Some aspects of fractional Brownian motion

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Introduction

fBm

A continuous square integrable centered process X defined on a probability space $(\Omega, \mathbb{F}, \mathbb{P})$, $X_0 = 0$, is a *fractional Brownian motion* with self-similarity index $H \in (0, 1)$ if it is a Gaussian process with zero mean and covariance function

$$\mathbb{E}\left(X_s X_t\right) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t - s|^{2H}\right).$$

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- fBm has stationary increments.
- Self-similarity: Law $(X_{a\cdot}) = $ Law $(a^H X_\cdot)$, $a > 0$.
- History: Kolmogorov, Molchan, Golosov, Mandelbrot, . . .

Introduction

fBm: more properties

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- X positively correlated increments, when $H > \frac{1}{2}$ and negatively correlated increments, when $H < \frac{1}{2}$.
- X is a standard Brownian motion, if $H = \frac{1}{2}$.
- X is not a semimartingale, if $H \neq \frac{1}{2}$.

Introduction
The transformations

Assume that B is a fBm with Hurst index H. Then

$$M_t = C_H \int_0^t s^{\frac{1}{2}-H}(t-s)^{\frac{1}{2}-H} dB_s$$

is a Gaussian martingale with bracket $c_H t^{2-2H}$. This was know to Molchan and Golosov, and rediscovered by several authors, including Norros, V., Virtamo. Moreover, one can transform M back to B: $B_t = \int_0^t m(t,s) dM_s$.

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Here the integrals are path wise, defined using integrations by parts [possible because of the Hölder continuity of the kernel/fBm].

Characterization of fBm

Extension of Lévy's characterization to fBm

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(a) The sample paths of the process X are Hölder continuous with any $\beta \in (0, H)$.

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(c) The process

$$M_t = \int_0^t s^{\frac{1}{2}-H}(t-s)^{\frac{1}{2}-H} dX_s$$

is a martingale with respect to the filtration $\mathbb{F}^X$.

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Theorem [Mishura-V.] Let X be a continuous centered square integrable process, $X_0 = 0$. Then the following are equivalent:

- The process X is a fBm .
- The process X satisfies (a), (b), (c)

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- The process X is a fBm .
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A version of this theorem in the book *Stochastic Calculus for Fractional Brownian Motion and Related Processes* by Yuliya Mishura, and a slightly different version will appear in *Annals of Probability*.

Characterization of fBm

Extension of Lévy's characterization to fBm, first result, on the proof

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Extension of Lévy's characterization to fBm, first result, on the proof

- If X is a fBm, then we have (a), (b), (c).
- To prove that (a), (b), (c) imply that X is a fBm is more difficult. The first step is to show that M in (c) is a Gaussian martingale with bracket $\langle M, M \rangle :_t = c_H t^{2-2H}$. After this everything will be easy, since we can go back from the martingale M to the process X using integral transformations.

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Characterization of fBm

An alternative characterization

In the characterization of fBm we used the condition

(b) For $t > 0$ we have

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Consider the alternative condition:

(bb) fBm satisfies

$$\sum_{k=1}^{n} |X_{T\frac{k}{n}} - X_{T\frac{k-1}{n}}|^{\frac{1}{H}} \overset{L^1(P)}{\to} E|X_1|^{\frac{1}{H}} T.$$

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$$\sum_{k=1}^{n}|X_{T\frac{k}{n}} - X_{T\frac{k-1}{n}}|^{\frac{1}{H}} \overset{L^1(P)}{\to} E|X_1|^{\frac{1}{H}}T.$$

Hu, Nualart and Song have shown that one can replace the condition (b) by the condition (bb) ; but for $H > \frac{1}{2}$ they additionally ask that the bracket of M is absolutely continuous with respect to Lebesgue measure for (a), (bb), (c) imply that the process X is fBm. [*Annals of Probability*, to appear.]

Characterization of fBm

Discussion

- Can one use the above characterizations?

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- Can one use the above characterizations?
- Can one simplify the proof of these results?
- Is it possible to obtain more intrinsic characterizations for fBm than the two already mentioned are?

