A universal signal process for control and stopping problems

Jenny Sexton

University of Manchester UK

October 2010

A representation for quasi-left continuous processes

- Let $X = (X_t)_{t \geq 0}$ be a right continuous process such that $\{X_\tau | \tau \in \mathcal{T}\}$ is a uniformly integrable family of random variables and $X_{+\infty} = 0$.
- The process X is quasi-left continuous so for any $(\tau_n)_{n \geq 1}$ announcing a stopping time τ

$$\limsup_{n \to \infty} X_{\tau_n} = X_\tau \qquad ; \qquad P - a.s.$$

- Therefore, the process X can only have discontinuities at completely inaccessible stopping times.

A representation for quasi-left continuous processes

Let $c(t,l)$ to be an increasing random function of l then there exists a progressively measurable **signal process** $(\xi_t)_{t\geq 0}$ such that

$$X_\tau = E\left[\int_\tau^\infty c\left(t, \sup_{v\in[\tau,t)} \xi_v\right) re^{-rt}\,dt \middle| \mathcal{F}_\tau\right] \qquad ; \qquad \forall\tau\in\mathcal{T}.$$

The existence and uniqueness of such a representation has been established by Bank & El Karoui.

Remark The term $re^{-rt}\,dt$ may be replace with a finite random measure $\mu(dt)$ with full support

Interpretation of signal process

1. **Deterministic case** - If $c(t, l) = -l$ and $X(t)$ is a deterministic function then ξ is the derivative of the largest convex minorant of X. Equivalently, ξ is the derivative of X_{**}.
2. **Stochastic case** - The process Y defined via

$$Y_\tau = E\left[\int_\tau^\infty c\left(t, \sup_{v\in[0,t)} \xi_v\right) re^{-rt}\, dt \,\middle|\, \mathcal{F}_\tau\right] \qquad ; \qquad \forall \tau \in \mathcal{T}.$$

is the largest semimartingale which does not exceed X and with drift that can be expressed as $c(t, A_t)\, re^{-rt}\, dt$ for an increasing process A. That is the flat-off condition is satisfied

$$\int_0^\infty (Y_t - X_t)\, dA_t = 0$$

Interpretation of signal process

3. Let $c(t,l) = -l$ and let $Z = (Z_t)_{t\geq 0}$ be a Markov process such that the process X can be expressed as $X = u(Z)$ where u is an excessive function. Then ξ coincides with the subadditive operator

$$\underline{D}u(x) \triangleq \inf \frac{u(x) - E_x\left[\,u(X_\tau)\right|\tau < T_r]}{P_x[\tau = T_r]}.$$

where $T_r \sim \exp(r)$. The inf is taken over exit times from open neighbourhoods of x.

4. Consider a family of auxiliary stopping problems indexed by $l \in \mathbb{R}$

$$Y^l(\sigma) = \operatorname{ess}\sup_{\tau\in\mathcal{T}([\sigma,+\infty])} E\left[X_\tau + \int_\sigma^\tau c(t,l)\, re^{-rt}\, dt \middle| \mathcal{F}_\sigma\right]$$

For $(t,\omega) \in \mathbb{R}\times\Omega$ then

$$\xi(\omega,t) = \sup\left\{\, l\in\mathbb{R} \middle|\, Y^l(\omega,t) = X(\omega,t)\right\}.$$

Optimal stopping using Snell envelope

The Snell envelope is defined as

$$U_\sigma = \text{ess} \sup_{\tau \in \mathcal{T}([\sigma,\infty])} E\left[X_\tau \middle| \mathcal{F}_\sigma \right] \quad ; \quad \forall \sigma \in \mathcal{T}.$$

