y))\frac{\bar G_{n\bar\lambda}(x,0)}{1 - \bar\beta \bar G_{n\bar\lambda}}, \quad n \geq 2,$$

and functions g_n are defined by

$$g_n(\bar C_1, \bar C_2, \dots, \bar C_{n-1}) = \sum_{r=2}^{n} \frac{\beta_r}{r!} \sum_{\substack{i_1,\dots,i_r>0\\ i_1+\dots+i_r=n}} \frac{n!}{i_1!\dots i_r!}\bar C_{i_1}\dots\bar C_{i_r}.$$

Theorem. Asymptotics of total moments for CBRW (2010)

If $\bar\beta > \bar\beta_c$ and $s = 1 - \bar a(0)(\alpha - 1)^{-1}$, then

$$m_n(t;x) \sim \bar C_n(x) e^{n\bar\lambda t}, \quad t \to \infty,$$

where

$$\bar C_1(x) = \frac{(1-s)\bar G_{\bar\lambda}(x,0)}{\bar\lambda\left(\|\bar G_{\bar\lambda}(\cdot,0)\|^2 - s\bar G^2_{\bar\lambda}\right)},$$

$$\bar C_n(x) = g_n(\bar C_1(0), \bar C_2(0), \ldots, \bar C_{n-1}(0))\frac{\bar G_{n\bar\lambda}(x,0)}{1-\bar\beta\bar G_{n\bar\lambda}}, \quad n \geq 2.$$

and functions g_n are defined (as above) by

$$g_n(\bar C_1, \bar C_2, \ldots, \bar C_{n-1}) = \sum_{r=2}^{n} \frac{\beta_r}{r!} \sum_{\substack{i_1,\ldots,i_r>0\\ i_1+\ldots+i_r=n}} \frac{n!}{i_1!\ldots i_r!} \bar C_{i_1} \ldots \bar C_{i_r}.$$

Lemma. Connections between $\bar{C}_n(x,y)$ and $\bar{C}_n(x)$ (2010)

If $\bar{\beta} > \bar{\beta}_c$ and $s = 1 - \bar{a}(0)(\alpha - 1)^{-1}$, then

$$\bar{C}_n(x,y) = \psi^n_{\bar{\lambda}}(y)\bar{C}_n(x) \quad \text{for every} \quad n \in \mathbf{N},$$

where

$$\psi_{\bar{\lambda}}(y) = \begin{cases} \bar{\lambda}\,\bar{G}_{\bar{\lambda}} & \text{for} \quad y = 0, \\ \bar{\lambda}(1-s)^{-1}\bar{G}_{\bar{\lambda}}(0,y) & \text{for} \quad y \neq 0. \end{cases}$$

Theorem. CBRW (2010)

If $\beta > \bar{\beta}_c$ then, in the sense of convergence of all moments,

$$\lim_{t\to\infty} \mu_t(y) e^{-\bar{\lambda} t} = \bar{\xi} \psi_{\bar{\lambda}}(y), \quad \lim_{t\to\infty} \mu_t e^{-\bar{\lambda} t} = \bar{\xi},$$

where $\bar{\xi}$ is a non-degenerate random variable such that

$$E_x \bar{\xi}^n = \bar{C}_n(x)$$

and functions $\bar{C}_n(x)$ and $\psi_{\bar{\lambda}}(y)$ are determined above.

Moreover, under the condition $\beta_n \leq O(n! n^{n-1})$ the moments $\bar{C}_n(x)$ uniquely determine the distribution of $\bar{\xi}$, so that the results are also valid in the sense of convergence in distribution.

Three Types of the Sources in BRW with a Few Sources

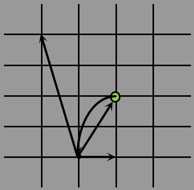
I type: SBRW. The sources with branching. BRW/r/0/0. The sources are situated at the points $z_s \in \mathbf{Z^d}$, $s = 1, \ldots, r$. The RW is assumed to be **symmetrical**. In particular, $a(z_s, y) = a(y, z_s)$.

II type: CBRW. The sources with branching. BRW/0/k/0. The sources are situated at the points $x_i \in \mathbf{Z^d}$, $i = 1, \ldots, k$. **An additional parameter** is introduced to intensify artificially the prevalence of branching or walk at the source. As a side effect, **it violates symmetry** of the random walk: $a(x_i, y) \neq a(y, x_i)$.