Path wise integrals with respect to fBm

Fractional Besov spaces

Definitions Fix $0 < \beta < 1$.
(i) Let $W_1^\beta = W_1^\beta[0, T]$ be the space of real-valued measurable functions $f : [0, T] \to \mathbb{R}$ such that

$$||f||_{1,\beta} := \sup_{0 \leq s < t \leq T} \left(\frac{|f(t) - f(s)|}{(t-s)^\beta} + \int_s^t \frac{|f(u) - f(s)|}{(u-s)^{1+\beta}} du \right) < \infty.$$

Path wise integrals with respect to fBm

Fractional Besov spaces

Definitions Fix $0 < \beta < 1$.
(ii) Let $W_2^{\beta} = W_2^{\beta}[0, T]$ be the space of real-valued measurable functions $f : [0, T] \to \mathbb{R}$ such that

$$||f||_{2,\beta} := \int_0^T \frac{|f(s)|}{s^{\beta}} ds + \int_0^T \int_0^s \frac{|f(u) - f(s)|}{(u - s)^{1+\beta}} duds < \infty.$$

Path wise integrals with respect to fBm

fBm and fractional Besov spaces

The Besov-spaces are closely related to the spaces of Hölder continuous functions. More precisely, for any $0 < \epsilon < \beta \wedge (1 - \beta)$,

$$C^{\beta+\epsilon}[0,T] \subset W_1^{\beta}[0,T] \subset C^{\beta-\epsilon}[0,T] \quad \text{and} \quad C^{\beta+\epsilon}[0,T] \subset W_2^{\beta}[0,T].$$

where $C^{\gamma}[0,T]$ stands for Hölder continuous functions of order γ.

Recall that the trajectories of B^H for a.s. $\omega \in \Omega$, any $T > 0$ and any $0 < \gamma < H$ belong to $C^{\gamma}[0,T]$. This follows from the Kolmogorov continuity theorem. We obtain that the trajectories of B^H for a.s. $\omega \in \Omega$, any $T > 0$ and any $0 < \beta < H$ belong to $W_1^{\beta}[0,T]$.

Path wise integrals with respect to fBm

Fractional integrals and fractional Besov spaces

Denote by $\Gamma(\beta)$ the Gamma-function. Recall the left-sided Riemann-Liouville fractional integral operator I_{+}^{β} of order $\beta > 0$:

$$I_{0+}^{\beta}(f)(s) = \frac{1}{\Gamma(\beta)} \int_0^s f(u)(s-u)^{\beta-1} du.$$

The corresponding right-sided fractional integral operator I_-^β is defined by

$$I_{t-}^\beta(f)(s) = \frac{1}{\Gamma(\beta)} \int_s^t f(u)(u-s)^{\beta-1} du.$$

If $f \in W_1^\beta[0,T]$, then its restriction to $[0,t] \subseteq [0,T]$ belongs to $I_-^\beta(L_\infty[0,t])$. Also, if $f \in W_2^\beta[0,T]$, then its restriction to $[0,t] \subseteq [0,T]$ belongs to $I_+^\beta(L_1[0,t])$, where $I_-^\beta(L_\infty[0,t])$ (resp. $I_+^\beta(L_1[0,t])$) stand for the image of $L_\infty[0,t]$ (resp. $L_1[0,t]$) by the fractional Riemann-Liouville operator I_-^β (resp. I_+^β).

Path wise integrals with respect to fBm

Fractional derivatives

We need one more definition to be able to define path wise integrals:

Let $f : [0, T] \to \mathbb{R}$ and $0 < \beta < 1$. If $f \in I_{+}^{\beta}(L_1[0, T])$ (resp. $f \in I_{-}^{\beta}(L_\infty[0, T])$) then the Weyl fractional derivatives are defined by

$$(D_{0+}^{\beta} f)(x) = \frac{1}{\Gamma(1-\beta)}\left(\frac{f(x)}{x^{\beta}} + \beta \int_0^x \frac{f(x)-f(y)}{(x-y)^{\beta+1}} dy\right) \mathbf{1}_{(0,T)}(x),$$

$$\left(\text{resp.} (D_{T-}^{\beta} f)(x) = \frac{1}{\Gamma(1-\beta)}\left(\frac{f(x)}{(T-x)^{\beta}} + \beta \int_x^T \frac{f(x)-f(y)}{(y-x)^{\beta+1}} dy\right) \mathbf{1}_{(0,T)}(x)\right).$$

Path wise integrals with respect to fBm

The following proposition clarifies the construction of the stochastic integrals using the farctional spaces. This approach is by Nualart and Rascanu.