Theorem

Consider the Doob-Meyer decomposition of the Snell envelope $U = M - A$ into a uniformly integrable martingale M and a predictable increasing process A with $A_0 = 0$. Then

$$\underline{T} = \inf \left\{ t \geq 0 \middle| X_t = U_t \right\} \quad ; \quad \overline{T} = \inf \left\{ t \geq 0 \middle| A_t > 0 \right\}$$

are the smallest and largest stopping times which attain the supremum in

$$V = \sup_{\tau \in \mathcal{T}} E\left[X_\tau \right].$$

Optimal stopping using a signal process

Theorem
(Bank & Föllmer 2005) *The stopping times*

$$\underline{T} = \inf \{ t \geq 0 | \, \xi_t \geq 0 \} \qquad ; \qquad \overline{T} = \inf \{ t \geq 0 | \, \xi_t > 0 \}$$

are the smallest and largest stopping times which attain the supremum in

$$V = \sup_{\tau \in \mathcal{T}} E \left[X_\tau \right].$$

Connecting the signal and the Snell envelope

To connect these two theorems consider the process

$$\zeta_s \triangleq \sup_{v \in [0,s)} \xi_v \vee 0.$$

and the supermartingale V defined via

$$V_t \triangleq E\left[\int_t^\infty \zeta_s r e^{-rs}\, ds \middle| \mathcal{F}_t \right] = E\left[\int_0^\infty \zeta_s r e^{-rs}\, ds \middle| \mathcal{F}_t \right] - \int_0^t \zeta_s r e^{-rs}\, ds$$

then $V \geq X$ because

$$V_t \triangleq E\left[\int_t^\infty \zeta_s r e^{-rs}\, ds \middle| \mathcal{F}_t \right] \geq E\left[\int_t^\infty r e^{-rs} \sup_{v \in [t,s)} \xi_s\, ds \middle| \mathcal{F}_t \right] = X_t$$

so V dominates the Snell envelope of X.

Connecting the signal and the Snell envelope

However, $V_t = U_t$ for all $t \in \left[0, \overline{T}\right)$ because for $t \leq \overline{T}$

$$\begin{aligned} V_t &= E\left[\int_t^{\overline{T}} \zeta_s r e^{-rs}\, ds + \int_{\overline{T}}^{\infty} \zeta_s r e^{-rs}\, ds \middle| \mathcal{F}_t \right] \\ &= E\left[\int_{\overline{T}}^{\infty} r e^{-rs} \sup_{v\in[0,s)} \xi_v\, ds \middle| \mathcal{F}_t \right] \\ &= E\left[E\left[\int_{\overline{T}}^{\infty} r e^{-rs} \sup_{v\in[0,s)} \xi_v\, ds \middle| \mathcal{F}_{\overline{T}} \right] \middle| \mathcal{F}_t \right] \\ &= E\left[X_{\overline{T}} \middle| \mathcal{F}_t \right] \leq U_t. \end{aligned}$$

Link to a BSDE

Recall $\zeta_s = \sup_{v\in[0,s)} \xi_v \vee 0$ then the supermartingale V

$$V_\tau = E\left[\int_\tau^\infty \zeta_t\, re^{-rt}\, dt \middle| \mathcal{F}_\tau\right]$$

solves the BSDE

$$\begin{aligned} -dY_t &= \zeta_t re^{-rt}\, dt - Z_t\, dW_t \qquad ; \qquad \lim_{t\to+\infty} Y_t = 0 \\ Y_t &\geq X_t \ \forall t \geq 0 \ P\text{-a.s.} \\ 0 &= \int_0^\infty (Y_t - X_t)\, d\zeta_t \end{aligned}$$

A BSDE for the Snell envelope

Let $Y \in \mathbb{S}^1$, $\int Z\, dW$ is a uniformly integrable martingale, K is an increasing process with $K_{0-} = 0$. If the triplet (Y, Z, K) satisfies

$$\begin{aligned} -dY_t &= -Z_t\, dW_t + dK_t \qquad ; \qquad \lim_{t \to +\infty} Y_t = 0 \\ Y_t &\geq X_t \ \forall t \geq 0 \ P\text{-a.s.} \\ 0 &= \int_0^\infty (Y_t - X_t)\, dK_t \end{aligned}$$

Then, the process Y has the following explicit representation for any $t \geq 0$

$$Y_t = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}} E\left[X_\tau \middle| \mathcal{F}_t \right].$$

