Analysis of stochastic flows

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VISIONS in STOCHASTICS , 2010

Stochastic flows

- (X, ρ) is a complete separable metric space
- $\{\phi_{s,t}; 0 \le s \le t\}$ is a family of random mappings in X
- For any $0 \le s_1 \le s_2 \le \ldots s_n < \infty$ ϕ_{s_1,s_2} ,... ,ϕ_{s_{n-1},s_n} are independent.
- For any $s, t, r \ge 0$ $\phi_{s,t}$ and $\phi_{s+r,t+r}$ are equidistributed.
- For any $r \le s \le t$ and $u \in X$ $\phi_{r,s}\phi_{s,t}(u) = \phi_{r,t}(u)$, $\phi_{r,r}$ is the identity map.
- The random process $\{x(u,t) = \phi_{0,t}(u); t \ge 0\}$ describes the motion of a particle, which starts from the point u

Stochastic flows

Example

- $X = R^d$, for every u the process $x(u,t)$ is a solution to the Cauchy problem for SDE

$$dx(u,t) = a(x(u,t))dt + b(x(u,t))dw(t),\ x(u,0) = u.$$

Example

- The Harris flow of Brownian particles

$X = R$, $\{x(u,t);\ u \in X,\ t \geq 0\}$ is a family of Brownian martingales with respect to a common filtration, x is order-preserving and

$$d < x(u_1, \bullet), x(u_2, \bullet) >= \varphi(x(u_1,t) - x(u_2,t))dt,$$

where φ is a positive definite function.

- The Arratia flow

$$\varphi(x) = 1_{\{x=0\}}$$

Properties

- The law of the one-point motions does not determine the flow
- Possibility of coalescence
- The map $x(\bullet, t)$ can be discontinuous

The relationships with SDE

$$dx_\varepsilon(u,t) = \int_{\mathbb{R}} \psi_\varepsilon(x_\varepsilon(u,t) - p) W(dp, dt)$$

$$\int_{\mathbb{R}} \psi_\varepsilon^2(u) du = 1, \; supp \psi_\varepsilon \subset [-\varepsilon, \varepsilon]$$

Theorem

A. A. Dorogovtsev, 2005. The $n-$point motions of x_ε converge to the $n-$point motions of the Arratia flow when $\varepsilon \to 0$.

Theorem

T. V. Malovichko, 2008. The same statement when

$$\psi_\varepsilon^2 \to p_1 \delta_{-1} + p_2 \delta_1$$

Questions

- Does the flow of Brownian particles inherit the properties of solutions to SDE with a Gaussian noise? (Girsanov theorem, Large Deviations Principle, Clark representation, Krylov-Veretennikov expansion)
- Properties of the mappings $\phi_{0,t} : X \to X$

Gaussian approach

$$dx_\varepsilon(u,t) = \sqrt{\varepsilon}\int \varphi(x_\varepsilon(u,t)-p)W(dp,dt)$$

$$x_\varepsilon(u,0) = u$$

Theorem

Dorogovtsev, Ostapenko, 2009. Suppose that $\varphi = \varphi_1 \star \varphi_2$, $\varphi_i \subset S(\mathbb{R})$, $\mu = N(0,1)$. x_ε *satisfies LDP in* $C([0;1], L_2(\mathbb{R},\mu))$ *with the rate function*

$$I(z) = \inf \frac{1}{2}\int\int_0^{+\infty} h^2(p,t)\,dp\,dt,$$

$$dz(u,t) = \int \varphi(z(u,t)-p)h(p,t)\,dp\,dt,\ z(u,0) = u,\ u \in \mathbb{R}$$

$I(z) = +\infty$ *if there is no such* h.

$$0 = u_0 < u_1 < \dots < u_n = 1,$$

$$\tau(u_0) = T,\ \tau(u_{k+1}) = \inf\{s : x(u_{k+1}, s) = x(u_k, s)\} \wedge T$$

Theorem

Dorogovtsev, 2006. The total time of free motion in the Arratia flow is finite.

