Numerical methods for LIBOR models

Antonis Papapantoleon

QP Lab & TU Berlin

Based on joint work with David Skovmand (Aarhus)

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Outline of the talk

Market size

According to the Bank for International Settlements:

	Jun 2006	Jun 2007	Jun 2008	Jun 2009
Foreign exchange	38,127	48,645	62,983	48,775
Interest rate	262,526	347,312	458,304	437,198
Equity-linked	6,782	8,590	10,177	6,619
Commodity	6,394	7,567	13,229	3,729
Credit default swaps	20,352	42,581	57,403	36,046
Unallocated	35,997	61,713	81,719	72,255
Total	370,178	516,408	683,815	604,622

Table: Notional amounts outstanding for OTC derivatives in billions of US$

Evolution of interest rates

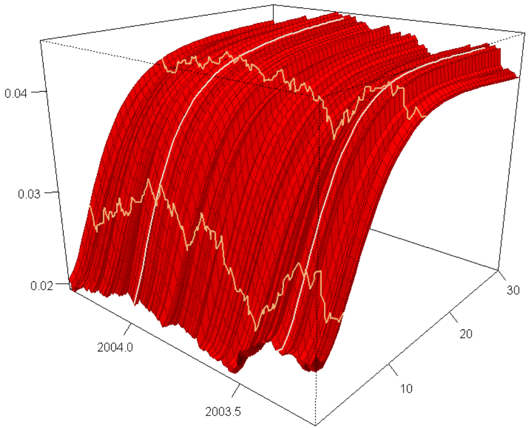

Figure: Evolution of interest rate term structure, 2003–2004

Interest rates

- Tenor: $0 < T_1 < T_2 < \cdots < T_N < T_{N+1} = T_*$, tenor length δ
- $B(t,T)$: value of a zero coupon bond for T, $B(T,T) = 1$
- $L(t,T)$: forward LIBOR rate for $[T, T+\delta]$

$$L(t,T) = \frac{1}{\delta}\left(\frac{B(t,T)}{B(t,T+\delta)} - 1\right)$$

- $F(t,T,U)$: forward price for T and U; $F(t,T,U) = \frac{B(t,T)}{B(t,U)}$

"Master" relation

$$F(t,T,T+\delta) = \frac{B(t,T)}{B(t,T+\delta)} = 1 + \delta L(t,T) \tag{1}$$

Calculation of LIBOR rate

~~*Barclays Bank plc*~~	~~*2.15*~~
~~*Bank of Tokyo-Mitsubishi UFJ Ltd*~~	~~*2.15*~~
~~*HSBC*~~	~~*2.12*~~
~~*Royal Bank of Scotland Group*~~	~~*2.11*~~
UBS AG	*2.105*
Abbey National	*2.1*
Bank of America	*2.1*
Citibank NA	*2.1*
Mizuho Corporate Bank	*2.1*
Rabobank	*2.1*
Royal Bank of Canada	*2.1*
WestLB AG	*2.1*
~~*BNP Paribas*~~	~~*2.05*~~
~~*LLoyds Banking Group*~~	~~*2*~~
~~*Deutsche Bank AG*~~	~~*1.95*~~
~~*JP Morgan Chase*~~	~~*1.95*~~

bbalibor Rate =

2.10063

- Question: “At what rate could you borrow funds by accepting interbank offers in a reasonable market size?”

Axiomatics for LIBOR models

Economic thought and practical applications lead to the following ...

Axioms:

1. LIBOR rates should be non-negative: $L(t, T_k) \geq 0$ for all t, k.
2. The model should be arbitrage-free: $L(\cdot, T) \in \mathcal{M}(\mathbb{P}_{T+\delta})$.
3. The model should be analytically tractable, easy to implement and calibrate.
4. The model should provide a good calibration to liquid derivatives (caps and swaptions).

Lévy processes

- A time-inhomogeneous Lévy process $X = (X_t)_{0 \le t \le T_*}$
- $\mathbb{R}$-valued stochastic process with independent increments
- the law of X_t is

$$\mathbb{E}\left[e^{iuX_t}\right] = \exp\left(\int_0^t \kappa_s(iu)\mathrm{d}s\right) \tag{2}$$

where

$$\kappa_s(iu) = iub_s - \frac{u^2 c_s}{2} + \int_{\mathbb{R}} (e^{iux} - 1 - iux) F_s(\mathrm{d}x) \tag{3}$$

with $b_s \in \mathbb{R}$, $c_s \in \mathbb{R}_{\geqslant 0}$ and F_s are Lévy measures, $\forall s \in [0, T_*]$