Example of a perpectual option

- The asset price denoted $P = (P_t)_{t \geq 0}$ is an exponential Lévy process: $P_t = P_0 e^{Y_t}$
- The payoff of an American put option with strike $k > 0$ is

$$X_t^k = e^{-rt} (k - P_t)^+ .$$

- Let $K_t = P_t / \alpha$ where $\alpha \in (0, 1)$ is defined via

$$\alpha \triangleq E\left[\int_0^\infty \inf_{0 \leq v \leq t} e^{Y_v} r e^{-rt} \, dt\right] = E\left[\inf_{0 \leq v \leq T_r} e^{Y_v}\right]$$

  where $T_r \sim \exp(r)$ independent of the filtration $\mathbb{F}$.
- The price process P has the representation

$$e^{-r\tau} P_\tau = E\left[\int_\tau^\infty r e^{-rt} \inf_{v \in [\tau, t)} K_v \, dt \middle| \mathcal{F}_\tau\right]$$

Example of a perpectual option

It can be directly verified that this process $K_t = P_t/\alpha$ satisfies the required representation:

$$
\begin{aligned}
& E\left[\int_\tau^\infty re^{-rt} \inf_{v\in[\tau,t)} K_v \, dt \middle| \mathcal{F}_\tau\right] \\
= \quad & \frac{e^{-r\tau}P_\tau}{\alpha} E\left[\int_\tau^\infty re^{-r(t-\tau)} \inf_{v\in[\tau,t)} e^{(Y_v - Y_\tau)} \, dt \middle| \mathcal{F}_\tau\right] \\
= \quad & \frac{e^{-r\tau}P_\tau}{\alpha} E\left[\int_0^\infty re^{-rs} \inf_{v\in[0,s)} e^{Y_v} \, dt\right] \\
= \quad & \frac{e^{-r\tau}P_\tau}{\alpha} E\left[\inf_{v\in[0,T_r)} e^{Y_v}\right] = e^{-r\tau}P_\tau
\end{aligned}
$$

Example of a perpectual option

Let $\xi_t^k = k - K_t$ then the payoff function can be represented as

$$X_\tau^k = E\left[\int_\tau^\infty re^{-rt} \sup_{v\in[\tau,t)} \xi_v^k \, dt \middle| \mathcal{F}_\tau \right].$$

Therefore, for any $k > 0$ the smallest and largest stopping times which attain

$$\sup_{\tau\in\mathcal{T}} E\left[e^{-r\tau} \left(k - P_\tau \right) \right]$$

are

$$\underline{T}^k = \inf \{ t \geq 0 | \, P_t \leq \alpha k \} \qquad ; \qquad \overline{T}^k = \inf \{ t \geq 0 | \, P_t < \alpha k \} .$$

Provisional example of a perpectual option

Let the asset price denoted $P = (P_t)_{t\geq 0}$ is a Lévy process. The payoff function of an American option $G(x)$ is such that for a continuous function g

$$G(x) = E_x\left[\int_0^\infty g(P_t)\ re^{-rt}\,dt\right].$$

In this case the signal coincides with the solution to the (non-standard) stopping problem

$$\underline{D}G(x) \triangleq \inf_{n\geq 1} \frac{E_x\left[\left.\int_0^{T_n} e^{-rs}g(P_s)\ ds\right| T_n < T_r\right]}{E_x\left[\int_0^{T_n} e^{-rs}\,ds\right]}.$$

where the inf is taken over exit times from open neighbourhoods of x. A different approach to this problem due to Deligiannidis et. al. (2009) suggests that if g is monotone then it may be possible to validate that

$$\underline{D}u(x) = g(x) \qquad \forall x \in \mathcal{D}.$$

Provisional example of a perpectual option

To identify $\underline{D}u(x)$ we consider a transformation similar to that used for a different purpose by Hobson & Klimmek (2010). Denoted τ_y the exit times from sets of the form $(-\infty, y)$ then

$$E_x\left[G\left(X_{\tau_y}\right)\middle|\tau_y < T_r\right] = G(y)\, E_x\left[e^{-r\tau_x}\right]$$

so letting

$$p(y) \triangleq E_x\left[e^{-r\tau_x}\right], \qquad \widetilde{G}(x,y) \triangleq \frac{G(x)}{1-p(y)}, \qquad f(y) \triangleq \frac{G(y)\,p(y)}{1-p(y)}$$

If $\widetilde{G} \in C^{2,2}$ and $\widetilde{G}_x$ and $\widetilde{G}_y$ are both increasing (resp. decreasing) then $\underline{D}G(x)$ coincides with

$$v(x) \triangleq \inf_{y\geq 0}\left\{\widetilde{G}(x,y) - \frac{G(y)\,p(y)}{1-p(y)}\right\}.$$

That is $\underline{D}G(x)$ is the $\widetilde{G}$-concave dual of $f(y)$.

