One-dimensional stochastic differential equations with generalized and singular drift

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Introduction

- stochastic differential equations (SDEs) with *generalized* drift, described by a *drift measure*:

$$X_t = X_0 + \int_0^t b(X_s)\,\mathrm{d}B_s + \int_{\mathbb{R}} L^X(t,y)\,\nu(\mathrm{d}y)$$

- $b : \mathbb{R} \to \mathbb{R}$ measurable, $B = (B_t)_{t \geq 0}$ Wiener process
- X continuous semimartingale
- L^X local time of X: $\int_0^t h(X_s)\,\mathrm{d}\langle X\rangle_s = \int_{\mathbb{R}} h(y)\,L^X(t,y)\,\mathrm{d}y$, h nonnegative, measurable
- quadratic variation:
$\langle X\rangle_t = \lim_{n\to\infty} \sum_{k=0}^{\infty} \left(X_{\frac{k+1}{2^n}\wedge t} - X_{\frac{k}{2^n}\wedge t} \right)^2$ in probability

Introduction

$$X_t = X_0 + \int_0^t b(X_s)\,\mathrm{d}B_s + \int_{\mathbb{R}} L^X(t,y)\,\nu(\mathrm{d}y)$$

- ν locally finite signed measure (with an additional condition)
- There exists a unique f such that

$$\nu(\mathrm{d}y) = \frac{1}{2f(y)}\,\mathrm{d}f(y)\,.$$

- f and $1/f$ are right-continuous, strictly positive and of locally bounded variation.

Introduction

$$\int_{\mathbb{R}} L^X(t,y)\,\nu(\mathrm{d}y) = \int_{\mathbb{R}} L^X(t,y)\,\frac{1}{2f(y)}\,\mathrm{d}f(y)$$

- New: Consider *drift functions* $f:\mathbb{R}\to[0,\infty)$ with:
- f right-continuous, of locally bounded variation.
- $1/f:\mathbb{R}\to(0,\infty]$ locally integrable.

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- New: Consider *drift functions* $f : \mathbb{R} \to [0,\infty)$ with:
- f right-continuous, of locally bounded variation.
- $1/f : \mathbb{R} \to (0,\infty]$ locally integrable.
- Do not assume that $1/f$ is of locally bounded variation.
- So, it is allowed that

$$F_+ := \{x \in \mathbb{R} : f(x) = 0\} \quad \text{and} \quad F_- := \{x \in \mathbb{R} : f(x-) = 0\}$$

are not empty

$$F := F_+ \cup F_- \neq \emptyset.$$

SDEs with generalized and singular drift

SDEs with generalized and singular drift

$$(*) \qquad X_t = X_0 + \int_0^t b(X_s)\,\mathrm{d}B_s + \int_{\mathbb{R}} L_m^X(t,y)\,\mathrm{d}f(y)$$

- L_m^X local time of the solution defined in dependence of f
- Drift function $f : \mathbb{R} \to [0, +\infty)$ is right-continuous and of locally bounded variation.
- $1/f : \mathbb{R} \to (0, +\infty]$ is locally integrable.

SDEs with generalized and singular drift

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Definition of a solution

$(X, \mathbb{F})$ is called a solution to equation $(*)$ over $(\Omega, \mathcal{F}, \mathbf{P})$, if the following conditions are satisfied:

- $(X, \mathbb{F})$ is a continuous semimartingale.
- L_m^X is a version of the local time of X in the sense of an occupation times formula w.r.t. the measure

$$m(\mathrm{d}x) := 2\, f(x)\, \mathrm{d}x\,,$$

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  i.e., L_m^X satisfies for all $t \geq 0$

$$\int_0^t h(X_s)\, \mathrm{d}\langle X\rangle_s = \int_{\mathbb{R}} h(x)\, L_m^X(t,x)\, m(\mathrm{d}x) \quad \mathbf{P}\text{-a.s.}$$

  for all nonnegative and measurable functions h.

