Limit Distributions Arising in Critical Branching Random Walks on $\mathbb{Z}^d$

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Outline

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2. Description of the model
3. Criticality of CBRW
4. Main results
5. Sketch of the proofs

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Introduction

Branching random walk (BRW) is a branching process where a particle population evolves in a certain metric space and the distribution of distances between the locations of offsprings and that of the parent particle, immediately upon branching, is given.

J.Barral, I.Benjamini, J.Biggins,
M.Birkner, T.Boidecki, M.Cranston,
F. den Hollander, P.Milos, S.Molchanov,
S.M$\ddot{u}$ller, S.Popov, V.Topchii, V.Vatutin,
E.Yarovaya, N.Yoshida, I.Z$\ddot{a}$hle, . . .

BRW on a graph, on $\mathbb{Z}^d$, in $\mathbb{R}^d$,
BRW with different types of particles,
State dependent BRW, with immigration,
BRW with random landscape,
BRW in random environment, . . .

R.A.Carmona, J.Gärtner,
S.A.Molchanov et al. (1990)
BRW in Random Environment

It turned out that the main contribution to the average number of particles on $\mathbb{Z}^d$ at time t comes from few small remote islands.

These islands are randomly located, dependent on t, not too far from the origin.

BRW with a finite number of sources – an approximation of the last model

BRW with a single source of branching – the simplest solvable model

E.B.Yarovaya (1991,. . .,2010)
symmetric BRW (SBRW)

V.A.Vatutin, V.A.Topchii, E.B.Yarovaya (2003)
catalytic BRW (CBRW)

Some applications

Approximation of the catalytic super-Brownian motion with a single point of catalyst
I.Kaj and S.Sagitov (1998)

Application to the queueing system with a random number of independent servers
V.A.Vatutin, V.A.Topchii and E.B.Yarovaya (2003)

Description of the model

- At the origin after exponentially distributed time (with parameter 1) a particle either branches or leaves the origin with probabilities α and $1 - \alpha$ respectively
- Outside the origin a particle performs a continuous time random walk on $\mathbb{Z}^d$ with an infinitesimal transition matrix A until it hits the origin

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Our assumptions

(I) At the initial time $t = 0$ there is a single particle at point $x \in \mathbb{Z}^d$

(II) When branching a particle produces children according to an offspring generating function

$$f(s) = \sum_{k=1}^{\infty} f_k s^k, \ 0 \le s \le 1,$$

$$\boxed{\alpha f'(1) + (1 - \alpha)(1 - h_d) = 1}$$

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(III) The underlying random walk with generator $A = (a(x,y))_{x,y\in\mathbb{Z}^d}$ is symmetric, homogeneous and irreducible, i.e.

$$a(x,y) = a(y,x) \stackrel{def}{=} a(y-x),$$

$$\sum_{y\in\mathbb{Z}^d} a(y) = 0,$$

$$a(y) \geq 0 \text{ if } y \neq 0, \quad a(0) < 0,$$

moreover, $\sum_{y\in\mathbb{Z}^d} \|y\|^2 a(y) < \infty.$

(IV) When a particle leaves the origin, the probability of its jump to point $y \neq 0$ equals

$$-(1-\alpha)a(y)a^{-1}(0)$$

(V) At the birth moment the newborn particles are located at the origin. They evolve according to the scheme described above independently of each other as well as of the parent particles

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Necessary notation

- μ_t is the total number of particles in the process at time t.
- $\mu_t(0)$ is the number of particles at the origin at time t.
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Criticality of CBRW

1. Bellman-Harris branching process

V.A.Vatutin, V.A.Topchii (2010)

τ_1 is the time spent by the particle at the origin until it leaves the origin

$G_1(t) = \mathsf{P}(\tau_1 \le t) = 1 - e^{-(1-\alpha)t}$

τ_2 is the time spent by the particle outside the origin until its first return to the origin. $G_2(t) = \mathsf{P}(\tau_2 \le t)$, $t \ge 0$.

$$\boxed{h_d := 1 - \lim_{t\to\infty} G_2(t) \ge 0.}$$

A particle of the first type has the life time distribution $G_1(t)$.
When dying the particle produces the offspring of three types in accordance with the probability generating function

$$\begin{aligned} f_1(s_1, s_2, s_3) &= \alpha f(s_1) + (1 - h_d)(1 - \alpha)s_2 \\ &+ h_d(1 - \alpha)s_3. \end{aligned}$$

The life time distribution of a particle of the second type is $G_2(t)/(1 - h_d) = \mathsf{P}(\tau_2 \le t | \tau_2 < \infty)$.

