Efficient parameter estimation for stochastic differential equations with jumps

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3 November 2010

Ornstein-Uhlenbeck (OU) Processes

Let $(L_t, t \geq 0)$ be a Lévy process on $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$, i.e. a càdlàg process with independent and stationary increments.
For every $a \in \mathbb{R}$

$$dX_t = -aX_t\, dt + dL_t, \quad t \in \mathbb{R}_+, \quad X_0 = x, \tag{1}$$

defines an Ornstein-Uhlenbeck process driven by the Lévy process L with initial distribution $\pi = \mathcal{L}(x)$.
Equivalently,

$$X_t = e^{-at} X_0 + \int_0^t e^{-a(t-s)} dL_s. \tag{2}$$

Problems to be studied:

1. Estimation of a from continuous observations $(X_t)_{t \geq 0}$ and unknown Lévy-Khintchine triplet of L.
2. Estimation of a from discrete observations $X_{t_1}, \ldots, X_{t_n}$.

Absolute Continuity/Singularity (ACS) Problem

The paths of X lie in the Skorokhod space

$$D(\mathbb{R}_+) = \{f : \mathbb{R}_+ \to \mathbb{R};\ f \text{ càdlàg}\}.$$

For every coefficient $a \in \mathbb{R}$ the process $X : \Omega \to D(\mathbb{R}_+)$ induces a solution measure P^a on $D(\mathbb{R}_+)$.
Let $P^a_t := P^a_{|\mathcal{F}_t}$ denotes the restriction of P^a to $\mathcal{F}_t$.
Local absolute continuity:

$$P^{a'} \overset{loc}{\ll} P^a \Longleftrightarrow P^{a'}_t \ll P^a_t \quad \forall t \in \mathbb{R}_+$$

1. Does
$$P^{a'} \overset{loc}{\ll} P^a \text{ hold for all } a, a' \in \mathbb{R}?$$
2. Radon-Nikodym density?

Absolute Continuity

Theorem

- *Let* $P^a, P^{a'}$ *be two solution measures of the OU equation for the driving Lévy process L with characteristic triplet* (b, σ^2, ρ) *and initial distributions* π *and* π'. *Suppose that* $\sigma^2 > 0$ *and* $\pi' \ll \pi$, *then we have*

$$P^{a'} \overset{loc}{\ll} P^a.$$

- *If* $\sigma^2 > 0$, *then* $P^{a'} \perp P^a$.

Absolute Continuity

Proof:

1. Derive the Hellinger process h of P^a and $P^{a'}$ for $\alpha \in (0,1)$ via martingale problems:

$$h_t(\alpha) = \frac{\alpha(1-\alpha)}{2\sigma^2} \int_0^t \left[\int_0^u \left(a' e^{-a'(u-s)} - a e^{-a(u-s)} \right) L(ds) \right]^2 du.$$

2. Use that

$$P_T^{a'} \ll P_T^a \quad \Leftrightarrow \quad \forall T > 0 : P^{a'}(h(\alpha)_T < \infty) = 1 \text{ and } P_0^{a'} \ll P_0$$

Radon-Nikodym Derivative

Proposition

For two solution measures $P^{a'} \overset{loc}{\ll} P^{a}$ of the OU equation the Radon-Nikodym derivative is

$$\frac{dP_t^{a'}}{dP_t^{a}} = \frac{dP_0^{a'}}{dP_0^{a}} \exp\left(\int_0^t \frac{(a'-a)}{\sigma^2} X_{s-}\, dX_s^c - \frac{(a'-a)^2}{2\sigma^2}\int_0^t X_s^2 ds\right),$$

where X^c denotes the continuous martingale part of X under P^a.

Maximum-Likelihood-Estimator (MLE)

For continuous observations of the Ornstein-Uhlenbeck process X the likelihood function $\mathcal{L}$ for the statistical experiment $(\Omega, \mathcal{F}, (\mathcal{F}_t), (P^a)_{a\in\mathbb{R}})$ takes the form

$$\mathcal{L}(a, X^T) = \frac{dP_t^a}{dP_t^0} = \exp\left(-\frac{a}{\sigma^2}\int_0^T X_{s-}\, dX_s^c - \frac{a^2}{2\sigma^2}\int_0^T X_s^2\, ds\right).$$

Hence, the MLE for a is explicitly given by

$$\hat{a}_T = -\frac{\int_0^T X_{s-} dX_s^c}{\int_0^T X_s^2 ds}.$$

Continuous martingale part X^c

By the Lévy-Itô decomposition of L we can write X as

$$X_t = X_0 - a \int_0^t X_s \, ds + \sigma W_t + J_t \quad , t \geq 0,$$

where W is a standard Wiener process and J a quadratic pure jump process in the sense of Protter given by

$$J_t = \int_{\{|x|<1\}} x(N_t(dx) - t\mu(dx)) + bt + \sum_{0 \leq s \leq t} \Delta X_s 1_{\{|\Delta X_s| \geq 1\}}.$$

N is the Poisson random measure associated with the jumps of L with intensity μ.