$$\sup \sum_{k=0}^{n} \tau(u_k) < +\infty \ a.s.$$

Corollary

There exist the integrals

$$\int_0^1 \int_0^{\tau(u)} a(x(u,s))ds = \lim_{\max u_{k+1}-u_k \to 0} \sum_{k=0}^{n} \int_0^{\tau(u_k)} a(x(u_k,s))ds$$

$$\int_0^1 \int_0^{\tau(u)} a(x(u,s))dx(u,s) =$$

$$= L_2 - \lim_{\max u_{k+1}-u_k \to 0} \sum_{k=0}^{n} \int_0^{\tau(u_k)} a(x(u_k,s))dx(u,s)$$

Theorem

Dorogovtsev, 2006. The distribution of the Arratia flow with the drift a is absolutely continuous with respect to the distribution of the Arratia flow with the density

$$\exp\{\int_0^1 \int_0^{\tau(u)} a(x(u,s))dx(u,s) - \frac{1}{2}\int_0^1 \int_0^{\tau(u)} a^2(x(u,s))ds\}$$

Gaussian approach for the Arratia flow

Theorem

Dorogovtsev, Ostapenko, 2009. The family $\{x_\varepsilon(u,t) = x(u,\varepsilon t),\ \varepsilon > 0\}$ *satisfies LDP with the rate function*

$$I(x) = \frac{1}{2}\int_0^1 \int_0^{\tau(u)} h_t^{'}(u,t)^2 dt$$

Theorem

Dorogovtsev, 2008. The linear combinations of functions

$$\exp\{\int_0^1 \int_0^{\tau(u)} a(u,s)dx(u,s) - \frac{1}{2}\int_0^1 \int_0^{\tau(u)} a^2(u,s)ds\},\ a \in C([0,1]^2)$$

are dence in the space of all square-integrable functionals from the Arratia flow.

Definition

(A. V. Skorokhod) Strong random operator is a continuous in probability linear map from the space H to the space of all random elements in H .

Example

- $H = L_2([0;1]),\ Af(t) = \int_0^t f(s)dw(s)$
- $H = L_2([0;1]),\ Af = f(\theta),\ \theta$ is uniformly distributed

In general a strong random operator is not a randomly chosen bounded linear operator. It can be unbounded with probability one.

A. V. Skorokhod. Random linear operators. Kiev: Nauk. dumka, 1978.

Semigroups related to stochastic flows

Definition

A family of strong random operators $\{G_{s,t},\ 0 \le s \le t < \infty\}$ is referred to as a semigroup if the following conditions hold:

1. For any $s, t, r \ge 0:\ \ G_{s,t}$ and $G_{s+r,t+r}$ are equidistributed.
2. For any $x \in H:\ \ E||G_{0,t}x - x||^2 \mapsto 0,\ n \mapsto \infty.$
3. For any $0 \le s_1 \le \cdots \le s_n < \infty:\ \ G_{s_1,s_2}\ , \ldots\ , G_{s_{n-1},s_n}$ are independent.
4. For any $r \le s \le t:\ \ G_{r,s}G_{s,t} = G_{r,t},\ G_{r,r} = I$, where I is the identity operator.

Semigroups related to stochastic flows

Example

Stochastic semigroup related to the stochastic flow

$$G_{s,t}f(u) = f(\phi_{s,t}(u))$$

Theorem

(Krylov-Veretennikov expansion) If

$$dx(u,t) = a(x(u,t))dt + b(x(u,t))dw(t),\ x(u,0) = u$$

then

$$f(x(u,t)) =$$

$$= \sum_{n=0}^{\infty} \int_{0 \le s_1 \le \ldots \le s_n \le t} T_{t-s_n} B T_{s_n - s_{n-1}} B \ldots T_{s_1} f(u) dw(s_1) \ldots dw(s_n).$$

Here $\{T_t\}$ is a transition semigroup and $B = b\frac{d}{du}$.