- Assumptions: exponential moments, abs. continuous characteristics

Lévy processes

- X is a special semimartingale

$$X_t = \int_0^t b_s \mathrm{d}s + \int_0^t \sqrt{c_s} \mathrm{d}W_s + \int_0^t \int_{\mathbb{R}} x(\mu^X - \nu)(\mathrm{d}s, \mathrm{d}x) \qquad (4)$$

- W: $\mathbb{P}$-Brownian motion
- μ^X: random measure of jumps of X
- ν: $\mathbb{P}$-compensator of μ^X
- The predictable characteristics (B, C, ν) are deterministic

$$B_t = \int_0^t b_s \mathrm{d}s, \quad C_t = \int_0^t c_s \mathrm{d}s, \quad \nu([0,t] \times A) = \int_0^t \int_A F_s(\mathrm{d}x) \mathrm{d}s$$

The Lévy LIBOR model (Eberlein & Özkan)

- Tenor: $0 < T_1 < T_2 < \cdots < T_N < T_{N+1} = T_*$, tenor length δ
- Associate forward measures $\mathbb{P}_{T_k}$ to tenor dates T_k
- Relations:

$$\frac{\mathrm{d}\mathbb{P}_{T_k}}{\mathrm{d}\mathbb{P}_{T_{k+1}}}\bigg|_{\mathcal{F}_t} = \frac{F(t, T_k, T_{k+1})}{F(0, T_k, T_{k+1})} \tag{5}$$

- Model: $\mathrm{d}L = L\,\mathrm{d}X$

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- Model: $\mathrm{d}L = L\,\mathrm{d}X$
- Problem: $F = 1 + \delta L$ yields

$$\begin{aligned} \mathrm{d}F = \delta\,\mathrm{d}L = \delta L\,\mathrm{d}X = F\frac{\delta L}{1+\delta L}\,\mathrm{d}X \\ \iff F = F_0\,\mathcal{E}\left(\int \frac{\delta L}{1+\delta L}\,\mathrm{d}X\right) \end{aligned} \tag{6}$$

The Lévy LIBOR model

Backward induction construction

The dynamics of the LIBOR rate $L(\cdot, T_k)$ under $\mathbb{P}_{T_{k+1}}$ is

$$L(t, T_k) = L(0, T_k) \exp\left(\int_0^t b^L(s, T_k) \mathrm{d}s + \int_0^t \lambda(s, T_k) \mathrm{d}X_s^{T_{k+1}} \right) \qquad (7)$$

where $X^{T_{k+1}}$ is a $\mathbb{P}_{T_{k+1}}$-semimartingale

$$X^{T_{k+1}} = \int_0^{\cdot} \sqrt{c_s} \mathrm{d}W_s^{T_{k+1}} + \int_0^{\cdot} \int_{\mathbb{R}} x(\mu^X - \nu^{T_{k+1}})(\mathrm{d}s, \mathrm{d}x) \qquad (8)$$

and

$$b^L(s, T_k) = -\frac{1}{2}\lambda^2(s, T_k) c_s - \int_{\mathbb{R}} \left(\mathrm{e}^{\lambda(s, T_k)x} - 1 - \lambda(s, T_k)x \right) F_s^{T_{k+1}}(\mathrm{d}x).$$

The Lévy LIBOR model

The $\mathbb{P}_{T_{k+1}}$-Brownian motion is

$$W_t^{T_{k+1}} = W_t^{T_*} - \int_0^t \left(\sum_{l=k+1}^{N} \frac{\delta L(t-, T_l)}{1 + \delta L(t-, T_l)} \lambda(t, T_l) \right) \sqrt{c_s} \mathrm{d}s, \qquad (9)$$

and the $\mathbb{P}_{T_{k+1}}$-compensator of μ^X is

$$\nu^{T_{k+1}}(\mathrm{d}s, \mathrm{d}x) = \prod_{l=k+1}^{N} \underbrace{\left(1 + \frac{\delta L(s-, T_l)}{1 + \delta L(s-, T_l)} \left(\mathrm{e}^{\lambda(s, T_l)x} - 1 \right) \right)}_{:=\beta(s,x,T_l)} \nu^{T_*}(\mathrm{d}s, \mathrm{d}x).$$

Problem 1: X **has state-dependent characteristics under** $\mathbb{P}_{T_{k+1}}$
Problem 2: The product term grows exponentially fast

