The Feynman checkerboard: discrete quantum mechanics

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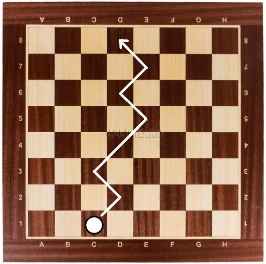
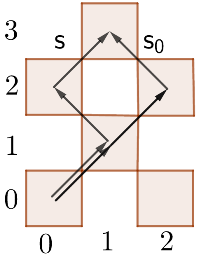
Overview

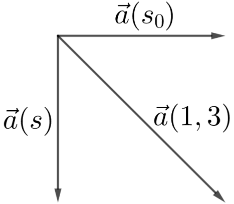
1. The simplest (and rough) particular case.
2. Feynman checkerboard.
3. Coupling to lattice gauge field.

The simplest particular case

Definition (R.Feynman, 1950s)







$$a(x,y) := 2^{(1-y)/2}\, i \sum_s (-i)^{t(s)}$$

is the sum over all checker paths s from $(0,0)$ to (x,y) with the first step to $(1,1)$, where $t(s)$ is the number of turns in s.

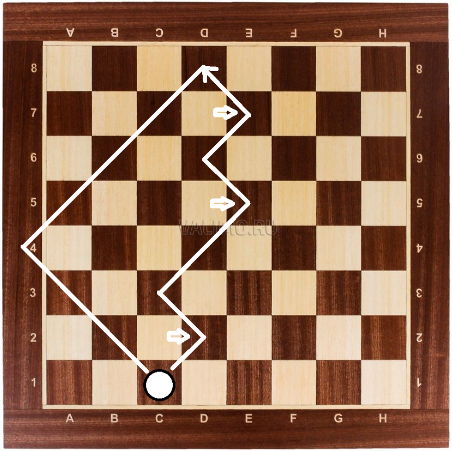
The probability to find an electron in the square (x, y), if it was emitted from the square $(0, 0)$:

$$P(x, y) := |a(x, y)|^2.$$

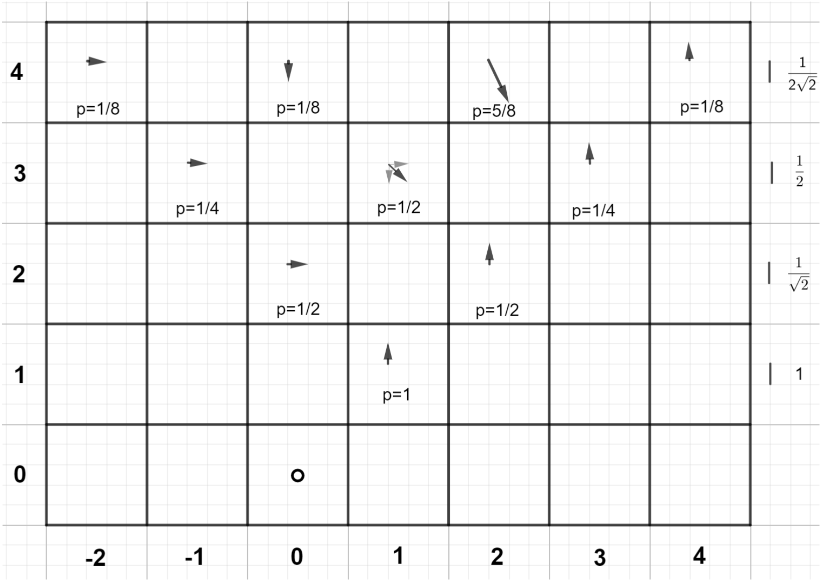
Denote

$$a_1(x, y) := \mathrm{Re}\, a(x, y),$$
$$a_2(x, y) := \mathrm{Im}\, a(x, y).$$

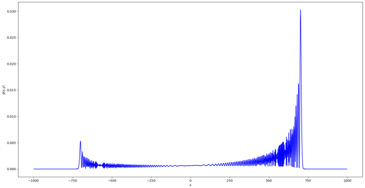
Young tableau interpretation

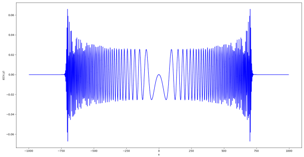


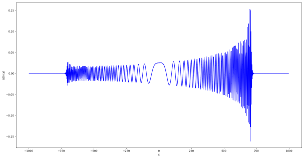
$a(x, y)$ and $P(x, y)$ for small x, y



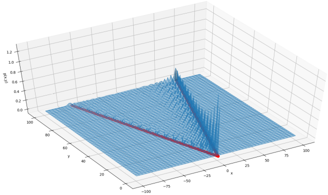
$P(x,1000)$, $\mathrm{Re}\, a(x,1000)$, $\mathrm{Im}\, a(x,1000)$ (A. Daniyarkhodzhaev–F. Kuyanov)

