Stochastic Delay Differential Equations

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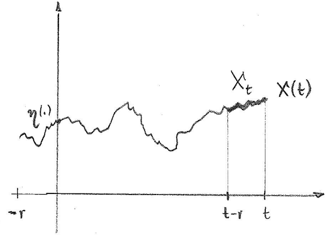
Outline

Stochastic Delay Differential Equations (SDDE)

$$dX(t) = \mu(t, X_t)dt + \sigma(t, X_t)dW(t), \quad t \geq 0. \tag{1}$$

- $X_t = (X(t+s), s \in [-r, 0])$: segment of $X(\cdot)$ at time t,
- $r \in [0, \infty]$: length of the memory,



- $\mu(t,x), \sigma(t,x) : [0,\infty) \times C([-r,0]) \to \mathbb{R}^1$ drift– and diffusion coefficients
- $W = (W(t), t \geq 0)$: standard Wiener process,
- $X_0 = (\eta(s), s \in [-r, 0])$: initial process. (1')

Existence and Uniqueness of Solutions

(1) and (1') mean:

$$X(t) = \eta(0) + \int_0^t \mu(s, X_s)ds + \int_0^t \sigma(s, X_s)dW(s), \quad t \geq 0,$$
$$X(s) = \eta(s), \quad s \in [-r, 0]. \tag{2}$$

Theorem (Mohammed (1984), Mao (2007))

Assume

- *some local Lipschitz conditions on $\mu(t,x)$ and $\sigma(t,x)$,*
- *some linear growth conditions on $\mu(t,x)$ and $\sigma(t,x)$,*
- *integrability of the initial process $\eta(\cdot)$.*

Then there exists a uniquely determined solution

$$(X(t), t \in [-r, \infty))$$

of (2).

diffusion-type processes

Stochastic Ordinary Differential Equations (SODE)

$$dX(t) = \mu(X(t))dt + \sigma(X(t))dW(t), \quad t \geq 0. \tag{3a}$$

$$X(0) = \eta \quad \text{initial value.} \tag{3b}$$

Available tools:

- $T_t f(x) := \mathbb{E}_x f(X_t)$ defines a semigroup $(T_t, t \geq 0)$ with infinitesimale operator $A = \frac{\sigma^2(x)}{2}\frac{d^2}{dx^2} + \mu(x)\frac{d}{dx}$,
- Feynman–Kac–formula,
- harmonic functions h: $Ah = 0$, Potential theory of A
- $(h(X(t)), t \geq 0)$ continuous local martingales,
- Itô–formula,
- Connections to parabolic and elliptic differential equations (Kolmogorov's differential equations).

Well developed:
Numerics and statistics of SODE
Kloeden, Platen (1999); Gilsing, Shadlow (2007); Kutoyants (2004)

Examples

1. Affine SODE, Ornstein-Uhlenbeck-Process

$$dX(t) = aX(t)dt + \sigma dW(t), \quad X(0) = X_0,$$

$$X(t) = X_0 \exp(at) + \int_0^t \exp[a(t-s)]dW(s), \quad t \geq 0.$$

2. Linear SODE, Geometric Brownian Motion

$$dX(t) = \mu X(t)dt + \sigma X(t)dW(t), \quad X(0) = X_0,$$

$$X(t) = X_0 \exp\left[\sigma W(t) + (\mu - \frac{\sigma^2}{2})t\right], \quad t \geq 0.$$

3. Stochastic logistic growth model

$$dX(t) = bX(t)(K - X(t))dt + \sigma X(t)dW(t),$$

$$X(0) = X_0 \in (0, K), \quad K > 0,$$

$$X(t) = \frac{X_0 \exp\left[(bK - \frac{\sigma^2}{2})t + \sigma W(t)\right]}{1 + bX_0 \int_0^t \exp\left[(bK - \frac{\sigma^2}{2})s + \sigma W(s)\right] ds}.$$

This great variety of mathematical tools to treat Stochastic Ordinary Differential Equations is paid by the assumption that the drift $\mu(X(t))$ and the diffusion $\sigma(X(t))$ depend on the present state $X(t)$ of X only, i.e. that$X$ has no memory, with other words,

X is a Markov process.

But: Many real phenomena include a memory or simply a time delay.

Financial industry: time between

- a claim occurs and the moment of settlement by an insurance company (settling delay),
- time lag for getting information.

Economics: time between

- taking a decision and occurrence of the feedback,
- time to build, to transport, to store.

Biology and Population dynamics:

- time to maturity,
- time to educate,
- incubation period.

Ecology: time between

- causing and occuring of damages, e.g. air pollution and changing of climate.

Delay and Memory are everywhere!

Instead of choosing a Markov model one can try to accept the memory. Then several questions arise:

1. How to model memories, and how to treat them mathematically?
2. Are there new effects caused by a memory?
3. Are there classes of memories, which can be treated mathematically in a reasonable way?

Two Examples of differential equations with time delay

Growth Models

T. Malthus (1798)

$$N(t) = N(0)\exp[at], \qquad \frac{dN(t)}{N(t)} = a dt.$$

a ... Malthusian coefficient of linear growth, growth rate.

P. F. Verhulst (1838)

$$\dot{N}(t) = a\left(1 - \frac{N(t)}{K}\right)N(t),$$

$$\frac{dN(t)}{N(t)} = a\left(1 - \frac{N(t)}{K}\right)dt.$$

K ... capacity of the habitat, determined by the food and the area of the habitat.

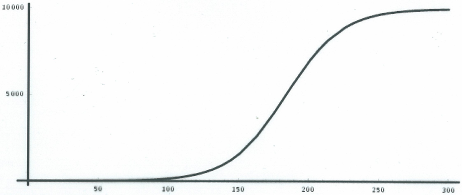
$$N(t) = \frac{N(0)Ke^{at}}{K + N(0)(e^{at} - 1)}, \quad t \geq 0, \quad N(0) \in (0, K).$$

Logistische Gleichung

$$\dot{N}(t) = rN(t)\left(1 - \frac{1}{K}N(t)\right)$$

- N Populationsgröße der Fliegen
- r Wachstumsrate
- K Parameter für bereitgestellte Nahrungsmenge

Lucilia cuprina



Quelle: A.J. Nicholson, 1954

1

Australische Schmeißfliege (1)

$$\dot{N}(t) = rN(t)\left(1 - \frac{1}{K}N(t)\right)$$

- N Populationsgröße der Fliegen
- r Wachstumsrate
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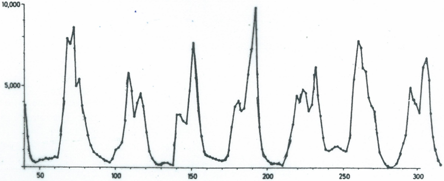

Quelle: A.J. Nicholson, 1954

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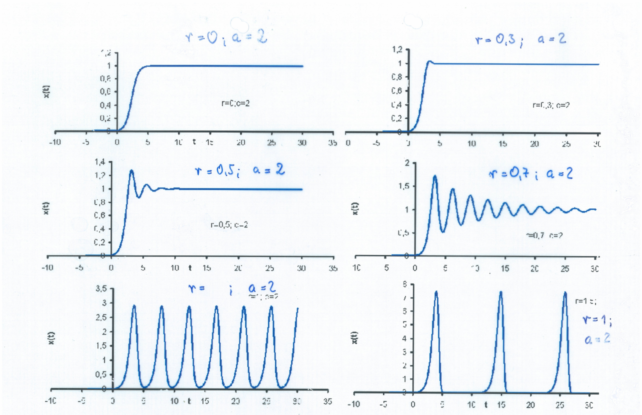
Fig. from Riedle (2005)

G. E. Hutchinson (1948)

$$\frac{dN(t)}{N(t)} = a\left[1 - \frac{N(t-r)}{K}\right]dt.$$

r...average age of sexually mature females.