Complexity of the problem

$L(t, T_1)$	$L(t, T_2)$	$\dots$	$L(t, T_k)$	$\dots$	$L(t, T_{N-2})$	$L(t, T_{N-1})$	$L(t, T_N)$
$L(t, T_N)$	$L(t, T_N)$	$\dots$	$L(t, T_N)$	$\dots$	$L(t, T_N)$	$L(t, T_N)$	
$L(t, T_{N-1})$	$L(t, T_{N-1})$	$\dots$	$L(t, T_{N-1})$	$\dots$	$L(t, T_{N-1})$		
$L(t, T_{N-2})$	$L(t, T_{N-2})$	$\dots$	$\dots$	$\dots$			
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$			
$\dots$	$\dots$	$\dots$	$L(t, T_{k+1})$				
$L(t, T_3)$	$L(t, T_3)$	$\dots$					
$L(t, T_2)$							

Table: Matrix of dependencies for LIBOR rates

- The dynamics of LIBOR rates depend on all subsequent rates
- The computation cannot be parallelized

Terminal measure dynamics

The dynamics of $L(t, T_k)$ under $\mathbb{P}_{T_*}$ is

$$L(t, T_k) = L(0, T_k) \exp\left(\int_0^t b(s, T_k) \mathrm{d}s + \int_0^t \lambda(s, T_k) \mathrm{d}X_s \right) \tag{10}$$

where X is the driving Lévy process,

$$b(s, T_k) = -\frac{1}{2}\lambda^2(s, T_k)c_s - c_s\lambda(s, T_k) \sum_{l=k+1}^{N} \frac{\delta L(s-, T_l)}{1 + \delta L(s-, T_l)} \lambda(s, T_l)$$
$$- \int_{\mathbb{R}} \left(\left(\mathrm{e}^{\lambda(s, T_k)x} - 1 \right) \prod_{l=k+1}^{N} \beta(s, x, T_l) - \lambda(s, T_k)x \right) F_s^{T_*}(\mathrm{d}x)$$

and

$$\beta(t, x, T_l) = \frac{\delta L(t, T_l)}{1 + \delta L(t, T_l)} \left(\mathrm{e}^{\lambda(t, T_l)x} - 1 \right) + 1.$$

Log-LIBOR rates

- Let $Z(\cdot, T_k) = \log L(\cdot, T_k)$ denote the log-LIBOR rate

$$Z(t, T_k) = \log L(0, T_k) + \int_0^t b(s, T_k; Z(s))\mathrm{d}s + \int_0^t \lambda(s, T_k)\mathrm{d}X_s,$$

where $b(s, T_k; Z(s)) := b(s, T_k)$.

- Hence, $Z(\cdot, T_k)$ satisfies the linear SDE

$$\mathrm{d}Z(t, T_k) = b(t, T_k; Z(t))\mathrm{d}t + \lambda(t, T_k)\mathrm{d}X_s \qquad (11)$$

with initial condition $Z(0, T_k) = \log L(0, T_k)$.

Picard approximations

Assume we didn't know the solution to

$$\mathrm{d}Z(t, T_k) = b(t, T_k; Z(s))\mathrm{d}t + \lambda(t, T_k)\mathrm{d}X_s, \quad Z(0, T_k) = \dots$$

Picard approximations

Assume we didn't know the solution to

$$dZ(t, T_k) = b(t, T_k; Z(s))dt + \lambda(t, T_k)dX_s, \quad Z(0, T_k) = \dots$$

Solution by Picard iterations:

0th iteration:

$$Z^{(0)}(t, T_k) = Z(0, T_k)$$

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Solution by Picard iterations:

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$$Z^{(0)}(t, T_k) = Z(0, T_k)$$

1st iteration:

$$Z^{(1)}(t, T_k) = Z(0, T_k) + \int_0^t b(s, T_k; Z^{(0)}(s))ds + \int_0^t \lambda(s, T_k)dX_s$$

Picard approximations

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2nd iteration:

$$Z^{(2)}(t, T_k) = Z(0, T_k) + \int_0^t b(s, T_k; Z^{(1)}(s))\mathrm{d}s + \int_0^t \lambda(s, T_k)\mathrm{d}X_s$$

Picard approximations – application

Remarks:

1. $Z^{(1)}(\cdot, T_k)$ is a Lévy process
2. $Z^{(1)}(\cdot, T_k)$ can be simulated independently of $Z^{(1)}(\cdot, T_l)$ $(k \neq l)$
3. $Z^{(2)}(\cdot, T_k)$ is our approximation for $Z(\cdot, T_k)$