Theorem

Dorogovtsev, 2010. If $\{G_{s,t}\}$ is a multiplicative functional from the Wiener process W, then

$$G_{0,t} = \sum_{n=0}^{\infty} \int_{0 \le s_1 \le \ldots \le s_n \le t} Q_{t-s_n} B Q_{s_n - s_{n-1}} B \ldots Q_{s_1} dW(s_1) \ldots dW(s_n),$$

$$Q_t = EG_{0,t},\ B = \lim_{t \to 0+} \frac{1}{t} EG_{0,t} W(t).$$

Remark. $G_{0,t}$ is a "solution" to the equation

$$dG_{0,t} = AG_{0,t}dt + BG_{0,t}dW(t)$$

Semigroups of finite-dimensional projections

Remark. Suppose that $\{G_t,\, 0 \le t < \infty\}$ is a strongly continuous semigroup of bounded operators in a separable Banach space $\mathscr{B}$. Assume that $\dim G_t(\mathscr{B}) < \infty$ for every $t > 0$. Then $\dim \mathscr{B} < \infty$.

Example

Let H be a Hilbert space with an orthonormal basis $\{e_k,\, k \ge 1\}$. Consider the sequence $\{n_k,\, k \ge 1\}$ of independent Poisson processes with intensities $\{\lambda_k,\, k \ge 1\}$. Suppose that

$$\forall \rho > 0: \quad \sum_{k=1}^{\infty} \exp(-\rho \lambda_k) < +\infty,$$

$$v^k_{s,t} = \begin{cases} 0, & n_k(t) - n_k(s) > 0, \\ 1, & n_k(t) - n_k(s) = 0. \end{cases}$$

Then

$$G_{s,t}(u) = \sum_{k=1}^{\infty} (u, e_k) v^k_{s,t} e_k.$$

Theorem

Let $\{G_{s,t},\, 0 \leq s \leq t < \infty\}$ be a semigroup of random finite-dimensional projections in a separable Hilbert space H. Then there exists an orthonormal basis $\{e_k,\, k \geq 1\}$ in H and Poisson processes $\{n_k,\, k \geq 1\}$ which have jointly independent increments, such that

$$G_{s,t}(u) = \sum_{k=1}^{\infty} (u, e_k) v_{s,t}^k e_k,$$

where for every k

$$v_{s,t}^k = \begin{cases} 0, & n_k(t) - n_k(s) > 0, \\ 1, & n_k(t) - n_k(s) = 0. \end{cases}$$

Let $\{G_{s,t}, 0 \leq s \leq t < \infty\}$ be a random semigroup of finite-dimensional projections and let K be a compact subset of H. The behavior of the value

$$\varsigma_K(t) = \max_{x \in K} ||x - G_{0,t}x||$$

as $t \to 0$ describes the geometry of the semigroup $\{G_{s,t}, 0 \leq s \leq t < \infty\}$.

Example

Suppose that $\lambda_n = n$, $n \geq 1$, Poissonian processes are independent and

$$K = \{x : (x, e_n)^2 \leq \frac{1}{n^2}, n \geq 1\}.$$

$$\varsigma_K(t)^2 = \sum_{n=1}^{\infty} \frac{\xi_n(t)}{n^2}$$

$$P - \lim_{t \to 0} \frac{\varsigma_K(t)}{\sqrt{t \ln t}} = 1$$

Widths of compact sets

Example

$$K = \{x : \sum_{n=1}^{\infty} n^2 (x, e_n)^2 \le 1\}.$$

$$\varsigma_K(t)^2 = \max_{n:\, \xi_n(t)=0} \frac{1}{n^2}$$

$$\liminf_{t\to 0} \varsigma_K(t)\varphi(t) \ge 1,\ \varphi(t) = \sqrt{\frac{2}{t} llnt}.$$

Dimension of operators in a semigroup

$$d_n(K) = \inf_{dimL=n} \max_{x \in K} \rho(x, K)$$

Example

Define $\alpha(t) = \dim G_{0,t}(H)$.

$$\limsup_{t \to 0} \frac{t\alpha(t)}{2|\ln t|} \leq 1$$

$$\liminf_{t \to 0} \alpha(t) t |\ln t| \geq \frac{1}{2}.$$