Time delay may generate oscillations, Fig. from Riedle (2005)

G. E. Hutchinson (1948)

$$\frac{dN(t)}{N(t)} = a\left[1 - \frac{N(t-r)}{K}\right]dt.$$

r...average age of sexually mature females.

Australische Schmeißfliege (2)

$$\dot{N}(t) = rN(t)\left(1 - \frac{1}{K}N(t-\alpha)\right)$$

- N Populationsgröße der Fliegen
- r Wachstumsrate
- K Parameter für bereitgestellte Nahrungsmenge
- α Entwicklungszeit von Larve zur Fliege (ca. 11 Tage)

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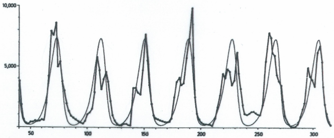

Quelle: A.J. Nicholson, 1954

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Time delay may generate oscillations, Fig. from Riedle (2005)

More realistic models take into account, that the birth rate underlies random fluctuations.
Thus leads e.g. to the randomized growth rate

$$a\left[1-\frac{N(t-r)}{K}\right]+\sigma\dot{W}(t).$$

Therefore, we get

$$dN(t)=a\left[1-\frac{N(t-r)}{K}\right]N(t)dt+\sigma N(t)dW(t).$$

If the age of sexually mature females is considered to be distributed with probability distribution $\beta(\cdot)$ say, then we obtain

$$dN(t)=a\left[1-\frac{1}{K}\int_0^r N(t-s)\beta(ds)\right]N(t)dt+\sigma N(t)dW(t).$$

Stable point: $N(t)\equiv K$
Introducing $n(t)=N(t)-K$ and neglecting terms of higher order we get

$$dn(t) = a(K + n(t))\left[-K^{-1}\int_0^r n(t-s)\beta(ds)\right]dt + \sigma K dW(t),$$

$$dn(t) = -a\int_0^r n(t-s)\beta(ds)dt + \sigma K dW(t).$$

This is an example of an

Affine SDDE

Discontinuous dependence of the solution from the initial process

Geometric Brownian Motion

$$dX(t) = X(t)dW(t), \quad t \geq 0,$$
$$X(0) = \eta \in \mathbb{R},$$

$$X^{\eta}(t,\omega) = \eta \exp\left[W(t,\omega) - \frac{t}{2}\right], \quad t \geq 0.$$

The mapping

$$\eta \mapsto X^{\eta}(t,\omega), \quad \eta \in \mathbb{R}^1$$

is for every fixed pair (t,ω) $(t > 0, \omega \in \Omega)$ a continuous function of the initial value η.

Delay in the diffusion: $r > 0$

$$dX(t) = X(t-r)dW(t), \qquad t \geq 0,$$
$$X(s) = \eta(s), \qquad s \in [-r, 0],$$

$\eta \in \mathbb{C}([-r,0])$, continuous function.

$$X^\eta(t) = \int_0^t \eta(s-r)dW(s), \quad t \in [0, r].$$

The mapping $\eta \longrightarrow X^\eta(t,\omega)$ is discontinuous in the following sense:

Choose any $\delta > 0$ and define

$$\eta_k(s) = \delta \sin\left(2k\pi \frac{s}{r}\right),\ s \in [-r, 0],$$

$$X_k(\omega) = X^{\eta_k}(r, \omega) = \int_0^r \eta_k(s-r)dW(s).$$

Then, $(X_k, k \geq 1)$ are centered, mutually independent Gaussian random variables having identical positive variance $\frac{\delta^2}{2}$

$$\left(\mathbb{E}\, X_k X_l = \mathbb{E} \int_0^r \eta_k(s-r)dW_s \int_0^r \eta_l(s-r)dW_s = \int_{-r}^0 \eta_k(s)\eta_l(s)ds = 0,\ k \neq l\right).$$

Thus for every $M > 0$

$$C_M := \mathbb{P}(|X_k| > M) > 0,\ k \geq 1,$$

and Borel–Cantelli lemma yields

$$\mathbb{P}(\sup_k |X_k| > M) = 1 \text{ for all } M > 0, \quad \text{i.e.,}$$

$$\sup_{k \geq 1} |X_k(\omega)| = \infty \quad \mathbb{P}\text{–a. s.}$$

(See Mohammed (1998).)

Affine SDDE

$$dX(t) = \int_{-r}^{0} X(t+s)a(ds)dt + dW(t), \;\; t \geq 0,$$
$$X(u) = \eta(u), u \in [-r, 0],$$

$a(\cdot)$... finite signed measure.

Autoregressive Schema with continuous time

Examples:
$$dX(t) = \sum_{k=0}^{N} a_k X(t - r_k)dt + dW(t),$$
$$dX(t) = \int_{-r}^{0} X(t+s)\alpha(s)dsdt + dW(t).$$

Example (Ornstein-Uhlenbeck-case: $r = 0$:)

$$dX(t) = aX(t)dt + dW(t), \qquad (a(ds) = a\mathbf{1}_{\{0\}}(ds))$$
$$X(t) = \eta e^{at} + \int^{t} e^{a(t-s)} dW(s).$$

The solution of the affine SDDE

$$\begin{aligned} dX(t) &= \int_{-r}^{0} X(t+s)a(ds)dt + dW(t),\ t \geq 0, \\ X(t) &= \eta(t), t \in [-r,0]. \end{aligned}$$

admits the representation

$$X(t) = \eta(0)x_0(t) + \int_0^t x_0(t-s)dW(s) + \int_{-r}^{0}\int_{u}^{0} \eta(s)x_0(t+u-s)ds\, a(du),$$

where $x_0(\cdot)$ denotes the fundamental solution:

$$\dot{x}_0(t) = \int_{-r}^{0} x_0(t+s)a(ds),$$
$$x_0(s) = \mathbf{1}_{\{0\}}(s), \quad s \in [-r,0].$$

Example (Ornstein-Uhlenbeck-case:)

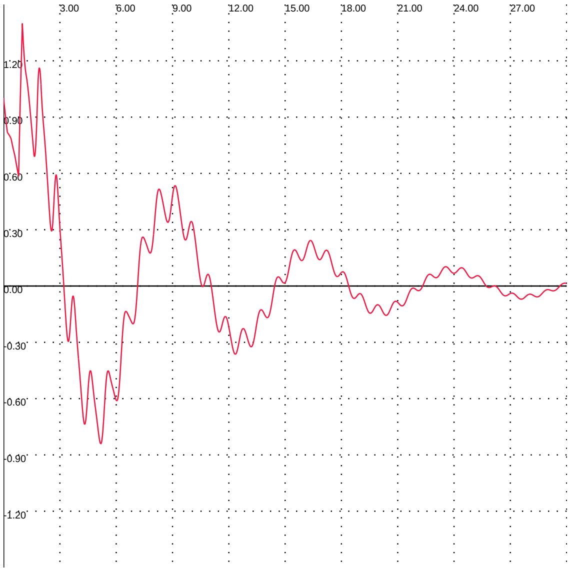
$$a(ds) = a\delta_{\{0\}}(ds) \quad , \quad \dot{x}_0(t) = ax_0(t),$$
$$x_0(t) = \exp[at].$$

In general, the fundamental solution $x_0(\cdot)$ of

$$\dot{x}_0(t) = \int_{-r}^{0} x_0(t+s)a(ds), \tag{4a}$$

$$x_0(s) = \mathbf{1}_{\{0\}}(s), \quad s \in [-r,0], \tag{4b}$$

- has no explicit expression,
- may be oscillating instead of being monotone.



$$a(du) = (-\delta_0 + 0.7\delta_{-0.2} - 0.3\delta_{-0.4} - 0.2\delta_{-0.6} + 5.5\delta_{-0.8} - 5.4\delta_{-1})(du)$$

Laplace transform of $x_0(t)$

$$\hat{x}_0(\lambda) = \int_0^\infty e^{-\lambda s} x_0(s) ds, \quad \text{Re } \lambda > \|a\|,$$

$$\hat{x}_0(\lambda) = \left[\lambda - \int_{-r}^{0} e^{\lambda s} a(ds)\right]^{-1}, \quad \text{Re } \lambda > \|a\|.$$

The denominator of the Laplace transform $\hat{x}_0(\lambda)$

$$h(\lambda) = \lambda - \int_{-r}^{0} e^{\lambda s} a(ds), \quad \lambda \in \mathbb{C},$$

is called the characteristic function of the deterministic differential equation (4).