A singular control problem

$$V(x) = \inf_{\theta \in \mathcal{A}^2} J(\theta)$$

$$J(\theta) = E\left[\int_0^\infty c\left(t, X_t^\theta\right)\, dt + \int_0^\infty k_t^1\, d\theta_t^1 + \int_0^\infty k_t^2\, d\theta_t^2\right]$$

1. The filtration $\mathbb{F}$ is assumed to be continuous.
2. Intervention costs k^1 and k^2 are positive processes.
3. Let $M = (M_t)_{t \geq 0}$ be a continuous martingale. The state process $X^\theta = \left(X_t^\theta\right)_{t \geq 0}$ is determined by the SDEs

$$dX_t^\theta = \left(a_t + b_t X_t^\theta\right)\, d[M]_t + \left(c_t + d_t X_t^\theta\right)\, dM_t + d\theta_t^1 - d\theta_t^2$$

where $(a_t, b_t, c_t, d_t)_{t \geq 0}$ are predictable bounded processes.

The set of controls

Take $\theta = \left(\theta^1, \theta^2\right)$ where $\theta^i \in \mathcal{A}$ and

$$\mathcal{A} \triangleq \{ \vartheta : \mathbb{R}_+ \times \Omega \to \mathbb{R}_+ | \, \vartheta \text{ non-decreasing, RCLL, } \vartheta_{0-} = 0 \} \, .$$

No additional constraints need be placed on the set of controls.

1. Under some additional assumptions on $c(t, x)$, taking $\theta^2 \equiv 0$ retrieves the problem studied by Bank (2005).
2. Taking the martingale M to be Brownian motion retrieves a class of problems studied by Cadenillas & Haussmann (1994) and Bahlali & Merzerdi (2005).

Assumptions

1. For any $\theta_t \equiv x \in R$ the process $c\left(t, X_t^x\right)$ is progressively measurable and $c\left(\cdot, X^x\right) \in L^1\left(P \otimes dt\right)$.
2. The process $c^*\left(t\right) = \inf_{\theta \in \mathbb{R}} c\left(t, X_t^\theta\right)$ is such that $c^* \in L^1\left(P \otimes dt\right)$.
3. For fixed $\left(t, \omega\right) \in R_+ \times \Omega$ the mapping $\theta \longmapsto c\left(t, \omega, \theta\right)$ is a convex functional mapping taking values in $\overline{\mathbb{R}}$.
4. The intervention costs $k = \left(k^1, k^2\right)$ are such that k^i for $i = 1, 2$ are positive processes such that the family of random variables $\left\{k_\tau^i \,|\tau \text{ is a stopping time}\right\}$ is uniformly integrable and $\lim_{t \to \infty} k_t^i = 0$.

Comment on Assumption 3

Bank substituted assumption 3 was substituted with:

3a. For fixed $(t, \omega) \in R_+ \times \Omega$ the function $c(t, \omega, \theta)$ only depends on $\theta_t(\omega)$ so the mapping $\theta \longmapsto c(t, \omega, \theta)$ is strictly convex function mapping $R_+ \longmapsto R$ with a continuous first derivative $c_\theta(t, \omega, \theta) \triangleq \frac{\partial}{\partial \theta} c(t, \omega, \theta)$ such that

$$c_\theta(t, \omega, +\infty) = +\infty \qquad ; \qquad c_\theta(t, \omega, -\infty) = -\infty.$$

This assumption restricts us to use $b = d \equiv 0$.