Under P^0 it follows that $X^c = W$ and under P^a

$$\tilde{W}_t = W_t - a \int_0^t X_s \, ds$$

defines a Wiener process such that $X^c = \tilde{W}$ under P^a.
Hence, given observations $(X_t(\omega), t \in [0, T])$

$$X_t^c = X_t - \int_{\{|x|<1\}} x(N_t(dx) - t\mu(dx)) - bt - \sum_{0 \le s \le t} \Delta X_s 1_{\{|\Delta X_s| \ge 1\}}.$$

which can be reconstructed in the finite variation case from continuous observations. Hence, the MLE can be rewritten as

$$\hat{a}_T = -\frac{\int_0^T X_{s-}(dW_s - aX_s ds)}{\int_0^T X_s^2 ds} = a - \frac{\int_0^T X_{s-} dW_s}{\int_0^T X_s^2 ds}.$$

under P^a.

Curved Exponential Families

Let $\{P^\theta, \theta \in \Theta\}$ be a family of measures on $(\Omega, \mathcal{F}, (\mathcal{F}_t))$.

Definition (Küchler and Sørensen (1997))

*A statistical experiment $\{P^\theta, \theta \in \Theta\}$ forms a **curved exponential family** if the likelihood function exists and is of the form*

$$\frac{dP_t^\theta}{dP_t^{\theta_0}} = \exp\left(\theta' A_t - \kappa(\theta) S_t\right).$$

- $\kappa : \Theta \to \mathbb{R}$, *for $\theta_0 \in \Theta$ arbitrary but fixed,*
- $A : \Omega \times \mathbb{R}_+ \to \mathbb{R}^d$ *is a càdlàg process,*
- $S : \Omega \times \mathbb{R}_+ \to \mathbb{R}$ *a non-decreasing continuous process with $S_0 = 0$ and $S_t \overset{t\to\infty}{\longrightarrow} \infty$.*

Asymptotic properties of MLE

Theorem

- *If* $\sigma^2 > 0$ *the MLE*

$$\hat{a}_T = -\frac{\int_0^T X_{s-} dX_s^c}{\int_0^T X_s^2 ds}.$$

exists and is strongly consistent.

- *If furthermore* X *is stationary and* $E_a[X_0^2] < \infty$ *then under* P^a

$$\sqrt{T}(\hat{a}_T - a) \to N\left(0, \frac{\sigma^2}{E_a[X_0^2]}\right) \quad \textit{weakly}$$

as $T \to \infty$.

Robustness against small jumps

A very interesting property is that the MLE is in fact robust to small jumps. Define

$$X_t^j(\epsilon) = \int_{|x|\leq\epsilon} x(N_t(dx) - t\mu(dx)),$$

then the resulting estimate remains strongly consistent.

Theorem

Let us assume that X is stationary with $E[X_0^2] < \infty$, $\sigma^2 > 0$ and set $X^{cj}(\epsilon) = X^c + X^j(\epsilon)$. If we define

$$\tilde{a}_T^\epsilon = -\frac{\int_0^T X_{s-} dX_s^{cj}(\epsilon)}{\int_0^T X_s^2 ds}, \tag{3}$$

then $\tilde{a}_T^\epsilon \to a$ with probability 1 as $T \to \infty$.

CLT when jumps are present

Theorem

Let X be a stationary Ornstein-Uhlenbeck process with $E_a[X_0^4] < \infty$, then

$$\sqrt{T}(\tilde{a}_T^{\epsilon} - a) \to N(0, \Sigma(\epsilon)) \text{ as } T \to \infty$$

where

$$\Sigma(\epsilon) = E_a[X_0^2]^{-1}\sigma^2 + E_a[X_0^2]^{-1} \int_{|x|<\epsilon} x^2 \, \mu(dx).$$

Discrete observations: high-frequency

Given non-equidistant observations $X_{t_1}, \ldots, X_{t_n}$ for $0 \leq t_1 < \ldots < t_n = T_n$ such that $T_n \to \infty$ for $n \to \infty$ and

$$\Delta(n) = \max\{t_{i+1} - t_i | 1 \leq i \leq n\} \to 0 \;\text{ as } n \to \infty$$

we consider

$$\breve{a}_{\Delta(n)} = \frac{\sum_{i=0}^{n-1} X_{t_i} \Delta X_i^c}{\sum_{i=0}^{n-1} X_{t_i}^2 \Delta t_i}.$$

Proposition

Assume that X is stationary and $E(X_0^4) < \infty$. If $\Delta(n) = o(T_n^{-2})$ then

$$\sqrt{T}(\breve{a}_{\Delta(n)} - a) \xrightarrow{\mathcal{D}} N(0, \sigma^2 E[X_0^2]^{-1}).$$

Hence, under these conditions the discretized MLE $\breve{a}_{\Delta(n)}$ and the MLE $\hat{a}_T$ based on continuous observations converge to the same asymptotic distribution as $T \to \infty$.