Properties:

- The characteristic function $h(\lambda) = \lambda - \int_{-r}^{0} e^{\lambda s} a(ds)$ is holomorphic.

Define $\Lambda := \{\lambda | h(\lambda) = 0\}$, the set of zeros of $h(\cdot)$.

- Λ is countable infinite, beside of $a(ds) = a \cdot \delta_{\{0\}}(ds)$,
- Λ has no finite accumulation point,
- the multiplicity m_λ of every $\lambda \in \Lambda$ is finite,
- $\#\{\lambda \in \Lambda | \operatorname{Re}\lambda \geq c\}$ is finite for every $c \in \mathbb{R}^1$.

Define the top coefficient

$$\upsilon_0 := \upsilon_0(a) = \max\{\operatorname{Re}\lambda | \lambda \in \Lambda\} < \infty$$

and further

$$\upsilon_{i+1} := \max\{\operatorname{Re}\lambda | \lambda, \operatorname{Re}\lambda < \upsilon_i\}, \ i \geq 0.$$

Series expansion of $x_0(\cdot)$:

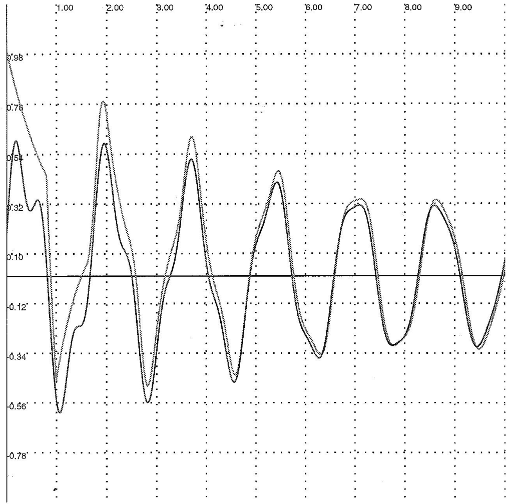
$$x_0(t) = \sum_{i: v_i \geq c} \left(\sum_{\substack{\lambda \in \Lambda \\ \lambda = v_i}} p_\lambda(t) e^{v_i t} + \sum_{\substack{\lambda \in \Lambda \\ \operatorname{Re}\lambda = v_i \\ \operatorname{Im}\lambda > 0}} \{ q_\lambda(t) \cos(t \cdot \operatorname{Im}\lambda) + r_\lambda(t) \sin(t \cdot \operatorname{Im}\lambda) \} e^{v_i t} \right) + o(e^{ct})$$

where

- $c < v_0$, $p_\lambda(t)$ are real polynomials in t with degree $m_\lambda - 1$,
- $q_\lambda(t), r_\lambda(t)$ are real polynomials in t with degree $\leq m_\lambda - 1$ (and at least one of them $= m_\lambda - 1$).

Example ($a(.) = a\dot{\delta}_0(.)$, O-U-case)

$$h(\lambda) = \lambda - a, \quad \Lambda = \{a\},$$
$$x_0(t) = e^{at}.$$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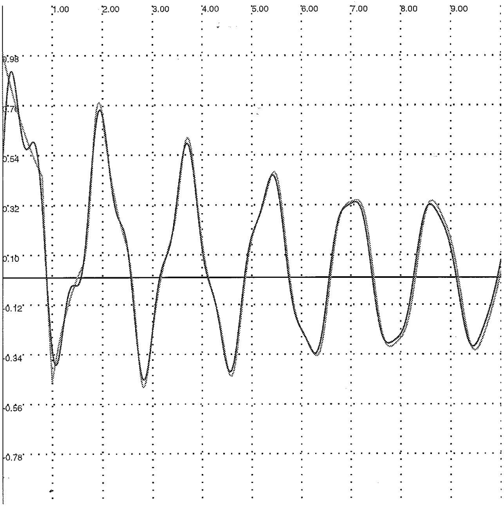
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Stationary Solution

Proposition

The affine SDDE

$$dX(t) = \int_{-r}^{0} X(t+s)a(ds)dt + dW(t)$$

admits a stationary solution X if and only if

$$v_0 = v_0(a) < 0.$$

This is equivalent to

$$\int_{0}^{\infty} x_0^2(t)dt < \infty.$$

Then,

$$X(t) = \int_{-\infty}^{t} x_0(t-s)dW(s).$$

In this case, X is Gaussian, with zero mean and covariance function K given by

$$K(t) = K(-t) = \int_0^\infty x_0(s)x_0(s+t)ds, \quad t \geq 0.$$

It has spectral density f with

$$f(\lambda) = |h(i\lambda)|^{-2} = |i\lambda - \int_{-r}^{0} \mathrm{e}^{\mathrm{i}\lambda s} a(ds)|^{-2}, \quad \lambda \in \mathbb{R}^1 .$$

The covariance function K satisfies the analogue of the Yule–Walker equation

$$\dot{K}(t) = \int_{-r}^{0} K(t+u)a(du),\ t > 0, \quad \text{and it holds}$$
$$\dot{K}(0+) = -\frac{1}{2}.$$

The covariance function decreases exponentially to zero as $\exp[v_0 h]$ for $h \to \infty$.

"Short-range dependence"

If $X = (X(t), t \geq 0)$ is the stationary solution of

$$dX(t) = \int_{-r}^{0} X(t+s)a(ds)dt + dW(t),$$

then X is absolutely regular (β-mixing), i.e.,

$$\beta(\tau) := \mathbb{E}\left[\sup_{A \in \sigma(X(s), s \geq t+\tau)} |\mathbb{P}(A|\sigma(X(s), s \leq t)) - \mathbb{P}(A)|\right] \leq C\mathrm{e}^{-\tau \cdot \upsilon},$$

where υ is any real number from $(0, -\upsilon_0)$ and $C = C(\upsilon) > 0$ is a constant, depending only of υ.