Directional derivative of the running cost functional

Fix $(t, \omega) \in \mathbb{R}_+ \times \Omega$ then the functional $\theta \mapsto c\left(t, X_t^\theta(\omega)\right)$ has a directional derivative at $\theta \in \mathcal{V}$ in the direction μ if $c\left(t, X_t^\theta\right) < +\infty$ and the limit

$$\nabla c_t(\theta; \mu) = \lim_{\lambda \downarrow 0} \frac{c\left(t, X_t^{\theta+\lambda\mu}\right) - c\left(t, X_t^\theta\right)}{\lambda}$$

exists, taking values in $\overline{\mathbb{R}}$.
For arbitrary $\eta \in \mathcal{A}^2$ the directional derivative of $c\left(t, X_t^\theta(\omega)\right)$ is

$$\nabla c_t(\theta; \eta) = -c_x\left(t, X_t^\theta\right) z_t$$

where z is the solution to the following SDE

$$dz_t = b_t z_t \, dA_t + d_t z_t \, dM_t + d\eta_t^1 - d\eta_t^2 \qquad ; \qquad z_0 = 0$$

Directional derivative of the running cost functional

In fact,

$$\nabla c_t(\theta;\eta) = -c_x\left(t, X_t^\theta\right)\Phi_t \int_0^t \frac{1}{\Phi_s}\, d\left(\eta_s^1 - \eta_s^2\right).$$

where

$$d\Phi_t = b_t \Phi_t \, d[M]_t + d_t \Phi_t \, dM_t \qquad ; \qquad \Phi_0 = 1.$$

The solution to this SDE coincides with

$$\frac{\partial X^0(t,\omega,x)}{\partial x}$$

where $X^\theta(t,\omega,x) = X_t^\theta$ with $X_0^\theta = x$. Recall that

$$dX_t^\theta = \left(a_t + b_t X_t^\theta\right) d[M]_t + \left(c_t + d_t X_t^\theta\right) dM_t + d\theta_t^1 - d\theta_t^2$$

A conditional expectation which is a subgradient

Denote by $\nabla J_i(\theta)_t$ for $i = 1, 2$ the continuous adapted process with the property that for each stopping time τ

$$\nabla J_1(\theta)_\tau = e^{-r\tau} k_\tau^1 + p_\tau^\theta \quad ; \quad \nabla J_2(\theta)_\tau = e^{-r\tau} k_\tau^2 - p_\tau^\theta$$

where

$$-p_t^\theta \triangleq \frac{1}{\Phi_t} E\left[\int_t^T c_x\left(s, X_s^\theta\right) \Phi_s \, ds \middle| \mathcal{F}_t \right].$$

Then, for each $\theta \in \mathcal{A}^2$ the process $\nabla J(\theta)$ is a subgradient of $J(\theta)$ i.e.

$$J(\varphi) - J(\theta) \geq E\left[\int_0^T \nabla J(\theta)_t \, d(\varphi_t - \theta_t) \right].$$

for all $\varphi \in \mathcal{A}^2$.

The objective function is Gâteaux differentiable

Lemma

The functional $J : \mathcal{V}^2 \to \overline{\mathbb{R}}$ is Gâteaux differentiable with Gâteaux differential given by

$$\nabla J(\theta) = (\nabla J_1(\theta), \nabla J_2(\theta))'$$

Lemma

Ekeland & Temam Proposition 2.2.1 *If $J : \mathcal{V}^2 \to \overline{\mathbb{R}}$ is a proper, convex and Gâteaux differentiable functional then the following two statements are equivalent*

1. $\theta^* \in \mathcal{A}^2$ *is a solution to* $\inf_{\theta \in \mathcal{A}^2} J(\theta)$
2. *For all* $\eta \in \mathcal{A}^2$

$$E\left[\int_0^\infty J'(\theta^*)_t \, d(\eta_t - \theta_t^*)\right] \geq 0.$$

where J' denotes the Gâteaux differential.