III type: CRW. The "sources" without branching. BRW/0/0/m. The pseudo "sources" are situated at the points $y_j \in \mathbf{Z^d}$, $j = 1, \ldots, m$. **An additional parameter** breakes symmetry of RW at every source y_j.

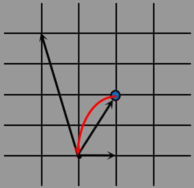
SBRW

BRW/1/0/0: $A+\beta\Delta_0$



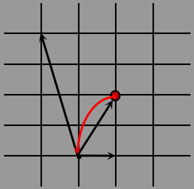
CBRW

BRW/0/1/0: $A+\zeta A\Delta_0+\eta\Delta_0$

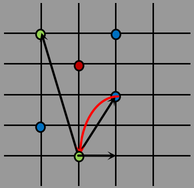


CRW

BRW/0/0/1: $A+\chi A\Delta_0$



BRW/2/3/1: $A+\beta_1\Delta_0+\beta_2\Delta_0+\zeta_1 A\Delta_0+\zeta_2 A\Delta_0+\zeta_3 A\Delta_0+\eta_1\Delta_0+\eta_2\Delta_0+\eta_3\Delta_0+\chi_1 A\Delta_0$



Generalization of BRW models

Three types of the sources in BRW/r/k/m :

$\{z_s\}_{s=1}^{r}$ are the sources of branching as in SBRW,

$\{x_i\}_{i=1}^{k}$ are the sources of branching as in CBRW,

$\{y_j\}_{j=1}^{m}$ are the points of $\mathbf{Z^d}$ where the symmetry of the random walk is broken.

The investigation of long-time behavior for the first moments $m_1(t,x,y)$ and $m_1(t)$ in BRW/r/k/m/ is based on spectral properties of the operator

$$Y = A + \sum_{i=1}^{k} \zeta_i \Delta_{x_i} A + \sum_{i=1}^{k} \eta_i \Delta_{x_i} + \sum_{j=1}^{m} \chi_j \Delta_{y_j} A + \sum_{s=1}^{r} \beta_s \Delta_{z_s}.$$

Theorem

For each $1 \leq p \leq \infty$, the operator $Y : l^p(\mathbf{Z^d}) \to l^p(\mathbf{Z^d})$ is bounded. All the points of the spectrum of Y lying outside of the circle $\{z \in \mathbf{C} : |z - a(0)| \leq |a(0)|\}$, if any, are isolated positive eigenvalues of finite multiplicity.

Theorem (BRW/r/k/m). *Let R_λ be the resolvent of A and $(\cdot,\cdot)$ be the Euclidean scalar product in $\mathbb{R}^d$. A number λ is an eigenvalue of the operator $Y : l^2(\mathbf{Z^d}) \to l^2(\mathbf{Z^d})$ iff the numbers $\{X_i\}_{i=1}^k$, $\{Y_j\}_{j=1}^m$ and $\{Z_s\}_{s=1}^r$ form a nontrivial set of solutions of the following system of linear equations.*

First Set of Equations:

$$X_{\bar{i}} + \sum_{i=1}^{k} \left(\zeta_{\bar{i}} \left(A\delta_{x_{\bar{i}}}, R_\lambda \delta_{x_i} \right) + \eta_{\bar{i}} \left(\delta_{x_{\bar{i}}}, R_\lambda \delta_{x_i} \right) \right) X_i$$

$$+ \sum_{j=1}^{m} \left(\zeta_{\bar{i}} \left(A\delta_{x_{\bar{i}}}, R_\lambda \delta_{y_j} \right) + \eta_{\bar{i}} \left(\delta_{x_{\bar{i}}}, R_\lambda \delta_{y_j} \right) \right) Y_j$$

$$+ \sum_{s=1}^{m} \left(\zeta_{\bar{i}} \left(A\delta_{x_{\bar{i}}}, R_\lambda \delta_{z_s} \right) + \eta_{\bar{i}} \left(\delta_{x_{\bar{i}}}, R_\lambda \delta_{z_s} \right) \right) Z_s = 0, \quad \bar{i} = 1, 2, \dots, k.$$

Theorem (BRW/r/k/m). *Let R_λ be the resolvent of A and $(\cdot,\cdot)$ be the Euclidean scalar product in $\mathbb{R}^d$. A number λ is an eigenvalue of the operator $Y : l^2(\mathbf{Z}^\mathbf{d}) \to l^2(\mathbf{Z}^\mathbf{d})$ iff the numbers $\{X_i\}_{i=1}^k$, $\{Y_j\}_{j=1}^m$ and $\{Z_s\}_{s=1}^r$ form a nontrivial set of solutions of the following system of linear equations.*