In particular, X is strong mixing (α-mixing) and ergodic.
(M. Reiß (2002))

Statistical Questions

$$dX(t) = \int_{-r}^{0} X(t+s)a(ds)dt + dW(t), \;\; t \geq 0$$

$$X(u) = \eta(u), u \in [-r, 0].$$

Log-Likelihood process

$$\ell_t = \ln L_t = \int_{-r}^{0} \underbrace{\Big(\int_{0}^{t} X(s+u)dX(s) \Big)}_{=V_t^0(u)} a(du)$$

$$- \frac{1}{2} \int_{-r}^{0} \Big(\int_{-r}^{0} \underbrace{\int_{0}^{t} X(s+u)X(s+v)ds \, a(du) \Big)}_{=I_t^0(u,v)} a(dv)$$

$$= \langle V_t^0, a \rangle - \frac{1}{2} \langle I_t^0 a, a \rangle \to \text{ max!}$$

I_t^0 ... Fisher information operator

The maximum likelihood equation

$$I_t^0 \hat{a}_t(\cdot) = V_t^0(\cdot),$$

i.e.,

$$\int_{-r}^{0} \int_{0}^{t} X(s+u)X(s+v)ds\hat{a}_t(du) = \int_{0}^{t} X(s+u)dX(s).$$

$\hat{a}_t$ is a nonparametric maximum likelihood estimator for the unknown signed measure a,

I_t^0 is a compact linear operator, thus

the maximum-likelihood-problem is an ill posed problem

Nonparametric cases, ill-posed-problem

(Reiss (2002) PhD-thesis)

Assume $a(dv)$ has a density $g(v), v \in [-r,0]$ belonging to the Sobolev space $H^s([-r,0])$ for some $s > 0$.
Introduce for $S > 0$ and $\delta > 0$
$M(s,S,\delta) :=$
$\{g \in H^s([-r,0]) | \|g\|_s \leq S, v_0(g) \leq -\delta\}$

Theorem: For $s > 0, S > 0, \delta > 0$ such that $M(s,S,\delta)$ has nonempty interior in $H^s([-r,0])$, the following lower bound holds for $T \to \infty$

$$\inf_{G_T} \sup_{g \in M(s,S,\delta)} E_g \left[\|G_T - g\|_{L^2}^2 \right]^{\frac{1}{2}} \gtrsim T^{-\frac{s}{2s+3}}$$

where the infimum is taken over all $\mathscr{F}_T^X$-measurable estimators G_T.
An optimal estimator $\hat{g}_T$ is constructed by Galerkins projection method.

Note: This rate is different from those for i.i.d. case and signal detection problem: $T^{-\frac{s}{2s+1}}$.

Discrete measure $a(\cdot)$

$$dX(t) = \sum_{i=0}^{m} \vartheta_i X(t - r_i)dt + dW(t), \quad t \geq 0,$$
$$X(s) = \eta(s), \quad s \in [-r, 0],$$

$0 = r_0 < r_1 < \cdots < r_m =: r.$

$$\ell_t(\vartheta) = \vartheta^* V_t^0 - \frac{1}{2}\vartheta^* I_t^0 \vartheta \quad \text{with}$$
$$V_t^0 = \left(\int_0^t X(s - r_i)dX(s), i = 0, \cdots, m \right)^*,$$
$$I_t^0 = \left(\int_0^t X(s - r_i)X(s - r_j)ds, i, j = 0, \cdots, m \right).$$

Maximum-Likelihood-estimator

$$\hat{\vartheta}_T = (I_T^0)^{-1} V_T^0 = \vartheta + (I_T^0)^{-1} V_T^W \qquad \text{with}$$
$$V_T^W = \left(\int_0^t X(s - r_i)dW(s), i = 0, \cdots m \right)^*.$$

Study case $N = 1$:

Typical properties of the estimator $\hat{\vartheta}_t$ for $T \to \infty$ can be studied at $N = 1$:

$$\hat{\vartheta}_T = (I_T^0)^{-1} V_T^0 = \begin{pmatrix} \int_0^T X^2(t)dt & \int_0^T X(t)X(t-1)dt \\ \bullet & \int_0^T X^2(t-1)dt \end{pmatrix}^{-1} \begin{pmatrix} \int_0^T X(t)dX(t) \\ \int_0^T X(t-1)dX(t) \end{pmatrix},$$

$$\hat{\vartheta}_T - \vartheta = (I_T^0)^{-1} V_T^W = \begin{pmatrix} \int_o^T X^2(t)dt & \int_0^T X(t)X(t-1)dt \\ \bullet & \int_0^T X^2(t-1)dt \end{pmatrix}^{-1} \begin{pmatrix} \int_0^T X(t)dW(t) \\ \int_0^T X(t-1)dW(t) \end{pmatrix}.$$

Asymptotics for $T \to \infty$ are determined by $X(t)$, $x_0(t)$, Λ.

- fundamental solution

$$\dot{x}_0(t) = ax_0(t) + bx_0(t-1), \quad x_0(s) = \mathbf{1}_{\{0\}}(s),\ s \in [-1,0].$$

- Laplace transform

$$\lambda \hat{x}_0(\lambda) = a\hat{x}_0(\lambda) + b\,\mathrm{e}^{-\lambda}\hat{x}_0(\lambda),$$
$$\hat{x}_0(\lambda) = \frac{1}{\lambda - a - b\,\mathrm{e}^{-\lambda}}.$$

- Characteristic function

$$h(\lambda) = \lambda - a - b\,\mathrm{e}^{-\lambda}, \quad \lambda \in K.$$

- Set of zeros of the characteristic function

$$\Lambda = \{\lambda \in K | h(\lambda) = 0\}.$$

$$x_0(t) = \psi_0(t)\mathrm{e}^{v_0 t} + \mathrm{o}(\mathrm{e}^{\gamma t}), \quad t \to \infty,$$

where γ is any real number with $v_1 < \gamma < v_0$, and

$$\psi_0(t) = \begin{cases} \dfrac{1}{v_0 - a + 1} & \text{if } v_0 \in \Lambda, m(v_0) = 1, \\ 2t + \frac{2}{3} & \text{if } v_0 \in \Lambda, m(v_0) = 2, \\ A_0 \cos(\xi_0 t) + B_0 \sin(\xi_0 t) & \text{if } v_0 \notin \Lambda \qquad (\lambda_0 = v_0 + i\xi_0). \end{cases}$$

In the first case ($v_0 \in \Lambda, m(v_0) = 1$) we get for every $\gamma < v_1$

$$x_0(t) = \frac{1}{v_0 - a + 1}\,\mathrm{e}^{v_0 t} + \psi_1(t)\mathrm{e}^{v_1 t} + \mathrm{O}(\mathrm{e}^{\gamma t})$$

with

$$\psi_1(t) = \begin{cases} \dfrac{1}{v_1 - a + 1} & \text{if } v_1 \in \Lambda, \\ A_1 \cos(\xi_1 t) + B_1 \sin(\xi_1, t) & \text{if } v_1 \notin \Lambda. \end{cases}$$

What can we expect?

Start with $b = 0, \quad a = \vartheta.$

Example (Ornstein-Uhlenbeck-case)

$$dX(t) = \vartheta X(t)dt + dW(t),$$
$$X(0) = \eta, \quad \text{independent of } W(\cdot).$$

$$X(t) = \eta e^{\vartheta t} + \int_0^t e^{\vartheta(t-s)} dW(s), \quad t \geq 0.$$

The solution is constructed from $\eta(\cdot)$, $x_0(t) = e^{\vartheta t}$, $W(t)$

$$\frac{d\,\mathbb{P}_a^T}{d\,\mathbb{P}_0^T} = \exp\left[\eta \int_0^T X(s)dX(s) - \frac{\eta^2}{2}\int_0^T X^2(s)ds\right].$$

In general, for $T \to \infty$, all integrals tend to infinity or behave irregular.