Complexity of the approximate problem

$L^{(2)}(t, T_1)$	$L^{(2)}(t, T_2)$	$\dots$	$L^{(2)}(t, T_k)$	$\dots$	$L^{(2)}(t, T_{N-1})$	$L(t, T_N)$
$Z^{(1)}(t, T_N)$	$Z^{(1)}(t, T_N)$	$\dots$	$Z^{(1)}(t, T_N)$	$\dots$	$Z^{(1)}(t, T_N)$	
$Z^{(1)}(t, T_{N-1})$	$Z^{(1)}(t, T_{N-1})$	$\dots$	$Z^{(1)}(t, T_{N-1})$	$\dots$		
$Z^{(1)}(t, T_{N-2})$	$Z^{(1)}(t, T_{N-2})$	$\dots$	$\dots$	$\dots$		
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		
$\dots$	$\dots$	$\dots$	$Z^{(1)}(t, T_{k+1})$			
$Z^{(1)}(t, T_3)$	$Z^{(1)}(t, T_3)$	$\dots$				
$Z^{(1)}(t, T_2)$						

Table: Matrix of dependencies for approximate LIBOR rates

- Approximate LIBOR rates depend on $Z^{(1)}$'s
- The computation can be parallelized!

Drift expansions

How to deal with

$$\mathbb{B}_i = \int_{\mathbb{R}} \left(\left(e^{\lambda_i x} - 1 \right) \prod_{l=i+1}^{N} \left(1 + \frac{\delta L_l}{1 + \delta L_l} \left(e^{\lambda_l x} - 1 \right) \right) - \lambda_i x \right) F_s^{T_*}(dx)\,?$$

The expansion has 2^N terms:

$$\prod_{k=1}^{N} (1 + a_k) = 1 + \sum_{k=1}^{N} a_k + \sum_{1 \le i < j \le N} a_i a_j + \sum_{1 \le \cdots \le N} a_i a_j a_k + \cdots + \prod_{k=1}^{N} a_k$$

This amounts to $2^N - 1$ evaluations of the cumulant ...

Drift expansions

Instead:

$$\prod_{l=1}^{N}\left(1+\frac{\delta L_l}{1+\delta L_l}\left(e^{\lambda_l x}-1\right)\right)$$
$$\approx 1+\sum_{l=1}^{N}\frac{\delta L_l}{1+\delta L_l}(\dots)+\sum_{1\le i<j\le N}\frac{\delta L_i}{1+\delta L_i}\frac{\delta L_j}{1+\delta L_j}(\dots)(\dots)+O(N^2\|L\|^3)$$

and the approximate drift term is

$$\mathbb{B}_i'' = -\kappa(\lambda_i)-\sum_{l=i+1}^{N}\frac{\delta L_l}{1+\delta L_l}\Big(\kappa(\lambda_i+\lambda_l)-\kappa(\lambda_i)-\kappa(\lambda_l)\Big)-\dots$$

which requires only $\approx N^2$ evaluations of the cumulant.

Data for the example

1. Tenor structure: 10 to 30 years, semi-annual
2. flat volatilities $\lambda(\cdot, T_i) = 18\%$
3. flat term structure of interest rates: bond prices $B(0, T_i) = \exp(-0.04T_i)$
4. driving process: NIG with $\alpha = \delta = 12$ and $\mu = \beta = 0$
5. caplets, swaptions and FRAs