Proposition Let $f \in W_2^{\beta}[0,T]$, $g \in W_1^{1-\beta}[0,T]$. Then for any $t \in (0,T]$ the Lebesgue integral

$$\int_0^t (D_{0+}^{\beta} f)(x)(D_{t-}^{1-\beta} g_{t-})(x)dx$$

exists, and we can define the *generalized Lebesgue-Stieltjes integral* by

$$\int_0^t fdg := \int_0^t (D_{0+}^{\beta} f)(x)(D_{t-}^{1-\beta} g_{t-})(x)dx.$$

Path wise integrals with respect to fBm

Properties of the integral

- If $f_1 = f_2 \quad a.e.$m then we have

$$\int_0^T f_1 dg = \int_0^T f_2 dg,$$

whenever both side are well-defined.

Path wise integrals with respect to fBm

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- One can show when $f \in C^\gamma[0, T]$ and $g \in C^\mu[0, T]$ with $\gamma + \mu > 1$, then the integral $\int_0^T f dg$ exists in the sense of the Proposition and coincides with the Riemann-Stieltjes integral. [Zähle]

Path wise integrals with respect to fBm

Properties of the integral, cont.

- Let $f \in W_2^{\beta}[0,T]$ and $g \in W_1^{1-\beta}[0,T]$. Then we have the estimation
$$|\int_0^t f dg| \le \frac{1}{\Gamma(\beta)} ||f||_{2,\beta} ||g||_{1,1-\beta}.$$

Path wise integrals with respect to fBm

Properties of the integral, cont.

- Let $f \in W_2^{\beta}[0,T]$ and $g \in W_1^{1-\beta}[0,T]$. Then we have the estimation
$$|\int_0^t f dg| \leq \frac{1}{\Gamma(\beta)} ||f||_{2,\beta} ||g||_{1,1-\beta}.$$
- Let $f, f^n \in W_2^{\beta}[0,T]$, $||f^n - f||_{2,\beta} \to 0$ as $n \to \infty$, and $g \in W_1^{1-\beta}[0,T]$. Then
$$\int f^n dg \to \int f dg.$$

Path wise integrals with respect to fBm

Composition of convex functions with fBm B

Theorem [Azmoodeh, Mishura, V.] Assume that F is a convex function with the right derivative F_x^+. Then we have the representation

$$F(B_T) = F(0) + \int_0^T F_x^+(B_s)dB_s.$$

Remark This extends the integration theory of fBm: if one applies the change of variables formula to the function $f(x) = |x|$ one obtains

$$|B_T| = \int_0^T \mathrm{sgn}(B_s)dB_s,$$

and the process $\mathrm{sgn}(B)$ has unbounded variation on compacts.

Path wise integrals with respect to fBm

Running maximum of a continuous bounded variation function A

The change of variables formula is the same for a continuous bounded variation functions A: if F is convex, then

$$F(A_T) = F(A_0) + \int_0^T F_x^+(A_s)dA_s.$$

- A natural question: to what extend fractional Brownian motion behaves as a continuous function with bounded variation?

Path wise integrals with respect to fBm

Running maximum of a continuous bounded variation function A

- Representation of the running maximum $A_t^* = \max_{s \leq t} A_s$ of a continuous bounded variation function A with $A_0 = 0$:

$$A_t^* = \int_0^t 1_{\{A_s^* = A_s\}} dA_s;$$

  this is a result by Azmoodeh, Tikanmäki, V..

Path wise integrals with respect to fBm

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  this is a result by Azmoodeh, Tikanmäki, V..
- We can show that here the analogy of the fBm to functions of bounded variation stops.