One method to study the asymptotic behaviour consists in localization: Fix a $\vartheta \in \mathbb{R}^1$, choose a function $\varphi_T(\vartheta)$ and introduce a new (local) parametrization by

$$\vartheta(u) = \vartheta + \varphi_T(\vartheta)u, \quad u \in \mathbb{R}^1 .$$

Consider

$$\begin{aligned}
\frac{d\,\mathbb{P}^T_{\vartheta+\varphi_T(\vartheta)u}}{d\,\mathbb{P}^T_\vartheta} &= \frac{d\,\mathbb{P}_{\vartheta+\varphi_T(\vartheta)u^T}}{d\,\mathbb{P}^T_0} \Bigg/ \frac{d\,\mathbb{P}^T_\vartheta}{d\,\mathbb{P}^T_0} \\
&= \exp\left[(\vartheta+\varphi_T(\vartheta)u)\,V_T^0 - \frac{(\vartheta+\varphi_T(\vartheta)u)^2}{2}\,I_T^0\right] \\
&= \exp\Bigg[\underbrace{\left(\varphi_T(\vartheta)\int_0^T X(s)dW(s)\right)}_{=V_T} u - \underbrace{\left(\varphi_T^2(\vartheta)\int_0^T X^2(s)ds\right)}_{=I_T} \frac{u^2}{2}\Bigg],
\end{aligned}$$

and hope the pair (V_T, I_T) tends to a limit for $T \to \infty$.
From this the limit behaviour of

$$\varphi_T^{-1}(\vartheta)(\hat{\vartheta}_T - \vartheta) = I_T^{-1} V_T \quad \text{for } T \to \infty$$

immediately follows. (Le Cam; Ibragimov, Khasminski)

$$\log \frac{d\,\mathbb{P}^T_{\vartheta+\varphi_T(\vartheta)u}}{d\,\mathbb{P}^T_\vartheta} = \underbrace{\varphi_T(\vartheta)\int_0^T X(s)dW(s)}_{=V_T^W}\cdot u - \underbrace{\varphi_T^2(\vartheta)\int_0^T X^2(s)ds}_{I_T}\cdot\frac{u^2}{2}$$

1. $\vartheta < 0$: Stationary case, LAN

$$\varphi_T = (2|\vartheta|T)^{-\frac{1}{2}}, \qquad (V_T^W, I_T) \xrightarrow{d} (Z,1) \text{ with } Z \sim N(0,1).$$

2. $\vartheta = 0$: LAQ, one cluster point

$$\varphi_T = T^{-1}, \qquad (V_T^W, I_T) \stackrel{d}{=} \left(\int_0^1 \widetilde{W}(s)d\widetilde{W}(s), \int_0^1 \widetilde{W}^2(s)ds\right).$$

3. $\vartheta > 0$: Exploding case, LAMN

$$\varphi_T = (2\vartheta)^{-\frac{1}{2}}\exp(-\vartheta T), \qquad (V_T^W, I_T) \xrightarrow{d} (Z\cdot I_\infty^{\frac{1}{2}}, I_\infty),$$

where $I_\infty \sim \chi^2(1)$, $\quad Z \sim N(0,1)$,
I_∞, Z independent random variables.

Back to the case $N = 1$:

$$(V_T^W)^* = \left(\int_o^T X(t)dW(t), \int_0^T X(t-1)dW(t) \right) \varphi_T,$$

$$\varphi_T = \varphi_T(\vartheta)... \quad \text{appropriate } 2 \times 2\text{-matrix.}$$

$$I_T = \varphi_T^* \begin{pmatrix} \int_0^T X^2(t)dt & \int_0^T X(t)X(t-1)dt \\ \int_0^T X(t)X(t-1)dt & \int_0^T X^2(t-1)dt \end{pmatrix} \varphi_T.$$

If we show that I_T for $T \to \infty$ tends in probability to some 2×2- matrix I_∞, then

the convergence of (V_T^W, I_T) to (V_∞, I_∞) for some random vector V_∞ follows from the stable limit theorem for martingales.

Theorem (Gushchin, Kü (1999, 2001))

For every $\vartheta = (a,b) \in \mathbb{R}^2$ *one can find a matrix function* $\varphi_T(\vartheta)$ *such that* (V_T^W, I_T) *with*

$$V_T^W = \varphi_T^*(\vartheta)\left(\int_0^T X(s)dW(s), \int_0^T X(s-1)dW(s)\right)^* \text{ and}$$

$$I_T = \varphi_T^* \begin{pmatrix} \int_0^T X^2(s)ds & \int_0^T X(s)X(s-1)ds \\ \bullet & \int_0^T X^2(s-1)ds \end{pmatrix} \varphi_T$$

tend for $T \to \infty$ *to a limit* (V_∞, I_∞) *at least in distribution or behaves asymptotically periodic.*

There are eleven different cases depending on the position of $v_0(\vartheta)$ and $v_1(\vartheta)$.
The limit distribution or cluster points respectively can be explicitly calculated for every $\vartheta \in \mathbb{R}^2$.

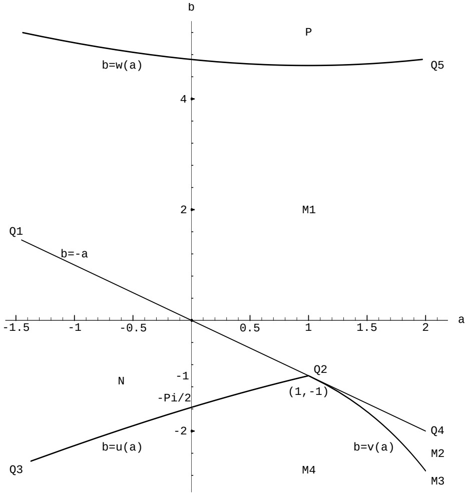


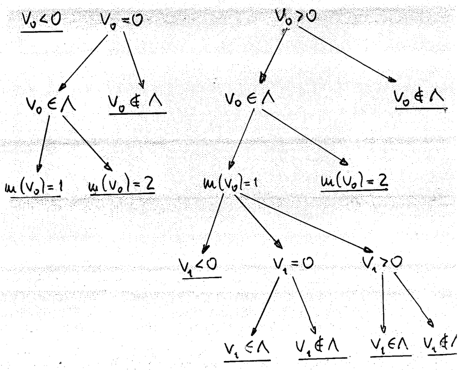
Fig: The eleven cases in the $(a, b)-$plane

$$\Lambda := \{\lambda \in \mathbb{C} \mid \lambda - a - b e^{-\lambda} = 0\}$$

$$v_0 := \max\{\operatorname{Re}\lambda \mid \lambda \in \Lambda\}$$

$$v_1 := \max\{\operatorname{Re}\lambda \mid \lambda \in \Lambda,\ \operatorname{Re}\lambda < v_0\}$$

THE ELEVEN DIFFERENT CASES



OF THE ASYMPT. BEHAVIOUR OF G(T)

(A. Gushchin, U. K. 1999)

<table>
<tr><td colspan="5">$v_0 < 0$</td><td>$a < 1,\ \ u(a) < b < -a$</td><td>N</td></tr>
<tr><td rowspan="3">$v_0 = 0$</td><td rowspan="2">$v_0 \in \Lambda$</td><td colspan="3">$m(v_0) = 1$</td><td>$a < 1,\ \ b = -a$</td><td>Q1</td></tr>
<tr><td colspan="3">$m(v_0) = 2$</td><td>$a = 1,\ \ b = -a$</td><td>Q2</td></tr>
<tr><td colspan="4">$v_0 \notin \Lambda$</td><td>$a < 1,\ \ b = u(a)$</td><td>Q3</td></tr>
<tr><td rowspan="7">$v_0 > 0$</td><td rowspan="6">$v_0 \in \Lambda$</td><td rowspan="5">$m(v_0) = 1$</td><td colspan="2">$v_1 < 0$</td><td>$-a < b < w(a)$</td><td>M1</td></tr>
<tr><td rowspan="2">$v_1 = 0$</td><td>$v_1 \in \Lambda$</td><td>$a > 1,\ \ b = -a$</td><td>Q4</td></tr>
<tr><td>$v_1 \notin \Lambda$</td><td>$b = w(a)$</td><td>Q5</td></tr>
<tr><td rowspan="2">$v_1 > 0$</td><td>$v_1 \in \Lambda$</td><td>$a > 1,\ \ v(a) < b < -a$</td><td>M2</td></tr>
<tr><td>$v_1 \notin \Lambda$</td><td>$b > w(a)$</td><td>P</td></tr>
<tr><td colspan="3">$m(v_0) = 2$</td><td>$a > 1,\ \ b = v(a)$</td><td>M3</td></tr>
<tr><td colspan="4">$v_0 \notin \Lambda$</td><td>$a < 1,\ b < u(a)$ or $a \geq 1,\ b < v(a)$</td><td>M4</td></tr>
</table>

Table: The eleven cases in terms of (v_0, v_1) and (a, b).