When dying a particle of the second type produces the offspring according to the probability generating function

$$f_2(s_1, s_2, s_3) = s_1.$$

The particles of the third type have infinite life times and do not produce any offsprings. We may set $f_3(s_1, s_2, s_3) \equiv 0$.

Denote the number of particles of type i, $i = 1, 2, 3$, by $Z_i(t)$. We see

$$\boxed{(\mu_t(0), \mu_t - \mu_t(0)) \stackrel{Law}{=} (Z_1(t), Z_2(t) + Z_3(t)).}$$

The mean matrix of the constructed Bellman-Harris process is

$$M = \left\| \frac{\partial f_i}{\partial s_j}(1,1) \right\|_{i,j=1,2} =$$

$$\begin{pmatrix} \alpha f'(1) & (1-h_d)(1-\alpha) & h_d(1-\alpha) \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

One has no difficulty verifying that matrix M has the Perron root 1 iff

$$\alpha(f'(1)-1) = h_d(1-\alpha).$$

Since a Bellman-Harris branching process is supercritical, critical or subcritical if the Perron root of M is > 1, $= 1$ or < 1 respectively,

CBRW is called supercritical, critical or subcritical if $\alpha(f'(1) - 1) > h_d(1 - \alpha)$, $\alpha(f'(1) - 1) = h_d(1 - \alpha)$ or $\alpha(f'(1) - 1) < h_d(1 - \alpha)$ respectively.

2. The criterion of criticality of CBRW on $\mathbb{Z}^d$ can be obtained by another method.

Namely, its basic idea is to derive differential equations in Banach spaces for mean numbers of particles by considering all possible evolutions of the particles population for time interval $[t, t+h)$ as $h \to 0$.

It turns out that if and only if
$\alpha(f'(1)-1) > h_d(1-\alpha)$
the linear operator at the right-hand side of the equation has a positive isolated eigenvalue which is responsible for exponential growth of the particles numbers
(see E.B.Yarovaya (2009)).

Previous results

Theorem (V.A.Vatutin et al.(2003))

Let assumptions (I) – (V) be satisfied for $d = 1$*. Then for any* $\lambda \in [0, \infty)$ *one has*

$$\lim_{t\to\infty} \mathsf{E}_0 \left[\exp\left\{ -\frac{\lambda \mu_t(0)}{\mathsf{E}_0(\mu_t(0)|\mu_t(0)>0)} \right\} \middle| \, \mu_t(0) > 0 \right] = \frac{1}{\lambda+1}$$

where the index 0 *indicates the starting point of CBRW.*

So the limit distribution obeys exponential law.

Theorem (E.Vl.Bulinskaya (2010))

Under assumptions (I) – (V) for $d = 2$ the following relation is valid

$$\lim_{t\to\infty} \mathsf{E}_0 \left[s^{\mu_t(0)} \middle| \mu_t(0) > 0 \right] = \frac{s - (H(0) - H(s))}{1 - H(0)}$$

where
$H(s) := H_0(s) = \alpha \int_0^\infty \left(f\left(\mathsf{E}_0 s^{\mu_t(0)}\right) - \mathsf{E}_0 s^{\mu_t(0)} \right) dt,$
$s \in [0,1]$, *and* $H(s) < 1 - s,\ s \in [0,1)$.

So the limit distribution is discrete.

