Seminar

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Characterization of product states on polynomial algebras in terms of scalar products of n−particle vectors

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Introduction

Quadratic quantization

The RSWN n–particle scalar product is given by

$$\left\langle \phi^{\otimes n}, \psi^{\otimes n} \right\rangle_n = \tag{1}$$

$$\sum_{i_1+2i_2+\cdots+ki_k=n} \frac{n!\beta^n c_2^{i_1+\cdots+i_k} \langle \phi, \psi \rangle^{i_1} \cdots \langle \phi^k, \psi^k \rangle^{i_k}}{i_1! \cdots i_k! \beta^{i_1} (2\beta)^{i_2} \cdots (k\beta)^{i_k}}$$

$$= c_2^n \langle \phi, \psi \rangle^n +$$

$$+ \sum_{i_1+2i_2+\cdots+ki_k=n,}^{i_1 \neq n} \frac{n!\beta^n c_2^{i_1+\cdots+i_k} \langle \phi, \psi \rangle^{i_1} \cdots \langle \phi^k, \psi^k \rangle^{i_k}}{i_1! \cdots i_k! \beta^{i_1} (2\beta)^{i_2} \cdots (k\beta)^{i_k}}$$

The complex polynomial algebra in d Hermitean indeterminates

$$\mathbb{C}[X_1, \ldots, X_d] \quad ; \quad (d \in \mathbb{N} \cup \{\infty\})$$

is a well studied object in mathematics since hundreds years.
For reasons that will be clear later, we want to give a **concrete representation** of this $*$–algebra **in terms of functions** on a real vector space.

The algebra $\mathcal{P}_V$ of polynomial functions on a real vector space V

V a **real** vector space.
V^* algebraic **dual** of V
natural embedding $V \hookrightarrow (V^*)^*$

$$X : v \in V \to X_v \in (V^*)^*$$

defined by

$$X_v : u^* \in V^* \to X_v(u^*) := \langle u^*, v\rangle \in \mathbb{R} \qquad (2)$$

$X_v(u^*) := \langle u^*, v\rangle :=$ **coordinate** of u^* along v

Thus we have the **coordinate maps**

$$X_v : V^* \to \in \mathbb{R}$$

The n–th powers of the X_v are the **monomial functions** (or simply **monomials**) on V of degree n

$$X_v^n : u^* \in V^* \to (X_v(u^*))^n \in \mathbb{R} \quad ; \quad v \in V \ , \ n \in \mathbb{N}$$

$$\mathcal{P}_V := \mathbb{C}\text{–linear span of monomials}$$

$:=$ **polynomial algebra** on V

$\mathcal{P}_V$ is a $*$**–algebra** for the pointwise operations.
Involution: complex conjugation.
X_v is real–valued as a function $\Rightarrow$ is **Hermitean** as element of $\mathcal{P}_V$:

$$X_v^* = X_v \qquad ; \qquad \forall v \in V$$

Classical random fields
as vector valued random variables

Denote $\mathcal{F}^*$ the $\sigma-$algebra on the real vector space V^* generated by the coordinate functions $\{X_v : V^* \to\in \mathbb{R} \quad ; \quad v \in V\}$.
If on $\mathcal{F}^*$ it is given a **probability measure** φ, we obtain a **probability space**

$$(V^*, \mathcal{F}^*, \varphi)$$

and the coordinate functions

$$X_v : u^* \in V^* \to X_v(u^*) := \langle u^*, v\rangle \in \mathbb{R}$$

become real valued random variables on $(V^*, \mathcal{F}^*, \varphi)$.
Therefore the map

$$X : v \in V \to X_v \tag{3}$$

can be interpreted as a
$V-$**valued classical random variable**.
or equivalently as a
real–valued classical random field on V.

If $V \equiv \mathbb{R}^d$ is **finite dimensional**, one usually identifies V with V^* fixing a linear basis on V. Through this identification, any probability measure φ **becomes a probability measure on** V.

Classical random fields with all moments

Definition 1 A classical random field X on V is said to have **finite moments of any order** if:

$$\varphi\left(|X_v|^n\right) < \infty \qquad ; \qquad \forall v \in V \ , \ \forall n \in \mathbb{N}$$

$$\varphi\left(|X_v|^n\right) := \int_{V^*} |X_v(\omega)|^n \varphi(d\omega)$$

Same symbol φ for measure and φ–integral.

From now on:
– **random field (or variable)** on $V \equiv$
$\equiv$**random field (or variable) on** V **with all moments**.

– We fix V (real vector space)

$$\mathcal{P} \equiv \mathcal{P}_V$$

States and pre–scalar products on $\mathcal{P}_V$

The restriction of the φ–integral on $\mathcal{P}_V$ gives a **state** φ on $\mathcal{P}_V$

$$\varphi(Q) := \int Q d\varphi \qquad ; \qquad Q \in \mathcal{P}_V$$

$\iff$

– φ is positive on positive elements in $\mathcal{P}$.
– $\varphi(1) = 1$.
– φ defines a **pre–scalar product on** $\mathcal{P}_V$:

$$\langle P, Q \rangle := \varphi(P^* Q) \qquad ; \qquad Q \in \mathcal{P}_V$$

The pair

$$(\mathcal{P}_V, \varphi)$$

is a standard example of a
classical algebraic probability space.
Classical because the $*$–algebra $\mathcal{P}_V$ is **commutative**.