Case N: $v_0(\vartheta_0) < 0$

Proposition

The family $(\mathbb{P}_\vartheta)$ *is locally asymptotically normal at* ϑ_0 *(LAN):*

$$(V_T^W, I_T) \xrightarrow{d} (V_\infty, I_\infty)$$

with $\varphi_T(\vartheta_0) = T^{-\frac{1}{2}} \cdot \mathscr{I}_2$,

$$V_\infty \sim N(0, I_\infty) \quad \textit{and}$$

$$I_\infty = \begin{pmatrix} \int_0^\infty x_0^2(s)ds & \int_0^\infty x_0(s)x_0(s+1)ds \\ \bullet & \int_0^\infty x_0^2(s)ds \end{pmatrix}.$$

Consequently,

$$\lim_{T\to\infty} \hat{\vartheta}_T = \vartheta_0 \quad \textit{in probability } \mathbb{P}_{\vartheta_0},$$

$$T^{\frac{1}{2}}(\hat{\vartheta}_T - \vartheta_0) \xrightarrow{d} N(0, I_\infty^{-1}).$$

Case M1: $\vartheta_0 = (a,b)^* \in \mathbb{R}^2$ with $-a < b < w(a)$.

This means

$$v_0(\vartheta_0) > 0, v_0 \in \Lambda, m(v_0) = 1, v_1(\vartheta_0) < 0, v_1(\vartheta_0) \in \Lambda$$

Proposition

$(\mathbb{P}_\vartheta, \vartheta \in \mathbb{R}_2)$ *is LAMN at* ϑ_0*:*

$$(V_T^W, I_T) \xrightarrow{d} (V_\infty, I_\infty), \quad \textit{where} \quad (V_\infty, I_\infty) \overset{d}{=} (I_\infty^{\frac{1}{2}} Z, I_\infty)$$

$Z \sim N(0, \mathscr{I}_2)$*, independent of* I_∞*,*

$$I_\infty = \begin{pmatrix} \frac{U_0^2}{2v_0(v_0-a+1)^2} & 0 \\ 0 & \int\limits_0^\infty (x_0(t) - e^{v_0} x_0(t-1))^2 dt \end{pmatrix} \quad \textit{with}$$

$$U_0 = \eta(0) + b \int\limits_{-1}^{0} e^{-v_0(s+1)} \eta(s) ds + \int\limits_0^\infty e^{-v_0 s} dW(s)$$

Case Q1: $\vartheta_0 = (a, b)^*$ with $b = -a$, $a < 1$

Then, $v_0 = 0$, $v_0 \in \Lambda$, $m(v_0) = 1$,

$$V_\infty = \left(\frac{1}{1-a} \int_0^1 \tilde{W}(t) d\tilde{W}(t), N \right)$$

and

$$I_\infty = \begin{pmatrix} \frac{1}{(1-a)^2} \int_0^1 \tilde{W}^2(t)dt & 0 \\ 0 & \sigma^2 \end{pmatrix}.$$

Here, N denotes a $N(0, \sigma^2)$-distributed r.v. independent of the standard Wiener process $(\tilde{W}(t))$ and $\sigma^2 = \int_0^\infty (x_0(t) - x_0(t-1))^2 dt$.

Proposition

The family $(\mathbb{P}_\vartheta, \vartheta \in \mathbb{R}^2)$, *is locally asymptotically quadratic at* $\vartheta_0 = (a, b)$:

$$(V_T^W, I_T) \xrightarrow{d} (V_\infty, I_\infty).$$

Case P1: $\vartheta_0 = (a,b)^* \in \mathbb{R}^2$ with $b > w(a)$

I.e.,

$$v_0,\ v_1 > 0,\ v_0 \in \Lambda,\ v_1 \notin \Lambda.$$

$$\varphi_T = \varphi_T^{(1)} \varphi_T^{(2)},\ \varphi_T^{(1)} = \begin{pmatrix} 1 & 1 \\ 0 & -\mathrm{e}^{v_0} \end{pmatrix},\ \varphi_T^{(2)} = \begin{pmatrix} \mathrm{e}^{-v_0 T} & 0 \\ 0 & \mathrm{e}^{-v_1 T} \end{pmatrix}.$$

For all $u \in [0, \Delta]$ with $\Delta = \frac{2\pi}{\xi_1}$, there exists $I_\infty(u)$ random regular 2×2-matrix:

$$I_\infty(u) = \begin{pmatrix} \frac{U_0^2}{2v_0(v_0-a+1)^2} & \frac{U_0}{v_0-a+1} \int\limits_0^\infty \mathrm{e}^{-(v_0+v_1)t} U(u-t)dt \\ \frac{U_0}{v_0-a+1} \int\limits_0^\infty \mathrm{e}^{-(v_0+v_1)t} U(u-t)dt & \int\limits_0^\infty \mathrm{e}^{-2v_1 t} U^2(u-t)dt \end{pmatrix},$$

where

- $U_0 = \int_0^\infty \mathrm{e}^{-v_0 s} dW(s)$, random variable,
- $U(t) = \int_0^\infty \Phi(t-s)\mathrm{e}^{-v_1 s} dW(s)$, random process,
- $\Phi(t) = A\cos(\xi_1 t) + B\sin(\xi_1 t)$, periodic function.

Proposition

The family $\mathbb{P}^{\vartheta}, \vartheta \in \mathbb{R}^2$, *is periodically local asymptotically mixed normal at* ϑ_0*:*

$$(V_{T_n}^W, I_{T_n}) \xrightarrow{d} (V_\infty(u), I_\infty(u)),$$

where $T_n = u + n\Delta$ *and* $u \in [0, \Delta]$.

A general parametric model

$$dX(t) = \int_{-r}^{0} X(t+s)a_\vartheta(ds)dt + dW(t), \qquad \vartheta \in \Theta \subseteq \mathrm{I\!R}^k, \tag{5a}$$

$$X(s) = \eta(s), \quad s \in [-r,0]. \tag{5b}$$

- $M = \{a : a$ finite variation signed measure on $[-r,0]\}$,
- $M_s = \{a \in M :$ (5) admits a stationary solution$\}$.

Assume

$$\mathscr{A} := \{a_\vartheta(ds) : \vartheta \in \Theta \subseteq \mathrm{I\!R}^k\} \subseteq M_s, \qquad M^k = \underbrace{M \times \cdots \times M}_{k-\text{times}},$$

$$M^k_\# = \{(a_1, \cdots, a_k) \in M^k : a_1, \cdots, a_k \text{ are linearly independent}\}.$$

To study the asymptotic properties of $\hat{\vartheta}_T$ for $T \to \infty$ we turn once again to the framework of asymptotic statistics of LeCam and Ibragimov, Khasminski.