First Set of Equations:

$$X_{\bar{i}} + \sum_{i=1}^{k} \left(\zeta_{\bar{i}} \left(A\delta_{x_{\bar{i}}}, R_\lambda \delta_{x_i} \right) + \eta_{\bar{i}} \left(\delta_{x_{\bar{i}}}, R_\lambda \delta_{x_i} \right) \right) X_i$$

$$+ \sum_{j=1}^{m} \left(\zeta_{\bar{i}} \left(A\delta_{x_{\bar{i}}}, R_\lambda \delta_{y_j} \right) + \eta_{\bar{i}} \left(\delta_{x_{\bar{i}}}, R_\lambda \delta_{y_j} \right) \right) Y_j$$

$$+ \sum_{s=1}^{m} \left(\zeta_{\bar{i}} \left(A\delta_{x_{\bar{i}}}, R_\lambda \delta_{z_s} \right) + \eta_{\bar{i}} \left(\delta_{x_{\bar{i}}}, R_\lambda \delta_{z_s} \right) \right) Z_s = 0, \quad \bar{i} = 1, 2, \ldots, k.$$

Theorem (BRW/r/k/m). *Let R_λ be the resolvent of A and $(\cdot,\cdot)$ be the Euclidean scalar product in $\mathbb{R}^d$. A number λ is an eigenvalue of the operator $Y : l^2(\mathbf{Z^d}) \to l^2(\mathbf{Z^d})$ iff the numbers $\{X_i\}_{i=1}^k$, $\{Y_j\}_{j=1}^m$ and $\{Z_s\}_{s=1}^r$ form a nontrivial set of solutions of the following system of linear equations.*

Second Set of Equations:

$$Y_{\bar{j}} + \sum_{i=1}^{k} \chi_{\bar{j}} \left(A\delta_{y_{\bar{j}}}, R_\lambda \delta_{x_i}\right) X_i + \sum_{j=1}^{m} \chi_{\bar{j}} \left(A\delta_{y_{\bar{j}}}, R_\lambda \delta_{y_j}\right) Y_j + \sum_{s=1}^{m} \chi_{\bar{j}} \left(A\delta_{y_{\bar{j}}}, R_\lambda \delta_{z_s}\right) Z_s = 0, \quad \bar{j} = 1,2,\ldots,m.$$

Theorem (BRW/r/k/m). *Let R_λ be the resolvent of A and $(\cdot,\cdot)$ be the Euclidean scalar product in $\mathbb{R}^d$. A number λ is an eigenvalue of the operator $Y : l^2(\mathbf{Z^d}) \to l^2(\mathbf{Z^d})$ iff the numbers $\{X_i\}_{i=1}^k$, $\{Y_j\}_{j=1}^m$ and $\{Z_s\}_{s=1}^r$ form a nontrivial set of solutions of the following system of linear equations.*

Third Set of Equations:

$$Z_{\bar{s}} + \sum_{i=1}^{k} \beta_{\bar{s}} \left(\delta_{z_{\bar{s}}}, R_\lambda \delta_{x_i}\right) X_i + \sum_{j=1}^{m} \beta_{\bar{s}} \left(\delta_{z_{\bar{s}}}, R_\lambda \delta_{y_j}\right) Y_j + \sum_{s=1}^{m} \beta_{\bar{s}} \left(\delta_{z_{\bar{s}}}, R_\lambda \delta_{z_s}\right) Z_s = 0, \quad \bar{s} = 1, 2, \ldots, r.$$

A Limit Behavior of the Number of Particles

From the above Theorem it follows that in the case when the spectrum of the operator Y contains a maximal isolated positive eigenvalue λ both the local numbers of particles and their total number grow exponentially at infinity:

$$\lim_{t\to\infty} \mu_t(y)\, e^{-\lambda t} = \xi\, \psi(y), \quad \lim_{t\to\infty} \mu_t\, e^{-\lambda t} = \xi,$$

where $\psi(y)$ is a function and ξ is a nondegenerate random variable.