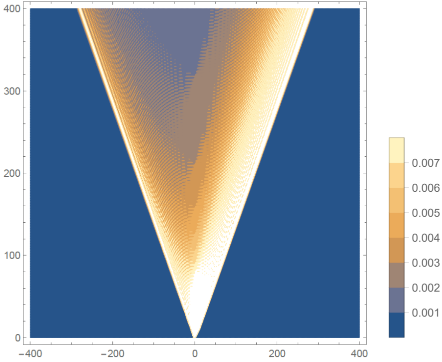






Contour plot for $P(x, y)$





Conjecture (Daniyarkhodzhaev-Kuyanov).
The sides of the "angle" are $y = \pm\sqrt{2}\,x$.

Proposition (folklore)

For each $x, y \in \mathbb{Z}$, $y \geq 2$, *we have*

$$a_1(x,y) = \frac{1}{\sqrt{2}} a_2(x+1, y-1) + \frac{1}{\sqrt{2}} a_1(x+1, y-1)$$
$$a_2(x,y) = \frac{1}{\sqrt{2}} a_2(x-1, y-1) - \frac{1}{\sqrt{2}} a_1(x-1, y-1)$$

Cf. $(1+1)$-dimensional Dirac equation

$$\begin{pmatrix} -m & i\,\partial/\partial t - i\,\partial/\partial x \\ i\,\partial/\partial t + i\,\partial/\partial x & -m \end{pmatrix} \begin{pmatrix} i\,a_2 \\ a_1 \end{pmatrix} = 0.$$

Proposition

For each integer $y \geq 1$ we have

$$\sum_{x \in \mathbb{Z}} P(x, y) = 1.$$

Proof

$$\sum_{x\in\mathbb{Z}} P(x,y{+}1) = \sum_{x\in\mathbb{Z}} a_1(x,y{+}1)^2 + \sum_{x\in\mathbb{Z}} a_2(x,y{+}1)^2 =$$

$$= \frac{1}{2}\sum_{x\in\mathbb{Z}} (a_1(x+1,y) + a_2(x+1,y))^2 +$$

$$+ \frac{1}{2}\sum_{x\in\mathbb{Z}} (a_2(x-1,y) - a_1(x-1,y))^2 =$$

$$= \frac{1}{2}\sum_{x\in\mathbb{Z}} (a_1(x,y){+}a_2(x,y))^2 + \frac{1}{2}\sum_{x\in\mathbb{Z}} (a_2(x,y){-}a_1(x,y))^2$$

$$= \sum_{x\in\mathbb{Z}} \left[a_1(x,y)^2 + a_2(x,y)^2\right] = \sum_{x\in\mathbb{Z}} P(x,y). \square$$

Proposition (folklore)

For each $y > |x|$ such that $x + y$ is even

$$a_1(x, y) = 2^{(1-y)/2} \sum_{r=0}^{\lfloor y/2 \rfloor} (-1)^r \binom{(x+y-2)/2}{r} \binom{(y-x-2)/2}{r},$$

$$a_2(x, y) = 2^{(1-y)/2} \sum_{r=1}^{\lfloor y/2 \rfloor} (-1)^r \binom{(x+y-2)/2}{r} \binom{(y-x-2)/2}{r-1}$$

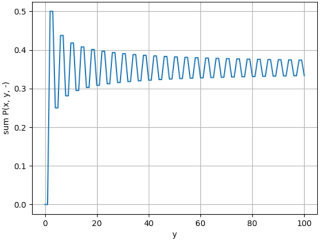
$a_1(x,y)^2$ is the probability to find a *left* electron in the square (x,y), if a *right* one was emitted from the square $(0,0)$.

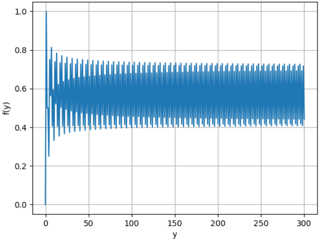
Theorem (S.-Ustinov'19)

We have $\sum_{x\in\mathbb{Z}} a_1(x,y)^2 = \frac{1}{2\sqrt{2}} + O\left(\frac{1}{\sqrt{y}}\right)$.

Continuum analogue of the sum is ill-defined!