Consider a family $(a_\vartheta, \vartheta \in \Theta)$ with an open $\Theta \subseteq \mathbb{R}^k$ of bounded signed measures on $[-r,0]$ and let $\mathbb{P}_\vartheta^T$ be the measure on $\mathcal{C}([-r,T])$ generated by the solution X^ϑ of

$$dX(t) = \int_{-r}^{0} X(t+s)a_\vartheta(ds)dsdt + dW(t) \text{ with}$$

$$X(s) = \eta(s), \quad s \in [-r,0].$$

Then one introduces the localization around a fixed $\vartheta_0 \in \Theta$:

$$\vartheta = \vartheta_0 + \varphi_T(\vartheta_0)u \quad , \quad u \in \mathbb{R}^k,$$

where $\varphi_T(\vartheta_0)$ is a $k \times k-$ matrix, regular, with

$$\varphi_T(\vartheta_0) \to 0 \quad \text{for} \quad T \to \infty. \qquad \text{(normalizing matrix)}$$

The likelihood process

$$dX(t) = \int_{-r}^{0} X(t+s)a_\vartheta(ds)dt + dW(t), \quad t \geq 0, \tag{6a}$$

$$X(t) = \eta(t), \quad t \in [-r, 0], \quad \vartheta \in \Theta \subseteq \mathbb{R}^k. \tag{6b}$$

In the examples above the expression for the log–likelihood function often (but not always) is a quadratic function of the parameters under consideration. Here we have

$$Z_{\vartheta,T}(u) = \frac{d\,\mathbb{P}^T_{\vartheta+\varphi_T u}}{d\,\mathbb{P}^T_\vartheta} = \exp\Bigg[\int_0^T \int_{-r}^0 X(s+v)\Big(a_{\vartheta+\varphi_T u}(dv) - a_\vartheta(dv)\Big)dX(s) - \frac{1}{2}\int_0^T \Bigg(\int_{-r}^0 \int_{-r}^0 X(s+v)X(s+w)\Big[a_{\vartheta+\varphi_T u}(dv)a_{\vartheta+\varphi_T u}(dw) - a_\vartheta(dv)a_\vartheta(dw)\Big]\Bigg)ds. \tag{7}$$

The further treatment very depends on the structure of the function $\vartheta \to a_\vartheta(\cdot)$.
Problem: Under which conditions the family $(\mathbb{P}^T_\vartheta, \vartheta \in \Theta)$ is locally asymptotically normal (LAN)?

$$dX(t) = \int_{-r}^{0} X(t+s)a_\vartheta(ds)dt + dW(t), \quad t \geq 0,$$
$$X(s) = \eta(s), \quad s \in [-r, 0].$$

Assumptions (A):

1) $\mathscr{A} = (a_\vartheta, \vartheta \in \Theta)$, Θ open, bounded subset of $\mathbb{R}^k$,
2) $\exists\, C < \infty : \|a_\vartheta - a_\eta\|_v \leq C\|\vartheta - \eta\|$,
3) $a_\vartheta \in M_s$, $\vartheta \in \bar{\Theta}$,
4) $\vartheta \mapsto \int_{[-r,0]} g(u)a_\vartheta(du)$ continuously differentiable in Θ with the gradient

$$\int_{[-r,0]} g(u)\dot{a}_\vartheta(du),$$

where $g \in C[-r,0]$, $\dot{a}_\vartheta = (\dot{a}_\vartheta^{(1)}, \dot{a}_\vartheta^{(2)}, \cdots, \dot{a}_\vartheta^{(k)})^* \in M_\#^k$, $\vartheta \in \Theta$,

5) $a_\vartheta \neq a_\eta$, $\vartheta \in \Theta$, $\eta \in \bar{\Theta}$, $\vartheta \neq \eta$.

$$\begin{aligned} X^{\vartheta}(t) &= x_0(t)\eta(0) + \int_{-r}^{0}\int_{u}^{0} \eta(s) x_0(t+u-s) ds a_{\vartheta}(du) \\ &+ \int_{0}^{t} x_0(t-s) dW(s), \quad t \geq 0, \\ K(t) &= \int_{0}^{\infty} x_0(t+s) x_0(s) ds, \quad t \geq 0. \end{aligned}$$

Example

$$dX(t) = \sum_{k=0}^{N} \vartheta_k X(t - r_k)dt + dW(t),$$

$$a_\vartheta(\cdot) = \sum_{i=0}^{N} \vartheta_i \cdot \delta_{r_i},$$

$$\int g(u)a_\vartheta(du) = \sum_{i=0}^{N} \vartheta_i g(-r_i),$$

$$\dot{a}_\vartheta(\cdot) = (\delta_{-r_0}(\cdot), \cdots, \delta_{-r_N}(\cdot))^T,$$

satisfies 1) - 5).

Example

$$dX(t) = b \cdot X(t-\vartheta)dt + dW(t),$$
$$a_\vartheta(du) = b\delta_{\{-\vartheta\}}(du),$$

does not satisfy the assumptions, because

$$\vartheta \to \int_{[-r,0]} g(u)a_\vartheta(du) = bg(-\vartheta)$$

is not differentiable for all $g \in C[-r,0]$.

Theorem (Gushchin, Kü, 2001)

Under the assumptions (A) for every compact set $K \subseteq \Theta$ it holds uniformly in $\vartheta \in K$:

$$\sqrt{T}(\vartheta_T - \vartheta) \xrightarrow{d} N(0, \Sigma^{-1}(\vartheta)), \quad T \to \infty,$$

where

$$\Sigma(\vartheta) = (\Sigma_{ij}(\vartheta))_{i,j=1,\cdots,k},$$
$$\Sigma_{ij}(\vartheta) = \int\int_{\mathscr{J} \times \mathscr{J}} K_\vartheta(u-v)\dot{a}_{\vartheta,i}(du)\dot{a}_{\vartheta,j}(dv).$$

All the moments of $\sqrt{T}(\hat{\vartheta}_T - \vartheta)$ under $\mathbb{P}_T^\vartheta$ tend as $T \longrightarrow \infty$ to the corresponding moments of the normal distribution with parameters $(0, \Sigma^{-1}(\vartheta))$.
The maximum likelihood estimator $\hat{\vartheta}_T$ is asymptotically efficient in K.

A counterexample

$$dX(t) = bX(t-\vartheta)dt + dW(t),$$

$$b < 0, \text{ fixed}, \quad \vartheta \in (0, \frac{1}{e|b|}) = \Theta.$$

(ensures $x_0(\cdot)$ is square integrable and does not oscillate)

$$Z_{T,\vartheta}(u) := \frac{d\,\mathbb{P}_{\vartheta+\varphi_T u}}{d\,\mathbb{P}_\vartheta} = \exp\left(b\int_0^T (X(t-\vartheta-\varphi_T u) - X(t-\vartheta))dX(t) \right.$$
$$\left. - \frac{b}{2}\int_0^T (X(t-\vartheta-\varphi_T u) - X(-\vartheta))^2 dt) \right).$$

Here, the log–likelihood function is not quadratic w.r.t. to the parameter. Moreover, $Z_T(u)$ depends on ϑ in a non–differentiable way.

Proposition (Kü, Kutoyants (2000))

For $\varphi_T = T^{-1}$ the marginal distributions of $Z_{T,\vartheta}(\cdot)$ converge to the marginal distribution of

$$Z(u) = \exp\left\{ b\widetilde{W}(u) - \frac{1}{2}|u|b^2 \right\}, \quad u \in \mathbb{R}^1,$$

uniformly over every compact set $K \subseteq \mathbb{R}^1$.

Here, $\widetilde{W}(\cdot)$ denotes a twosided standard Wiener process.

No LAQ

Remember:

$$Z_T(u) = \frac{d\,\mathbb{P}^T_{\vartheta+\varphi_T(\vartheta)u}}{d\,\mathbb{P}^T_\vartheta} \longrightarrow Z(u) = \exp\left[b\tilde{W}(u) - \frac{1}{2}|u|b^2\right],$$

in case $a_\vartheta(dv) = b\mathbf{1}_{\{-\vartheta\}}(dv)$.

$$Z_T(u) = \frac{d\,\mathbb{P}^T_{\vartheta+\varphi_T(\vartheta)u}}{d\,\mathbb{P}^T_\vartheta} \longrightarrow Z(u) = \exp\left[uW - \frac{c^2}{2}u^2\right]$$

in case $a_\vartheta(dv) = \vartheta\mathbf{1}_{\{-1\}}(dv)$.