Caplets: Picard approximation

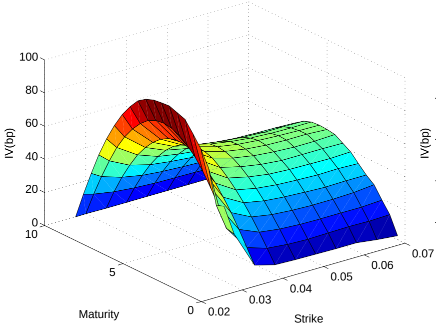

Figure: frozen ($Z^{(1)}$) vs true

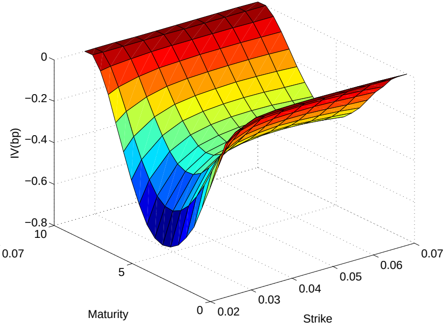

Figure: Picard ($Z^{(2)}$) vs true

Difference in caplet implied volatilities

Caplets: drift expansions

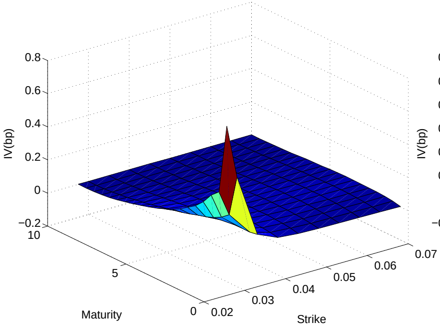

Figure: 1st order vs true

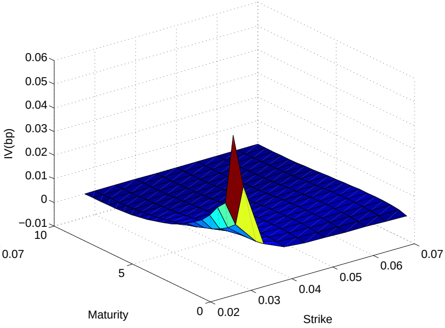

Figure: 2nd order vs true

Difference in caplet implied volatilities

Computational times

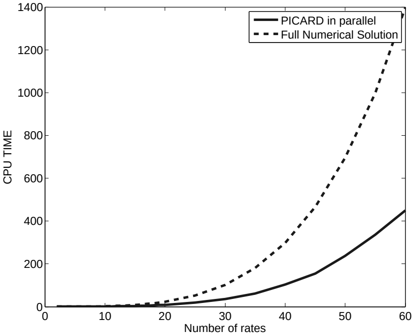

Figure: Time vs. rates

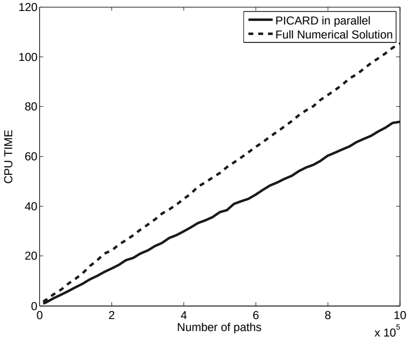

Figure: Time vs. paths

Swaptions

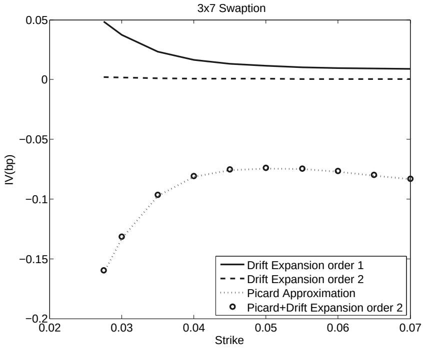

Figure: true vs. . . .

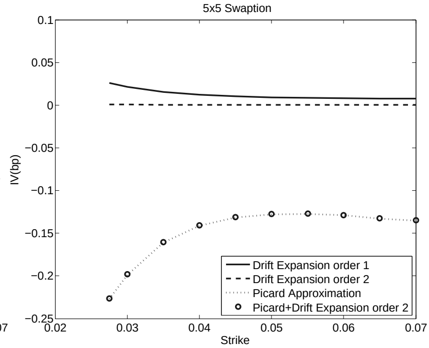

Figure: true vs. . . .

Difference in swaption implied volatilities

FRAs

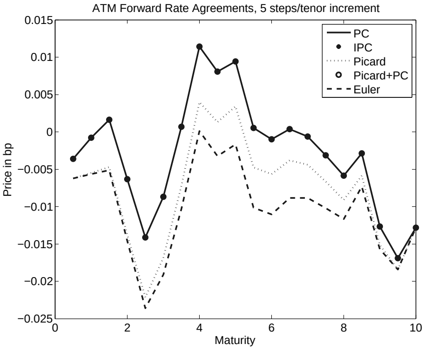

Figure: 5 time steps

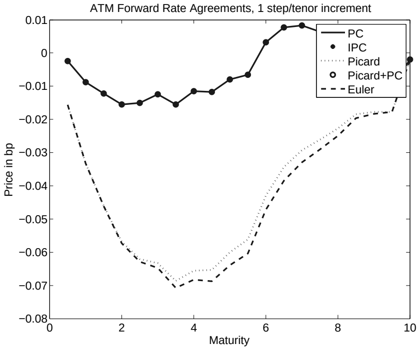

Figure: 1 time step

Difference in atm FRA prices

Summary and Outlook

1. We have presented . . .
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   - applicable for many driving processes
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3. A. Papapantoleon, D. Skovmand
   Picard approximation of stochastic differential equations and application to LIBOR models
   Preprint, 2010, `arXiv/1007.3362`

Thank you for your attention!