Theorem (Y.Hu, V.A.Vatutin, V.A.Topchii (2010))

If (I) – (V) are true for $d = 4$ *then for* $z > 0$

$$\lim_{t\to\infty} \mathsf{P}_0\left(\frac{\mu_t(0)}{\mathsf{E}_0(\mu_t(0)|\mu_t(0)>0)} \le z \,\middle|\, \mu_t(0) > 0\right)$$

$$= \frac{1}{3} + \frac{2}{3}\left(1 - e^{-2z/3}\right).$$

The limit distribution is a mixture of exponential law and atom at zero.

Main results. One-dimensional case

Theorem (E.Vl.Bulinskaya (2010))

Given (I) – (V) for $d = 3$ or $d \geq 5$, the following relation holds for each $\lambda \in [0, \infty)$

$$\lim_{t\to\infty} \mathsf{E}_0 \left[\exp\left\{ -\frac{\lambda \mu_t(0)}{\mathsf{E}_0(\mu_t(0)|\mu_t(0)>0)} \right\} \middle| \, \mu_t(0) > 0 \right] = \frac{1}{\lambda+1}.$$

Again the limit distribution is exponential

Theorem (E.Vl.Bulinskaya (2010))

The previous theorems hold true if E_0 *and* P_0 *at the left-hand sides of the stated formulae are replaced by* E_x *and* P_x *respectively, whereas the right-hand sides of those formulae do not change. Here* x *indicates the starting point of CBRW on* $\mathbb{Z}^d$*.*

Sketch of the proofs

1. For $d \neq 2$, it is easy to check that

$$\lim_{t\to\infty} \mathrm{E}_x \left[\exp\left\{ -\frac{\lambda\mu_t(0)}{\mathrm{E}_x(\mu_t(0)|\mu_t(0)>0)} \right\} \middle| \, \mu_t(0) > 0 \right] = 1 - \lim_{t\to\infty} \frac{q_x(t;s(t;\lambda))}{q_x(t)}$$

where $q_x(t;s) := 1 - \mathrm{E}_x s^{\mu_t(0)}$,

$$s(t;\lambda) := \exp\left\{ -\frac{\lambda}{\mathrm{E}_x(\mu_t(0)|\mu_t(0)>0)} \right\}.$$

2. An important role belongs to the distribution function

$G_x(t) := \mathsf{P}_x(\tau \le t),\ t \ge 0,\ x \in \mathbb{Z}^d,$

of the time τ spent by the parent particle outside the origin until the first hitting the origin if the particle starts walking at point x.

We derive the crucial formula

$$\boxed{q_x(t;s) = \int_0^t q_0(t-u;s)\,dG_x(u).}$$

Really, due to the Markovian property

$$
\begin{aligned}
q_x(t;s) &= 1 - \mathsf{E}_x s^{\mu_t(0)} \mathbb{I}(\tau > t) - \mathsf{E}_x s^{\mu_t(0)} \mathbb{I}(\tau \le t) \\
&= 1 - \mathsf{P}_x(\tau > t) - \int_{\{\omega:\tau(\omega)\le t\}} s^{\mu_t(0)}\, d\mathsf{P}_x \\
&= \mathsf{P}_x(\tau \le t) - \int_{\{\omega:\tau(\omega)\le t\}} \mathsf{E}_x \left(s^{\mu_t(0)} \middle| \tau \right) d\mathsf{P}_x \\
&= G_x(t) - \int_0^t \mathsf{E}_x \left(s^{\mu_t(0)} \middle| \tau = u \right) dG_x(u) \\
&= G_x(t) - \int_0^t \mathsf{E}_0 s^{\mu_{t-u}(0)}\, dG_x(u) \\
&= G_x(t) - \int_0^t (1 - q_0(t-u;s))\, dG_x(u).
\end{aligned}
$$