SDEs with generalized and singular drift

For all $x \in \mathbb{R}$ $L_m^X(\,.\,, x)$ is $\mathbb{F}$-adapted.

L_m^X is in t **P**-a.s. continuous and increasing as well as in x right-continuous with limits from the left.

Example

- Equation of a Bessel process of dimension $\delta \in (1, 2)$

(BES) $$X_t = x_0 + B_t + \int_0^t \frac{\delta - 1}{2\, X_s}\, ds$$

- Equation (BES) coincides with equation

($*$ BES) $$X_t = x_0 + B_t + \int_{\mathbb{R}} L_m^X(t, y)\, df(y)$$

- where $f(x) = |x|^{\delta - 1}$, $x \in \mathbb{R}$, and therefore $F = F_- = \{0\}$.

Transformation of the equation

$$(*) \qquad X_t = X_0 + \int_0^t b(X_s)\,\mathrm{d}B_s + \int_{\mathbb{R}} L_m^X(t,y)\,\mathrm{d}f(y)$$

- Want to find propositions on existence and uniqueness of solutions of $(*)$.
- Aim: Drift shall be removed as far as possible.
- Define the strictly increasing and continuous function

$$G(x) := \int_0^x \frac{1}{f(y)}\,\mathrm{d}y, \qquad x \in \mathbb{R}.$$

Transformation of the equation

- $F = \emptyset$: Then, $1/f$ is of locally bounded variation.
- Therefore, G is difference of convex functions.
- Can apply the generalized Itô formula
- X solution of $(*)$, then G leads to a solution $Y = G(X)$ of

$$Y_t = Y_0 + \int_0^t \sigma(Y_s)\,\mathrm{d}B_s\,,$$

where $\sigma = \frac{b}{f} \circ H$, and H is the inverse of G.

Transformation of the equation

- $F \neq \emptyset \iff 1/f$ is not of locally bounded variation.
- Then, G is not the difference of convex function.
- Cannot apply the generalized Itô formula.
- It is not ensured that $Y = G(X)$ is a semimartingale.
- Beside the continuous local martingale part also another part (drift) remains.
- This remaining drift does not need to be of locally bounded variation.

Transformation of the equation

Example: Bessel equation with $\delta \in (1,2)$

$$(* \text{ BES}) \qquad X_t = x_0 + B_t + \int_{\mathbb{R}} L_m^X(t,y)\, \mathrm{d}f(y)$$

- $f(x) = |x|^{\delta-1}$, $x \in \mathbb{R}$, $\qquad F = F_- = \{0\}$
- Let X be a solution of $(*$ BES$)$.
- Apply G to X: $\qquad Y := G(X)$.

Transformation of the equation

Example: Bessel equation with $\delta \in (1,2)$

- Use an approximation $G_n(x) \xrightarrow[n\to\infty]{} G(x)$, $x \in \mathbb{R}$
- G_n difference of convex functions, $n \in \mathbb{N}$
- Set $Y^n := G_n(X),\ n \in \mathbb{N}$, apply the generalized Itô formula and $n \to +\infty$

$$\begin{aligned} Y_t &= y_0 + \int_0^t \sigma(Y_s)\,\mathrm{d}B_s + L_m^X(t,0) - L_m^X(t,0-) \\ &= y_0 + \int_0^t \sigma(Y_s)\,\mathrm{d}B_s + \frac{1}{2}\Big(L^Y(t,0) - L^Y(t,0-)\Big) \\ &= y_0 + \int_0^t \sigma(Y_s)\,\mathrm{d}B_s + \int_0^t \mathbb{1}_{\underbrace{G(F_-)}_{\{0\}}}(Y_s)\,\mathrm{d}Y_s \end{aligned}$$

Transformation of the equation

Theorem

- Let F be finite. (or only $|F \cap [-n, n]| < \infty$, $n \in \mathbb{N}$)
- Let X be a solution to equation $(*)$ with generalized and singular drift.