Consider a model, where the randomness comes from mixed Brownian – fractional Brownian motion $X^\epsilon = \epsilon W + B^H$, where W is a standard Brownian motion and B^H, $H > \frac{1}{2}$, is a fractional Brownian motion, independent of W

$$X = \epsilon W + B^H.$$

The corresponding geometric process is $S = e^{\epsilon W + B^H - \frac{1}{2}\epsilon t}$ and the Samuelson model is $\tilde{S} = e^{\epsilon W - \frac{1}{2}\epsilon t}$. We have that

$$d\langle S, S\rangle_t = \epsilon^2 S_t^2 dt \text{ and } d\langle \tilde{S}, \tilde{S}\rangle_t = \epsilon^2 \tilde{S}_t^2 dt$$

Path wise integrals with respect to fBm

How to find new results?

With the conditional full support property and

$$d\langle S, S\rangle_t = \epsilon^2 S_t^2 dt \text{ and } d\langle \tilde{S}, \tilde{S}\rangle_t = \epsilon^2 \tilde{S}_t^2 dt$$

one can show the robust hedging result [Kloeden-Schumachers, Bender-Sottinen-V.]

- Assume that

$$f(\tilde{S}_T) = f(s_0) + \int_0^T h(t, \tilde{S}_s, \tilde{g}_1(s), \ldots, \tilde{g}_p(s)) d\tilde{S}_s,$$

  where h is continuous w.r.t. uniform topology, $\tilde{g}_i$ are hindsight factors constructed from $\tilde{S}$.

Path wise integrals with respect to fBm

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  where h is continuous w.r.t. uniform topology, $\tilde{g}_i$ are hindsight factors constructed from $\tilde{S}$.
- Then we have robust replication result:

$$f(S_T) = f(s_0) + \int_0^T h(t, S_s, g_1(s), \ldots, g_p(s)) dS_s,$$

  and g_i are hindsight factors constructed from S.

Path wise integrals with respect to fBm

How to obtain new results?

We have the replication formula

$$f(S_T) = f(s_0) + \int_0^T h(t, S_s, g_1(s), \ldots, g_p(s))dS_s$$

with $S = e^{\epsilon W + B^H - \frac{1}{2}\epsilon^2 t}$. Let now $\epsilon \to 0$. What happens?
For example, if $f(x) = (x - K)^+$, then $f(s_0) \to (s_0 - K)^+$ and $h(t, S_t) \to 1_{\{\hat{S}_t > K\}}(t)$, where $\hat{S} = e^{B^H}$. Moreover, it was possible to show that the integral

$$\int_0^T 1_{\{\hat{S}_t > K\}}(t)d\hat{S}_t$$

exists as a path wise integral.

Applications to finance

Write $S = e^{B^H}$.
From the change of variables formula we get

$$(S_T - K)^+ = (S_0 - K)^+ + \int_0^T 1_{\{S_u > K\}} dS_u.$$

We make two observations:

- If the option is out-of-the-money, this is an arbitrage example with geometric fractional Brownian motion with continuous trading.

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- If the option is out-of-the-money, this is an arbitrage example with geometric fractional Brownian motion with continuous trading.
- The strategy in the above replication formula is so called stop-loss-start-gain strategy. This strategy is self-financing with geometric fractional Brownian motion, but it is well-known that it is not self-financing in the Samuelson model.

Applications to finance

Zero problem

Let B be a fBm. Fix $T = 1$ and assume that the process g is such that

$$\int_0^1 g_u dB_u = 0,$$

where the integral is a Riemann-Stieltjes integral, and g is adapted to $\mathbb{F}^B$.

When we can conclude that $g = 0$? The following answers are known:

- If g is deterministic, then $g = 0$ [the random variable $\int_0^1 g_u dB_u = 0$, is a Gaussian random variable].
- If g is a simple predictable process, then $g = 0$ [true at least when predictable process is defined with respect to deterministic time grid. . .].

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A related problem: *strong arbitrage*:

Fix $c > 0$. Is there an adapted process g such that one could replicate c with

$$c = \int_0^1 g_u dS_u.$$

Open problems

The hard ones

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- Let $\mathcal{T}$ be the collection of all stopping times τ such that $\tau \leq 1$. Find
$$\sup_{\tau\in\mathcal{T}} \mathbb{E}B_\tau.$$
- In connection to transaction models: find the ϵ - semimartingale S^ϵ of geometric fractional Brownian motion: $1-\epsilon \leq \frac{e^B}{S^\epsilon} \leq 1+\epsilon$.

GOOD LUCK WITH THIS AGENDA

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That's all, folks