Example

For the operator $Y = A + \chi\Delta_y A + \beta\Delta_z$ the system takes the form

$$Y + \chi(A\delta_y, R_\lambda \delta_y)Y + (A\delta_y, R_\lambda \delta_z)Z = 0,$$
$$Z + \beta(\delta_z, R_\lambda \delta_y) + \beta(\delta_z, R_\lambda \delta_z) = 0.$$

The system of linear equations has a nontrivial solution if its determinant is equal zero:

$$(1 + \chi(A\delta_y, R_\lambda \delta_y))(1 + \beta(\delta_z, R_\lambda \delta_y)) - \beta\chi(A\delta_y, R_\lambda \delta_y)(\delta_z, R_\lambda \delta_z) = 0.$$

Example. The scalar products

The scalar products in this system of linear equations are of the form:

$$(A\delta_y, R_\lambda \delta_y) = 1 - \frac{\lambda}{(2\pi)^d} \int\limits_{[-\pi,\pi]^d} \frac{1}{\lambda - \phi(\theta)}\, d\theta,$$

$$(\delta_z, R_\lambda \delta_z) = -\frac{1}{(2\pi)^d} \int\limits_{[-\pi,\pi]^d} \frac{1}{\lambda - \phi(\theta)}\, d\theta,$$

$$(A\delta_y, R_\lambda \delta_z) = \delta(y-z) - \frac{\lambda}{(2\pi)^d} \int\limits_{[-\pi,\pi]^d} \frac{\cos(y-z,\theta)}{\lambda - \phi(\theta)}\, d\theta,$$

$$(\delta_z, R_\lambda \delta_y) = -\frac{\lambda}{(2\pi)^d} \int\limits_{[-\pi,\pi]^d} \frac{\cos(y-z,\theta)}{\lambda - \phi(\theta)}\, d\theta.$$

BRW in Random Media

The Anderson Hamiltonian. Random Media

Much attention in the theory of random media has been devoted to the study of spectral properties of the operator

$$\varkappa\Delta + V(x), \quad \varkappa > 0,$$

where Δ (as above) is the discrete Laplacian on $\mathbf{Z}^{\mathbf{d}}$ acting in variable x as

$$\Delta\psi(x) = \frac{1}{2d} \sum_{|x'-x|=1} \psi(x') - \psi(x),$$

and where the potential $V(x) = V(x, \omega)$, $x \in \mathbf{Z}^{\mathbf{d}}$, $d \geq 1$, is a random function determined by the random branching medium.

The Parabolic Anderson Problem. Random Media

If in BRW

- the transport of particles is governed by the law of **a simple SRW**,
- the random branching environment is defined by **random** birth and death intensities **at every lattice point**,

then **the expected total number of particles** (the first order moment) satisfies the Cauchy problem with a random potential:

$$\partial_t m_1(t,x) = \varkappa \Delta m_1(t,x) + V(x) m_1(t,x), \quad m_1(0,x) \equiv 1.$$

Here $\partial_t := \partial / \partial t$ stands for the partial derivative with respect to the time t.

Intermittency Phenomenon

Mathematical theory of intermittency in random environments was developed by Zeldovich, Molchanov, Gärtner, Carmona et al.

It has been discovered that the evolution of the field $m_1(t,x)$ leads to the formation of highly irregular spatio-temporal structures, characterized by **the generation of rare high peaks on a low-profile background**.

The Study of Intermittency

The study of intermittency in the works of J. Gärtner, S. Molchanov are based on **asymptotic analysis of the moments** $\langle m_1 \rangle$ obtained by averaging the random moment m_1 over medium's realizations, where the angular brackets $\langle \cdot \rangle$ denote expectation with respect to the random environment.

For instance, the second moments grow much faster than the squared first moments, the fourth moments behave in the same way with respect to the squared second moments, and so on:

$$\langle m_1^2 \rangle \gg \langle m_1 \rangle^2, \quad \langle m_1^4 \rangle \gg \langle m_1^2 \rangle^2, \quad \dots .$$

Main Objectives. Preliminary Remarks

The evolution of the mean number of particles m_1 in a **nonhomogeneous random environment** is determined by the operator

$$A + V(0)\Delta_0,$$

where the RW generator A is a bounded self-adjoint operator in $l^2(\mathbf{Z^d})$ and $\Delta_0 = \delta_0 \delta_0^T$, while $V(0)$ is a random variable characterizing the source intensity.