3. Asymptotical behavior of $G_x(t)$

Lemma

If condition (I) – (V) are true then $G_x(t)$ has a density $g_x(t)$ and for $x \neq 0$, $t \to \infty$,

$$1 - G_x(t) \sim \frac{\rho_1(x)}{\gamma_1\,\pi\,\sqrt{t}},\ \ g_x(t) \sim \frac{\rho_1(x)}{2\gamma_1\,\pi\,t^{3/2}} \text{ for } d = 1,$$

$$1 - G_x(t) \sim \frac{\rho_2(x)}{\gamma_2\,\ln t} \text{ for } d = 2,$$

$$\frac{G_0(x,0)}{G_0(0,0)} - G_x(t) \sim \frac{2\,\gamma_d\,\rho_d(x)}{(d-2)\,G_0^2(0,0)\,t^{d/2-1}},$$

$$g_x(t) \sim \frac{\gamma_d\,\rho_d(x)}{G_0^2(0,0)\,t^{d/2}} \text{ for } d \geq 3 \text{ where } \rho_d(x) :=$$

$$G_0(0,0) - G_0(x,0) = \frac{1}{(2\pi)^d} \int\limits_{[-\pi,\pi]^d} \frac{\cos(x,\theta)-1}{\phi(\theta)}\, d\theta < \infty,$$

$$\phi(\theta) = \sum_{x \in \mathbb{Z}^d} a(x,0)\cos(x,\theta).$$

Here $G_\lambda(x,y) = \int_0^\infty e^{-\lambda t} p(t;x,y)\,dt$ and $p(t;x,y)$ is a transition probability of the random walk generated by A.

4. We need one more formula (E.Vl.Bulinskaya (2010))

$$q_x(t;s) = (1-s)m_x(t) - \int_0^t m_x(t-u)h(q_0(u;s))\,du$$

where

$$h(s) := \alpha(f(1-s) - 1 + f'(1)\,s),\ s \in [0,1],$$

and $m_x(t) := \mathsf{E}_x \mu_t(0)$.

Main results. Joint distribution

Now we discuss another interesting research direction, namely, conditional limit distribution of the numbers of particles at the origin and outside it assuming the non-degeneracy at the origin.

Theorem (V.A.Vatutin, V.A.Topchii (2004))

If (I)-(V) are true for $d = 1$ *and* $h''(z) = \sigma^2 + o(|\ln^{-1} z|),\ z \to 0+$, *then* $\mathsf{E}_0\left[\exp\left\{-\frac{\lambda_1\mu_t(0)}{\mathsf{E}_0(\mu_t(0)|\mu_t(0)>0)} - \frac{\lambda_2\mu_t}{c_1\sqrt{t}}\right\}\Big|\,\mu_t(0) > 0\right] \to \frac{D(\lambda_2)}{1+\lambda_1}$ *as* $t \to \infty$ *for* $\lambda_1,\ \lambda_2 \geq 0$.

Here $c_1 := \gamma_1 a(0)/(\alpha - 1)$ *and* $D(\lambda_2),\ \lambda_2 \geq 0$, *is the unique bounded solution of the equation*

$$D(\lambda_2) = 1 - \alpha\sigma^2\lambda_2 \int_0^1 \frac{D(\lambda_2\sqrt{y})\varphi(\lambda_2^2 y)}{\sqrt{y(1-y)}}\, dy$$

where $\varphi(\lambda_1),\ \lambda_1 \geq 0$, *is a continuous positive solution of the integral equation*

$$\varphi(\lambda_1) = 1 - \alpha\sigma^2\lambda_1^{1/2} \int_0^1 \varphi^2(\lambda_1(1-v))\, dv^{1/2}.$$

Theorem (E.Vl.Bulinskaya (2010))

Under assumptions (I)-(V) when $d = 2$ *one has for* $s \in [0,1], \ \lambda \geq 0$

$$\lim_{t\to\infty} \mathsf{E}_0 \left[s^{\mu_t(0)} \exp\left\{ -\frac{\lambda \mu_t}{c_2 \ln t} \right\} \middle| \, \mu_t(0) > 0 \right] =$$

$$\frac{s - (H(0) - H(s))}{1 - H(0)} \, \frac{4}{\sqrt{1+2\sigma^2\lambda}(\sqrt{1+2\sigma^2\lambda}+1)^2}$$

where $c_2 := \frac{\gamma_2 a(0)}{(\alpha - 1)}$.

Thus for $d = 1$ and $d = 2$ the properly normed random variables $\mu_t(0)$ and μ_t are asymptotically independent given $\mu_t(0)$ as $t \to \infty$.

To prove the last theorem we follow the

Thank you!