Transformation of the equation

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Then $(Y, \mathbb{F})$, defined by $Y = G(X)$, is a solution to the equation

$$(**) \qquad Y_t = Y_0 + \int_0^t \sigma(Y_s)\,\mathrm{d}B_s + \int_0^t \mathbb{1}_{G(F_-)}(Y_s)\,\mathrm{d}Y_s$$

with $L_m^X(t, x) = \frac{1}{2} L^Y(t, G(x))$, $t \geq 0$, $x \in \mathbb{R}$, $\mathbf{P}$-a.s.

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with $L_m^X(t, x) = \frac{1}{2} L^Y(t, G(x))$, $t \geq 0$, $x \in \mathbb{R}$, **P**-a.s.

- $\sigma = \frac{b}{f} \circ H$, H inverse of G
- solution: $(Y, \mathbb{F})$ is a continuous semimartingale.

Transformation of the equation

$$(**) \qquad Y_t = Y_0 + \int_0^t \sigma(Y_s)\,\mathrm{d}B_s + \int_0^t \mathbb{1}_{G(F_-)}(Y_s)\,\mathrm{d}Y_s$$

Theorem

- Let $(Y, \mathbb{F})$ be a solution to equation $(**)$.
- Put $X = H(Y)$.

Then $(X, \mathbb{F})$ is a solution to equation $(*)$ with generalized and singular drift. Moreover,

$$L_m^X(t,x) = \frac{1}{2} L^Y(t, G(x)), \qquad t \geq 0,\ x \in \mathbb{R},\ \mathbf{P}\text{-a.s.}$$

idea of the proof. generalized Itô formula

Existence and uniqueness of solutions

Definition

The solution to equation $(*)$ with generalized and singular drift is called unique (in law), if the following is satisfied: Let $(X^1, \mathbb{F}^1)$ and $(X^2, \mathbb{F}^2)$ be two solutions to equation $(*)$ with the same initial distribution, then X^1 and X^2 have the same law.

Example: Bessel equation with $\delta \in (1,2)$

- X solution of ($*$ BES), $Y = G(X)$
- Then

$$
\begin{aligned}
Y_t &= y_0 + \int_0^t \sigma(Y_s)\,\mathrm{d}B_s + \frac{1}{2}\Big(L^Y(t,0) - L^Y(t,0-)\Big) \\
&= y_0 + \int_0^t \sigma(Y_s)\,\mathrm{d}B_s + \Big(L^X_m(t,0) - L^X_m(t,0-)\Big)
\end{aligned}
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Existence and uniqueness of solutions

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Existence and uniqueness of solutions

Example: Bessel equation with $\delta \in (1,2)$

- X solution of $(* \text{ BES})$, $Y = G(X)$
- Then

$$
\begin{aligned}
Y_t &= y_0 + \int_0^t \sigma(Y_s)\,\mathrm{d}B_s - \frac{1}{2}L^Y(t,0-) \\
&= y_0 + \int_0^t \sigma(Y_s)\,\mathrm{d}B_s - L_m^X(t,0-)
\end{aligned}
$$

Existence and uniqueness of solutions

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- Then

$$\begin{aligned} Y_t &= y_0 + \int_0^t \sigma(Y_s)\,\mathrm{d}B_s \\ &= y_0 + \int_0^t \sigma(Y_s)\,\mathrm{d}B_s \end{aligned}$$

Existence and uniqueness of solutions

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- X solution of ($*$ BES), $Y = G(X)$
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$$= y_0 + \int_0^t \sigma(Y_s)\,\mathrm{d}B_s + \Big(L_m^X(t,0) - L_m^X(t,0-)\Big)$$