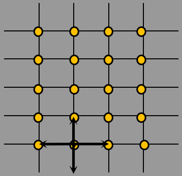
Objectives

1. to extend the results obtained earlier for the discrete Laplacian $\varkappa\Delta$ in the model of BRW in a spatially homogeneous branching random environment to a wider class of symmetric RW with the RW generator A, in particular to solve the Cauchy problem for the operator $A+V(x)$,

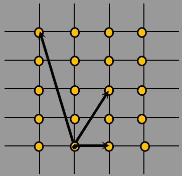
Objective 1. Random **Homogeneous** Environment. Generalization for SBRW

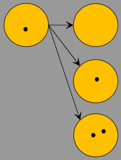
$\mathbf{Z}^d$, $d \geq 1$, **t=0**

$\kappa\Delta_0 + V(\cdot)$



$A + V(\cdot)$





``death'' or ``jump''

nothing

``birth''

Objectives

2 to study the long-time behavior of the moments $\langle m_n^p \rangle$ ($p \geq 1$, $n \geq 1$) for the local and total particle populations for BRW in a nonhomogeneous branching random environment, in particular to solve the Cauchy problem for the operator $A + V(0)\Delta_0$,

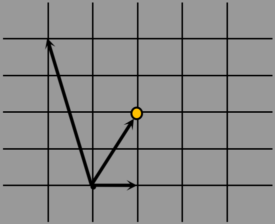
3 to determine **conditions** enabling the long-time behavior of the moments $\ln \langle m_n^p \rangle$ for the numbers of particles at an arbitrary site of the lattice and on the entire lattice to coincide for both models of BRW in spatially **homogeneous** and **nonhomogeneous** random environments,

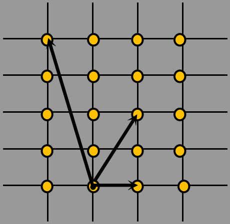
4 to construct examples where the distributions of the random potential V satisfy these **conditions**.

Objectives 2-3. Random Nonhomogeneous and Homogeneous Environments.

$\mathbf{Z}^d$, $d \geq 1$, $t=0$

SBRW: $A+V(0)\Delta_0$ $A+V(\cdot)$



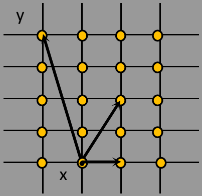


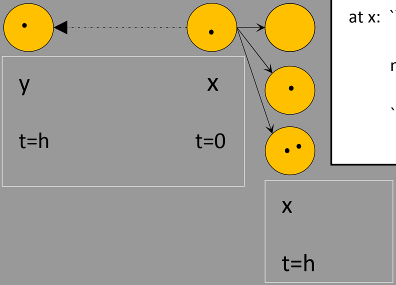
BRW in Homogeneous Random Environments

Suppose now that a branching random environment is formed by pairs of non-negative random variables, $\xi(x) := (\xi^-(x), \xi^+(x))$, $x \in \mathbf{Z}^d$, defined on a probability space $(\Omega, \mathscr{F}, \mathbf{P})$. The sample point $\omega \in \Omega$ represents sample realizations of the field $\xi(\cdot)$. In particular, we can assume that $\Omega = (\mathbf{R}_+^2)^{\mathbf{Z}^\mathbf{d}}$. The expectation with respect to the probability measure $\mathbf{P}$ will be denoted by angular brackets, $\langle\cdot\rangle$. We assume that the random field ξ is spatially homogeneous, that is, the distribution $\mathbf{P}$ of the field is invariant with respect to translations $x \mapsto x + y$, $x, y \in \mathbf{Z}^\mathbf{d}$ (see, e.g., S. Albeverio, L. Bogachev, S. Molchanov and E. Yarovaya (2000)).

BRW in a Random **Homogeneous** Environment

$\mathbf{Z}^d$, $d \geq 1$, $A+V(\cdot)$, $t=0$





at x: ``death'' $\xi^-(x)h+o(h)$ or ``jump'' $a(x,y)h+o(h)$

nothing $1+a(0)h-(\xi^+(x)+\xi^-(x))h+o(h)$

``birth'' $\xi^+(x)h+o(h)$

Homogeneous Environment. The Moments Equations

Let

$$V(x) := \xi^+(x) - \xi^-(x), \quad x \in \mathbf{Z^d}.$$

Then the moment functions $m_n(t,x,y)$, $m_n(t,x)$ satisfy the chain of linear differential equations

$$\begin{aligned} \partial_t m_1 &= A m_1 + V(x) m_1, \\ \partial_t m_n &= A m_n + V(x) m_n + \xi^+(x) g_n[m_1,\ldots,m_{n-1}], \quad n = 1,2,\ldots \end{aligned}$$

with the initial conditions $m_n(0,\cdot,y) = \delta_y(\cdot)$ and $m_n(0,\cdot) \equiv 1$, where

$$g_n[m_1,\ldots,m_{n-1}] := \sum_{i=1}^{n-1} \binom{n}{i} m_i m_{n-i}, \quad n \geq 2.$$