A bridge between these two cases

Consider two finite signed measures $a(dv)$ and $b(dv)$ on some interval $[-r,0]$. Assume $a(dv) \in M_s$ and denote by $b_\vartheta(dv)$ the translated $b(dv)$:

$$b_\vartheta(B) = b(B-\vartheta).$$

Define

$$a_\vartheta = a + b_\vartheta - b \tag{8}$$

and assume $a_\vartheta \in M_s, \vartheta \in (\vartheta_0, \vartheta_1)$ with $\vartheta_0 < 0 < \vartheta_1$.

Proposition (Gushchin, Kü (2010))

The Normalized Likelihood Ratio

$$Z_{\vartheta,T}(u) = \frac{d\,\mathbb{P}^T_{\vartheta+\tilde{\delta}_T u}}{d\,\mathbb{P}^T_{\vartheta}}$$

converges for $T \to \infty$ and for some normalizing function $\tilde{\delta}_T$

$$Z_{\vartheta,T}(u) \longrightarrow Z_{\vartheta}(u), \quad u \in \mathbb{R}^1$$

($Z_{\vartheta}(\cdot) \not\equiv 1$, continuous in probability at zero) if and only if the function

$$\psi_b(x) = \int_{-x}^{x} |\varphi_b(\lambda)|^2 d\lambda, \qquad x \geq 0$$

is regularly varying at infinity. In this case

$$\tilde{\delta}_T \sim c\delta_T, \quad T \to \infty,$$

for some $c > 0$ and

$$\delta_T^{-1} = inf\{x \geq 1 | Tx^{-2}\psi_b(x) < 1\}.$$

Define

$$H = \sup\left\{\gamma \le 1 : \int_{-\infty}^{\infty} (1+\lambda^2)^{-1}|\lambda|^{2\gamma}|\varphi_b(\lambda)|^2 d\lambda < \infty\right\}.$$

We have $\frac{1}{2} \le H \le 1$ and $\psi_b(\cdot)$ is regularly varying with index $2-2H$, δ_T is regularly varying with index $-\frac{1}{2H}$:

$$\delta_T \sim \ell(T)T^{-\frac{1}{2H}}.$$

$$Z_T(u) = \frac{d\,\mathbb{P}^T_{\vartheta+\varphi_T(\vartheta)u}}{d\,\mathbb{P}^T_\vartheta} \longrightarrow Z(u) = \exp\left[B^H(u) - \frac{\mathbb{E}(B^H(u))^2}{2}\right], \qquad (9)$$

where B^H is a fractional BM, i.e. a centered Gaussian process with covariance function given by

$$K(u,v) = C(u|^{2H} + |v|^{2H} - |u-v|^{2H}).$$

Proposition (Gushchin, Kü (2010))

For every $H \in [\frac{1}{2}, 1]$ one can find examples $(a_\vartheta(dv))$ with property (9).

The extrem cases:
$H = \frac{1}{2}$: $\delta_T = T^{-1}$, and $B^H(u)$ is a twosided standard Wiener process $W(u)$: (Kutoyants-Kü-case)

$H = 1$: $\delta_T = T^{-\frac{1}{2}}$, and $B^H(u)$ equals uW with a centered Gaussian r.v. W: LAN-case.

Fractional affine DDE

$$D_0^\alpha y(t) = \int_{-r}^{0} y(t+s)a(ds) + f(t), \tag{10a}$$

$$y(s) = \xi(s), \quad s \in [-r,0], \quad \xi(\cdot) \in C([-r,0]). \tag{10b}$$

Solution of (10):

$$y(t) = \xi(0)x_0(t) + D^{1-\alpha} \int_{-r}^{0} \left(\int_{u}^{0} x_0(t+u-v)\xi(v)dv \right) a(du) + \int_{0}^{t} x_0(t-s)f(s)ds.$$

Here $x_0(\cdot)$ denotes the fundamental solution of (10), i.e.,

$$\begin{aligned} D_0^\alpha x_0(t) &= \int_{-r}^{0} x_0(t+s)a(ds), \\ x_0(s) &= \mathbf{1}_{\{0\}}(s), \quad s \in [-r,0]. \end{aligned}$$

Fundamental solution

$$D_0^{\alpha} x_0(t) = \int_{-r}^{0} x_0(t+s)a(ds),$$
$$x_0(s) = \mathbf{1}_{\{0\}}(s), \quad s \in [-r, 0].$$

Laplace-transform:

$$\hat{x}_0(\lambda) = \frac{\lambda^{\alpha-1}}{\lambda^{\alpha} - \int_{-r}^{0} \mathrm{e}^{\lambda s} a(ds)}, \text{ Re } \lambda > \|a\|^{\frac{1}{\alpha}}.$$

Characteristic function:

$$h_{\alpha}(\lambda) = \left[\lambda^{\alpha} - \int_{-r}^{0} \mathrm{e}^{\lambda s} a(ds)\right] \cdot \lambda^{1-\alpha} = \lambda - \lambda^{1-\alpha} \int_{-r}^{0} \mathrm{e}^{\lambda s} a(ds).$$

Fundamental solution

$$D_0^{\alpha} x_0(t) = \int_{-r}^{0} x_0(t+s)a(ds),$$

$$x_0(s) = \mathbf{1}_{\{0\}}(s), \quad s \in [-r, 0].$$

There is a qualitative jump from $\alpha = 1$ to $\alpha < 1$:
The characteristic function

$$h_\alpha(\lambda) = \lambda - \lambda^{1-\alpha} \int_{-r}^{0} \mathrm{e}^{\lambda s} a(ds)$$

is

- holomorphic only on $\mathbb{C} \setminus \mathbb{R}_-$,
- discontinuous on $\mathbb{R}_-$,

Fundamental solution

$$D_0^{\alpha} x_0(t) = \int_{-r}^{0} x_0(t+s)a(ds),$$

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Theorem (Krol (2008))

For $v_0 < 0$ we have $\quad x_0(t) = p(t)e^{v_0 t} + g(t), \; t \geq 0,$

with $p(t)$ polynomially bounded function and

$$g(t) = \begin{cases} O(t^{-\alpha}), & \text{if } a([-r,0]) \neq 0, \\ O(t^{-\mu}), & \text{for all } \mu \in (0,\alpha), \text{ if } a([-r,0]) = 0. \end{cases}$$

Thus the asymptotic of the fundamental solution $x_0(t)$ for $t \to \infty$ is not exponentially but polynomially.

Stochastic case

$$X(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \int_{-r}^0 X(s+u)a(du)ds + B(t),$$
$$X(s) = \eta(s), \quad s \in [-r,0].$$

Theorem (Krol (2008))

There exists a stationary solution iff $v_0 < 0$ *and* $\alpha > \frac{1}{2}$.

Stochastic case

$$
\begin{aligned}
X(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \int_{-r}^0 X(s+u)a(du)ds + B(t), \\
X(s) &= \eta(s), \quad s \in [-r, 0].
\end{aligned}
$$

Corollary

The covariance function $K(t)$ of the stationary solution is given by

$$
K(t) = \int_0^t x_0(s)\, x_0(s+t) ds.
$$

$K(t)$ tends polynomially to zero for $t \to \infty$.

Thus, $(X(t), t \geq 0)$ has

"long-range dependence"

Thank you for your attention!

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