Existence and uniqueness of solutions

- In general we cannot expect uniqueness of solutions of $(*)$.
- The jumps $L_m^X(t,a) - L_m^X(t,a-)$ (resp. $L^Y(t,G(a)) - L^Y(t,G(a)-)$) in $a \in F_-$ are not determined by the equation.
- We control the jumps of L_m^X in the points of F_-.
- Therefore, let ν be a locally finite signed measure.
- ν has no mass in F_-^c and satisfies $\nu(\{x\}) < \frac{1}{2}$.

controlled equation

$$(* \, \nu) \quad \begin{cases} \text{(i)} \quad X_t = X_0 + \int_0^t b(X_s)\, \mathrm{d}B_s + \int_{\mathbb{R}} L_m^X(t,y)\, \mathrm{d}f(y), \\ \text{(ii)} \quad L_m^X(t,a) - L_m^X(t,a-) = 2\, L_m^X(t,a)\, \nu(\{a\}),\ a \in F_- \end{cases}$$

Existence and uniqueness of solutions

- F finite
- $(*\,\nu)$(ii) gives us an additional relation in the remaining drift of the transformed equation:

$$(**\,\nu)\qquad \begin{aligned} Y_t &= Y_0 + \int_0^t \sigma(Y_s)\,\mathrm{d}B_s + \int_0^t \mathbb{1}_{G(F_-)}(Y_s)\mathrm{d}Y_s \\ &= Y_0 + \int_0^t \sigma(Y_s)\,\mathrm{d}B_s + \int_{\mathbb{R}} L^Y(t,y)\,\nu^G(\mathrm{d}y), \end{aligned}$$

  where $\nu^G = \nu \circ G^{-1}$.
- Equations with generalized drift, described by a drift measure, are well-known.
- There are results about existence and uniqueness of solutions [Engelbert/Schmidt].

Existence and uniqueness of solutions

- We need the following sets:

$$N_b := \{x \in \mathbb{R} : b(x) = 0\}$$

  and

$$E_{b/\sqrt{f}} := \left\{ x \in \mathbb{R} : \int_U f(y)\, b^{-2}(y)\, \mathrm{d}y = \infty \quad \forall\, U \subseteq \mathbb{R} \text{ with } x \in U \right\}$$

- Applying G and H, we get

Existence and uniqueness of solutions

Theorem

Let $|F| < \infty$. For every initial distribution there exists a solution to the system $(* \, \nu)$ if and only if

$$E_{b/\sqrt{f}} \subseteq N_b \, .$$

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Remark

Sufficiency of $E_{b/\sqrt{f}} \subseteq N_b$ also holds without any condition on F.

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Remark

Sufficiency of $E_{b/\sqrt{f}} \subseteq N_b$ also holds without any condition on F.

Theorem

Let $|F| < \infty$. For every initial distribution there exists a unique solution to the system $(* \, \nu)$ if and only if

$$E_{b/\sqrt{f}} = N_b$$

is satisfied.

Existence and uniqueness of solutions

Example: Bessel equation with $\delta \in (1,2)$

- Control of the jumps of L_m^X in Zero

$$(* \text{ BES } \nu_\alpha) \begin{cases} \text{(i)} \quad X_t = x_0 + B_t + \int_{\mathbb{R}} L_m^X(t,y)\, \mathrm{d}f(y), \\ \\ \text{(ii)} \quad L_m^X(t,0) - L_m^X(t,0-) = 2\, L_m^X(t,0)\, \underbrace{\nu_\alpha(\{0\})}_{\alpha}, \end{cases}$$

- with $\nu_\alpha = \alpha\delta_0$, where $\alpha < \frac{1}{2}$.
- We have: $E_{b/\sqrt{f}} = N_b = \emptyset$.
- By variation of the parameter α we find uncountably many different solutions to equation ($*$ BES) and therefore to (BES).

Example: Bessel equation with $\delta \in (1, 2)$

- $\alpha = 0$: The solution X of (BES) has a continuous local time L_m^X (*symmetric Bessel process*).
- $\alpha \neq 0$: We find a lot of *skew Bessel processes*.

Thank you!