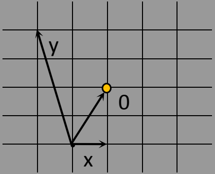
BRW in Nonhomogeneous Random Environments

Suppose now that a branching random environment is formed by only the one pair of non-negative random variables, $\xi(0) := (\xi^{-}(0), \xi^{+}(0))$ defined on a probability space $(\Omega, \mathscr{F}, \mathbf{P})$. It is assumed that $\Omega = \mathrm{R}_{+}^{2}$. In this case, the random environment is spatially non-homogeneous, since the branching medium formed of birth-and-death process only at the origin of the lattice.

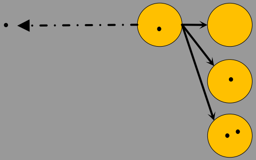
BRW in a Random **Nonhomogeneous** Environment

$\mathbf{Z}^d$, $d \geq 1$, $t=0$

$A+V(0)\Delta_0$



At the origin:



Nonhomogeneous Environment

Let

$$V(0) := \xi^{+}(0) - \xi^{-}(0).$$

Here, the first-order moments satisfy the homogeneous equation in operator form

$$\frac{dm_1}{dt} = Am_1 + V(0)\Delta_0 m_1,$$

where $\Delta_0 = \delta_0 \delta_0^T$.

Evolutionary Operators of BRW in Random Environments

Simple SBRW in a Homogeneous Random Environment:

$$\varkappa\Delta + V(x)$$

$$\Downarrow$$

SBRW in a Homogeneous Random Environment:

$$A + V(x)$$

$$\Downarrow$$

SBRW in a Nonhomogeneous Random Environment: $A + V(0)\Delta_0$

Kolmogorov's backward equation

Suppose that x_t is a continuous-time "jumping" trajectory of a continuous-time symmetric random walk on $\mathbf{Z^d}$ with the generator A, and E_x is the expectation under the condition that the random walk starts from x.

Theorem (Kolmogorov's backward equation)

Define $p(t,x,y) = E_x\delta_y(x_t)$*. Then* $p(t,\cdot,y) \in l^2(\mathbf{Z^d})$ *for each* $t > 0$ *and*

$$\partial p_t = Ap, \quad p(0,x,y) = \delta_y(x),$$

where the right-hand side is interpreted as a linear operator A applied to the function $x \mapsto p(t,x,y)$ *by the formula:*
$Ap(t,x,y) = \sum_{x'} a(x,x')p(t,x',y)$*.*
Moreover, if $p^*(t,x,y)$ *satisfies the Cauchy problem* (2)*, then* $p^*(t,x,y) = p(t,x,y)$ *with* $p(t,x,y) = E_x\delta_y(x_t)$*.*

Theorem (J. Gärtner and S. Molchanov, 1990)

Assume that $V(x)$ i.i.d. Then the Cauchy problem has a unique non-negative solution if

$$\left\langle \left(\frac{V(0)}{\ln_+ V(0)} \right)^d \right\rangle < \infty, \tag{1}$$

where $\ln_+ V(0) := \ln \max(V(0), e)$.

Theorem (Homogeneous Random Environment)

Assume that (1) *holds and*

$$m_1(t,x,y) = E_x\left[\exp\left(\int_0^t V(x_s)\,ds\right)\delta_y(x_t)\right],$$

$$m_1(t,x) = E_x\left[\exp\left(\int_0^t V(x_s)\,ds\right)\right].$$

Then $m_1(t,x,y)$ *and* $m_1(t,x)$ **P**-*a.s. satisfy the Cauchy problem:* $\partial_t m_1 = Am_1 + V(x)m_1$, *with the initial conditions* $m_n(0,\cdot,y) = \delta_y(\cdot)$ *and* $m_1(0,\cdot) \equiv 1$, *respectively.*

Theorem (Nonhomogeneous Random Environment)

Assume that (1) *holds for* $V(0)$ *and*

$$\begin{aligned} m_1(t,x,y) &= E_x\left[\exp\left(V(0)\int_0^t \delta_0(x_s)\,ds\right)\delta_y(x_t)\right], \\ m_1(t,x) &= m_1(t,x) = E_x\left[\exp\left(V(0)\int_0^t \delta_0(x_s)\,ds\right)\right]. \end{aligned}$$