Discrete Observations: long time asymptotics

Given discrete observations $X_\Delta, X_{2\Delta}, \ldots, X_{n\Delta}$ with fixed step size Δ a discretized version of $\hat{a}$ is

$$\hat{a}_n = -\frac{\sum_{m=0}^{n-1} X_{m\Delta}\ \delta X_m^c}{\Delta \sum_{m=0}^{n-1} X_{m\Delta}}$$

with increments $\delta X_m^c = X_{(m+1)\Delta}^c - X_{m\Delta}^c$.

Theorem

Under the assumption that X is stationary and $K_a(t) = E_a(X_t X_0)$ is continuously differentiable in t the MLE satisfies

$$\hat{a}_n + \frac{\hat{a}_n^2}{2}\Delta \xrightarrow{n\to\infty} a + O(\Delta^2) \qquad P_a\text{-}a.s.$$

Recovering X^c

Observations $X_{t_1^n}, \ldots, X_{t_{m_n}^n}$ such that $t_{m_n}^n \stackrel{n\to\infty}{\longrightarrow} \infty$ and

$$\max\{|t_{i+1}^n - t_i^n|, 1 \leq i \leq m_n - 1\} \stackrel{n\to\infty}{\longrightarrow} 0.$$

MLE with truncated increments:

$$\bar{a}_n := \frac{\sum_{i=1}^n X_{t_i^n} \Delta_i X \mathbf{1}_{\{\Delta_i X^2 \leq v_n\}}}{\sum_{i=1}^n X_{t_i^n}^2 \Delta_i^n}$$

Question: How to choose the cut-off sequence v_n?

Asymptotics of truncated MLE

Theorem

Let X be stationary and assume that $\sigma^2 > 0$ and that the jump part J of L is of finite activity. If $T_n \Delta_n^{\frac{1}{2}} \to 0$ and $v_n = \Delta_n^{\gamma}$ for $\gamma \in (0,1)$ then

$$T_n^{1/2}(\bar{a}_n - a) \xrightarrow{\mathcal{D}} N\left(0, \frac{\sigma^2}{E_a[X_0^2]}\right) \text{ as } n \to \infty.$$

Hence, the truncated MLE is **asymptotically efficient** in the sense of Hájek-Le Cam.

Sketch of proof:

1. Jump filtering by cutting large increments:

$$\left| \sum_{i=1}^{n} X_{t_i^n} (\Delta_i X \mathbf{1}_{\{\Delta_i X^2 \le v_n\}} - \Delta_i X(u_n)) \mathbf{1}_{A_n} \right| = o_p(1)$$

2. CLT for the discrete estimate with small jumps limit:

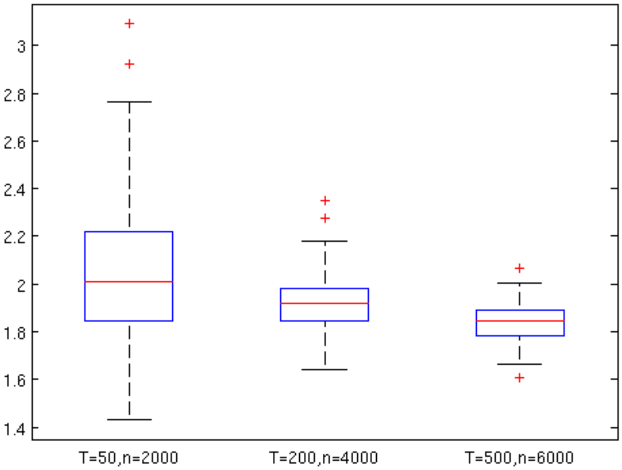
$$T_n^{-1/2} \sum_{i=1}^{n} X_{t_i^n} \Delta_i X(u_n) \xrightarrow{\mathcal{D}} N\left(0, \sigma^2 E_a[X_0^2]\right) \text{ as } n \to \infty.$$

3. $$T_n^{1/2}(\bar{a}_n - \hat{a}_n) = T_n^{1/2} \left(\frac{\sum_{i=1}^{n} X_{t_i^n} (\Delta_i X \mathbf{1}_{\{\Delta_i X^2 \le v_n\}} - \Delta_i X(u_n))}{\sum_{i=1}^{n} X_{t_i^n}^2 \Delta_i^n} \right)$$

and apply Slutzky's lemma.

Simulations

Boxplot for $\hat{a}_n$ from a Wiener process plus compound Poisson (intensity $\lambda = 4$, N(0,1)-jumps) driver and true parameter $a = 2$.



Summary

Continuous observations:

- The MLE takes an explicit form and is asymptotically normal and efficient.
- The influence of jumps on the asymptotic variance is well understood.

Discrete observations:

- Efficient jump filtering via truncation method.
- Discrete non-equidistant data yields asymptotically efficient estimator in the sense of Hájek-Le Cam.
- Good finite sample behavior has been demonstrated by a simulation example.

Bibliography

Uwe Küchler and Michael Sørensen. *Exponential families of stochastic processes.* Springer Series in Statistics. New York, 1997.