Necessary and sufficient conditions

Theorem
The control $\theta^* \in \mathcal{A}^2$ is optimal for the singular stochastic control problem if and only if both

$$\nabla J\left(\theta^*\right)_s \geq 0 \qquad ; \qquad \forall s \geq 0$$

and

$$\int_0^\infty \nabla J\left(\theta^*\right)_s \, d\theta^*_s = 0 \qquad P\text{-a.s.}$$

The optimal control for the control problem can be characterised in the usual way:

1. 'Initial jump' - if $g_0^i p_0 > k_0^i$ for any i, then $\triangle\theta^i_{0+} > 0$.
2. 'No action region' - the flat-off condition ensures that $d\theta^i_t = 0$ on the set $\left\{t : k_t^i > p_t g_t^i\right\}$ for each $i = 1, 2$.
3. 'Local time' - the measure $d\theta^i$ has the same points of increase as the local time of p_t, on the curve k_t^i / g_t^i.

Conditional expectations are linear BSDEs

For $\theta \in \mathrm{dom}\,(J)$ the conditional expectation

$$-p_t^\theta = \frac{1}{\Phi_t} E\left[\int_t^\infty c_x\left(s, X_s^\theta\right) \Phi_s \, ds \middle| \mathcal{F}_t \right].$$

is the first component to the solution of a linear BSDE. That is the process p^θ solves

$$dp_t^\theta = \left[c_x\left(t, X_t^\theta\right) - b_t p_t^\theta - d_t q_t^\theta \right] dt + q_t^\theta \, dM_t + dL_t^\theta.$$

with terminal condition $\lim_{T\to\infty} p_T^\theta = 0$ under the assumption that $q^\theta \in L\,(M)$; $\int q^\theta \, dM$ and L^θ are uniformly integrable martingales with $\left[M, L^\theta\right] = 0$.

A dual problem in terms of a BSDE

Take $\widehat{\theta} \in \mathrm{dom}\,(J)$ and let $(\widehat{p}, \widehat{q})$ denote the corresponding solution to the adjoint BSDE. Then, $\widehat{\theta} \in \mathcal{A}$ is optimal for the singular stochastic control problem if and only if $\left(\widehat{p}, \widehat{q}, \widehat{\theta}\right)$ is a solution to the following BSDE with obstacle constraint

$$\begin{aligned} dp_t &= \left[c_x\left(t, X_t^{\theta}\right) - b_t p_t - d_t q_t\right] dt + q_t\, dM_t + dL_t \\ \lim_{t \to +\infty} p_t &= 0 \\ e^{-rt} k_t^1 &\leq p_t \leq e^{-rt} k_t^2 \qquad \text{a.s.} \qquad \forall t \in [0, T] \\ 0 &= \int_0^{\infty} \left(e^{-rt} k_t^1 - p_t\right) d\theta_t^1 = \int_0^{\infty} \left(e^{-rt} k_t^2 + p_t\right) d\theta_t^2 \end{aligned}$$

Recall the obstacle problem linked to the stopping problem

Recall $\zeta_s = \sup_{v\in[0,s)} \xi_v \vee 0$ then the supermartingale V

$$V_\tau = E\left[\left.\int_\tau^\infty \zeta_t \, re^{-rt}\, dt\right| \mathcal{F}_\tau\right]$$

solves the BSDE

$$\begin{aligned} dY_t &= -\zeta_t re^{-rt}\, dt + Z_t\, dM_t \qquad ; \qquad \lim_{t\to+\infty} Y_t = 0 \\ Y_t &\geq X_t \ \forall t \geq 0 \ P\text{-a.s.} \\ 0 &= \int_0^\infty (Y_t - X_t)\, d\zeta_t \end{aligned}$$

Literature on this type of representation

- Bank (2005) ‘Optimal control under a dynamic fuel constraint’ SIAM J. Control Optim. 44 pp. 1529–1541.
- Bank & El Karoui (2004) ‘A stochastic representation theorem with applications to optimization and obstacle problems’ Annals of Prob. 32 pp. 1030-1067
- Bank & Föllmer (2003) ‘American options, multi-armed bandits, and optimal consumption plans: a unifying view, Paris-Princeton Lectures on Mathematical Finance, Vol. 1814, Springer.
- Föllmer & Knispel (2006) ‘A representation of excessive functions as expected suprema’ Prob & Math Stats, 26, Fasc. 2