Then $m_1(t,x,y)$ *and* $m_1(t,x)$ $\mathbf{P}$*-a.s. satisfy the Cauchy problem:* $\partial_t m_1 = Am_1 + V(0)\Delta_0 m_1$, *with the initial conditions* $m_n(0,\cdot,y) = \delta_y(\cdot)$ *and* $m_1(0,\cdot) \equiv 1$, *respectively.*

Preliminary Remarks

Now we are able to give our main result on the long-time behavior of the moments $\langle m_n^p \rangle$ where $n \in \mathbf{N}$, $p \geq 1$. Under the assumption that the analyzed Cauchy problems **P**-a.s. have a unique non-negative solutions, the following theorem holds.

Theorem (**Homogeneous and Nonhomogeneous** Random Environments (2010))

Let $V := V(0)$. Assume that

$$\lim_{t\to\infty} \frac{t}{\ln\langle e^{Vt}\rangle} = 0.$$

Then for all integer moments $\langle m_n^p\rangle$, where m_n is the solution of the Cauchy problems for BRW in ***homogeneous or nonhomogeneous*** *random environments) with the initial conditions $m_n(0,\cdot,y) = \delta_y(\cdot)$ and $m_n(0,\cdot) \equiv 1$, respectively, we obtain*

$$\lim_{t\to\infty} \frac{\ln\langle m_n^p\rangle}{\ln\langle e^{pnVt}\rangle} = 1.$$

Conclusion

In this way, condition

$$\lim_{t\to\infty} \frac{t}{\ln\left\langle e^{Vt}\right\rangle} = 0 \qquad (2)$$

appears ensuring that the long-time behavior of the moments $\langle m_n^p \rangle$, $n \geq 1$, for the numbers of particles at an arbitrary site of the lattice and on the entire lattice coincide for both models of BRW in spatially **homogeneous and nonhomogeneous random** environments.

Nonhomogeneous non-random environments

If the spectrum of the operator $A+\beta\Delta_0$ contains a maximum eigenvalue $\lambda>0$, then both the local numbers of particles and their total number grow exponentially as $t\to\infty$:

$$\lim_{t\to\infty}\mu_t(y)\,e^{-\lambda t}=\xi\,\psi(y),\quad \lim_{t\to\infty}\mu_t\,e^{-\lambda t}=\xi. \tag{3}$$

Here, $\psi(y)$ is a function and ξ is a non-degenerate random variable. This case is referred to as supercritical. Relations (3) hold in the sense of convergence in distribution. In particular, for the first moment if $\beta>G_0^{-1}(0,0)$, then for $n\in\mathbf{N}$, as $t\to\infty$,

$$m_1(t,x,y)\sim C_1(x,y)e^{\lambda t},\quad m_1(t,x)\sim C_1(x)e^{\lambda t}.$$

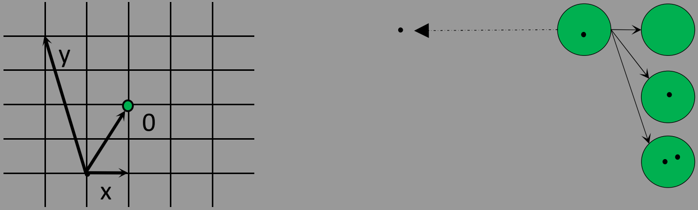
Nonhomogeneous non-random environments

Hence for supercritical BRW in an **nonhomogeneous non-random** environment the analog of (2) has the form

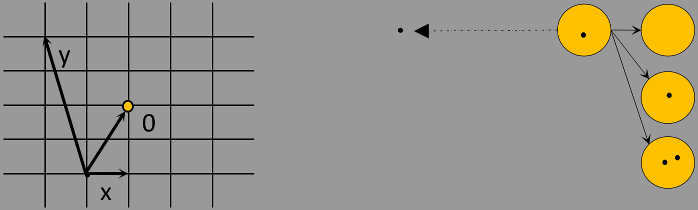
$$\lim_{t\to\infty} \frac{t}{\ln m_1} = \lim_{t\to\infty} \frac{t}{\ln e^{\lambda t}} = \frac{1}{\lambda}$$

and (2) is not valid. The validity of the condition (2) means that distribution of the potential V has **the heavier tail then exponential.**

Nonhomogeneous non-random environment: $A+\beta\Delta_0$



Nonhomogeneous random environment: $A+V(0)\Delta_0$



"Heavy Tails" of Distributions

In this section, we construct examples of distributions of the random potential V satisfying condition

$$\lim_{t\to\infty} \frac{t}{\ln\langle e^{Vt}\rangle} = 0.$$

Distributions with "heavy tails" have numerous applications in the catastrophe theory. It can partially be explained by the fact that their tails decay more slowly than any exponential tail. Therefore these distributions are often used to model disasters and other rare events.