Probabilities of spin reversal without/with a magnetic field (G. Minaev–I. Russkikh)





Feynman checkerboard

R.Feynman (1950s, published in 1965): definition & problems (still unsolved!)
J.Narlikar(1972),T.Jacobson–L.Schulman(1984): Dirac propagator in the continuum limit (physical derivation)
H.Gersch(1981): Ising model at imaginary T
G.Ord (1997): adding electromagnetic field
P.Jizba (2015): adding mass matrix
T.Jacobson etal.(2017): a 4D generalization (one of the several existing ones)

Definition (R.Feynman, 1950s)

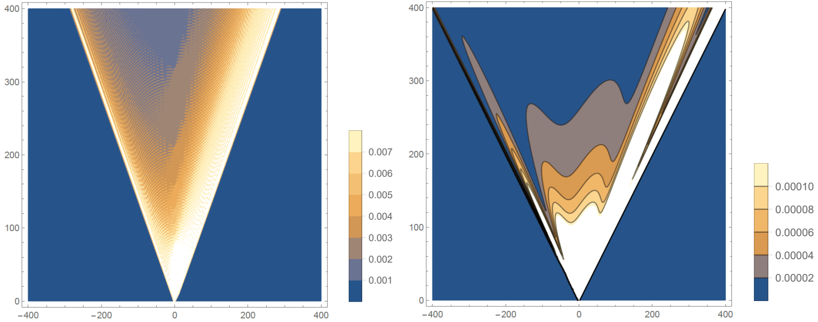
Fix $\varepsilon, m > 0$ (*lattice step* and *particle mass*).

$$a(x, y, m\varepsilon) := (1+m^2\varepsilon^2)^{(1-y)/2}\, i \sum_s (-im\varepsilon)^{t(s)}$$

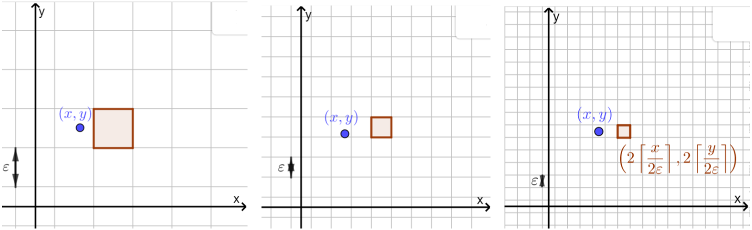
is the sum over all checker paths s from $(0,0)$ to (x,y) with the first step to $(1,1)$.

$$P(x, y, m\varepsilon) := |a(x, y, m\varepsilon)|^2.$$

Contour plots for $P(x, y, 1)$ and $P(x, y, 0.02)$



Continuum limit



Uniform continuum limit

Corollary (S.-Ustinov'19)

For each fixed $m > 0$, as $\varepsilon \to 0$ we have

$$\frac{1}{2\varepsilon}\, a\left(2\left\lceil\frac{x}{2\varepsilon}\right\rceil, 2\left\lceil\frac{y}{2\varepsilon}\right\rceil, m\varepsilon\right) \rightrightarrows$$

$$\frac{m}{2} J_0(m\sqrt{y^2-x^2}) - i\,\frac{m}{2}\cdot\frac{y+x}{\sqrt{y^2-x^2}}\, J_1(m\sqrt{y^2-x^2})$$

uniformly on compact subsets of angle $y > |x|$.

Here we use *Bessel functions of the 1st kind*

$$J_0(z) := \sum_{k=0}^{\infty}(-1)^k\frac{(z/2)^{2k}}{(k!)^2}, \quad J_1(z) := \sum_{k=0}^{\infty}(-1)^k\frac{(z/2)^{2k+1}}{k!(k+1)!}.$$

Theorem (S.-Ustinov'19)

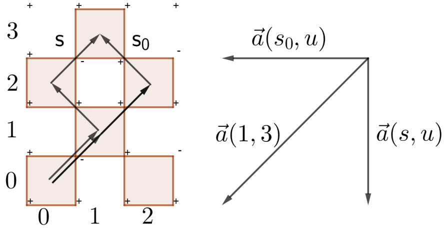
For each $x, y, m, \varepsilon, \delta$ such that $y - |x| \geq \delta > 0$ and $0 < \varepsilon < \delta^2 e^{-3z-\delta}/16$, where $z := m\sqrt{y^2 - x^2}$, we have

$$a\left(2\left\lceil \frac{x}{2\varepsilon} \right\rceil, 2\left\lceil \frac{y}{2\varepsilon} \right\rceil, m\varepsilon\right) =$$

$$= m\varepsilon \left(J_0(z) - i\frac{y+x}{\sqrt{y^2-x^2}} J_1(z) + \right.$$

$$\left. + O\left(\frac{\varepsilon \log^2 \varepsilon}{\delta} (my+1)\, I_0(z)\right)\right).$$

Coupling to lattice gauge field



An *electromagnetic vector-potential* u is a map from the set of all vertices of the squares to $\{+1, -1\}$.

$$a(x, y, u) := 2^{(1-y)/2}\, i \sum_s (-i)^{t(s)} u(s_1) \dots u(s_y),$$

where $s_1, \dots, s_y$ are the vertices passed by s.

References

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J. Narlikar, Path Amplitudes for Dirac particles, J. Indian Math. Society, 36, (1972) 9–32.

Acknowledgements

THANKS!