Example: Weibull-type upper tail

We begin with a theorem where the tail of the distribution of the branching potential V has a Weibull type upper tail:

$$\ln \mathbf{P}\{V > r\} \sim -cr^{\gamma}, \quad \gamma > 1, c > 0. \quad r \to \infty, \tag{4}$$

Theorem

Under assumption (4), *we have for every* $p \geq 1$

$$\ln \left\langle e^{pVt} \right\rangle \sim (\gamma - 1) \left(\frac{pt}{\gamma c^{1/\gamma}} \right)^{\gamma/(\gamma-1)}, \quad t \to \infty.$$

Condition (2) *also holds in this case.*

If $\gamma = 2$, we have an immediate corollary for the case where the upper tail is of **Gaussian type**.

Example: Gumbel-type upper tail

The upper tail of the distribution of the branching potential V has the following form:

$$\ln \mathbf{P}\{V > r\} \sim -\exp(r/c), \quad c > 0. \quad r \to \infty, \tag{5}$$

Theorem

Under assumption (5), *we have for every* $p \geq 1$

$$\ln \left\langle e^{pVt} \right\rangle \sim cpt \ln t, \quad t \to \infty.$$

Condition (2) *also holds in this case.*

Albeverio, S., Bogachev, L., Yarovaya, E. (1998): Asymptotics of branching symmetric random walk on the lattice with a single source. *Comptes Rendus de l'Academie des Sciences, Paris, series I.*, *326*, 975–980.

Bogachev, L., Yarovaya, E. (1998): Moment analysis of a branching random walk on a lattice with a single source. *Doklady Akademii Nauk*, *363*, 439–442.

Bogachev, L., Yarovaya, E. (1998): A limit theorem for supercritical branching random walk on $\mathbf{Z}^{\mathbf{d}}$ with a single source. *Russian Math. Surveys*, *53*, 1086–1088.

Cranston, M., Molchanov, S. (2007): On Phase Transitions and Limit Theorems for Homopolymers. *CRM Proceedings and Lecture Notes*, *42*, 97–113.

Vatutin, V.A., Topchii, V.A., Yarovaya, E.B. (2003): Catalytic branching random walk and queueing systems with random number of independent servers. *Teoriya Jmovirnostej ta Matematichna Statistika 69*, 158–172.

Yarovaya, E. (1991): Use of the spectral methods to study branching processes with diffusion in a noncompact phase space. *Theoretical and Mathematical Physics*, *88*, 25–30.

Yarovaya, E. (2007): *Branching Random Walks in Nonhomogeneous Medium.* MSU, Moscow.

Yarovaya, E. (2009): On investigations of branching random walks on multidimensional lattices. In: A. Shiryaev (Ed.): *Contemporary Problems of Mathematics and Mechanics.* Moscow, MSU, *4*, 119–136.

Albeverio, S., Bogachev, L., Molchanov, S., and Yarovaya, E., "Annealed Moment Lyapunov Exponents for a Branching Random Walk in a Homogeneous Random Branching Environment" ", *Markov Processes and Related Fields* 6, 473–516 (2000).

Carmona, R.,and Lacroix J., "Spectral Theory of Random Schroedinger Operators", *Probability and Its Applications* Birkhäuser Verlag, Basel (1990).

Gärtner, J., König, W., and Molchanov S., "Geometric characterization of intermittency in the parabolic Anderson model", *The Annals of Probability*, 35, 339–499 (2007).

Gärtner, J., and Molchanov S., "Parabolic problems for the Anderson model. I: Intermittency and related topics", *Commun. Math. Phys.*, 132, 613–655 (1990).

Molchanov, S., "Lectures on random media", *Lect. Notes Math.* 1581, 242–411 (1994).

Yarovaya, E., "Intermittency of leading moments in the model of branching process with diffusion in a random medium", *Mosc. Univ. Math. Bull.*, 45, 49–51 (1990).

Zeldovich, Ya., Molchanov, S., Ruzmaikin, A., Sokoloff, D., "Intermittency, diffusion and generation in a nonstationary random medium" *Sov. Sci. Rev., Sect. C: Math. Phys. Rev.* 7, 1–110.