Theorem 1 For a semi–scalar product $\langle \cdot , \cdot \rangle$ on $\mathcal{P}$ the following statements are **equivalent**:
(i) There exists a state φ on $\mathcal{P}$ such that:

$$\varphi(f^* g) = \langle f, g \rangle \qquad ; \qquad f, g \in \mathcal{P} \tag{4}$$

(ii) The semi–scalar product $\langle \cdot , \cdot \rangle$ satisfies

$$\langle 1_{\mathcal{P}}, 1_{\mathcal{P}} \rangle = 1 \tag{5}$$

and, for each $v \in V$, multiplication by the coordinate X_v is a **Hermitean** linear operator on $\mathcal{P}$ with respect to $\langle \cdot , \cdot \rangle$, i.e.:

$$\langle X_v f, g \rangle = \langle f, X_v g \rangle \tag{6}$$

Taking the **cyclic representation** of the pair

$$(\mathcal{P}_V, \varphi) \mapsto (\mathcal{H}_\varphi, \eta_\varphi, \pi_\varphi, \Phi_0)$$

one obtains:
– a Hilbert space $\mathcal{H}_\varphi \equiv [\mathcal{P}_V \cdot \Phi_0]$,
– a quotient map $\eta_\varphi : \mathcal{P}_V \to \mathcal{H}$,
– a $*$–representation π_φ of $\mathcal{P}_V$,
– a $\pi_\varphi(\mathcal{P}_V)$–cyclic vector (**vacuum vector**)

$$\Phi_0 := \eta_\varphi(1_{\mathcal{P}_V}) = \eta_\varphi(\text{ constant function } = 1)$$

Simplified notations.
we **omit the representation symbol** π_φ, η_φ and, for $Q \in \mathcal{P}_V$, we use the notations:

$$Q(X) = \pi_\varphi(Q) \text{ multiplication } \textbf{operator}$$

$$Q \cdot \Phi_0 = \eta_\varphi(Q) = \pi_\varphi(P) \cdot \Phi_0 \textbf{ vector}$$

Note that

$$\mathcal{P}_V \cdot \Phi_0 := \{P \cdot \Phi_0 \ : \ P \in \mathcal{P}_V\} \subseteq \mathcal{H}$$

is a dense sub–space invariant under the action of $\pi_\varphi(\mathcal{P}_V)$.
Its closure

$$\mathcal{H}_\varphi = [\mathcal{P}_V \cdot \Phi_0] =: L^2_{pol}(V^*, \varphi) \subseteq L^2(V^*, \varphi)$$

The degree filtration in $\mathcal{P}_V$

is the family of sub–spaces of $\mathcal{P}_V$: $n \in \mathbb{N}$

$$\mathcal{P}_{V,n]} := \tag{7}$$

$:= \{$linear span of **monomials of degree** $\leq n\}$

Closed if $\dim(V) = +\infty$.

It is increasing in the sense that

$$\mathcal{P}_{V,n]} \subset \mathcal{P}_{V,n+1]} \subset \mathcal{P}_V \quad ; \quad \forall n \in \mathbb{N} \tag{8}$$

moreover

$$\mathcal{P}_V = \bigcup_{n \in \mathbb{N}} \mathcal{P}_{V,n]} \tag{9}$$

Closed if $\dim(V) = +\infty$.

Since V is fixed, we write $\mathcal{P}_{n]}, \mathcal{P}, \ldots$

instead of $\mathcal{P}_{V,n]}, \mathcal{P}_V, \ldots$.

Abstract roots of generalized quantizations

Theorem 2 Let $\mathcal{H}$ be an Hilbert space and let be given:

– an **increasing filtration** (sequence) $(\mathcal{H}_{n]})$ of closed sub–spaces of $\mathcal{H}$ with corresponding orthogonal projectors

$$P_{n]} : \mathcal{H} \to \mathcal{H}_{n]} \qquad ; \qquad n \in \mathbb{N}$$

– a set D is and, for each $j \in D$, a linear Hermitean operator

$$Y_j = Y_j^* : \mathcal{H} \to \mathcal{H}$$

filtration increasing of degree $+1$

$$Y_j \left(\mathcal{H}_{n]}\right) \subseteq \mathcal{H}_{n+1]} \tag{10}$$

Then the sequence of mutually orthogonal projectors:

$$P_n := P_{n]} - P_{n-1]} \quad ; \quad \forall n \in \mathbb{N} \tag{11}$$

$$\Big(P_{-1} := P_{-1]} := 0\Big) \tag{12}$$

satisfies the 3−**diagonality relation**

$$P_m Y_j P_n = 0 \quad \text{if} \quad m \notin \{n-1, n, n+1\} \tag{13}$$

If, in addition

$$\lim_{n\to\infty} P_{n]} = 1_{\mathcal{H}} \tag{14}$$

in any operator topology, then (P_n) is a **partition of the identity** in the same topology and:

$$Y_j P_n = P_{n+1} Y_j P_n + P_n Y_j P_n + P_{n-1} Y_j P_n \quad (15)$$

or equivalently:

$$Y_j = \sum_{n \in \mathbb{N}} P_{n+1} Y_j P_n + \sum_{n \in \mathbb{N}} P_n Y_j P_n + \sum_{n \in \mathbb{N}} P_{n-1} Y_j P_n \quad (16)$$

In particular, defining the operators

$$\begin{cases} a^{+}_{Y_j} := \sum_{n \in \mathbb{N}} P_{n+1} Y_j P_n \ (\textbf{creation}) \\ a^{-}_{Y_j} := \sum_{n \in \mathbb{N}} P_{n-1} Y_j P_n \ (\textbf{annihilation}) \\ a^{0}_{Y_j} := \sum_{n \in \mathbb{N}} P_n Y_j P_n \ (\textbf{preservation}) \end{cases} \quad (17)$$

(16) becomes:

$$Y_j = a^{+}_{Y_j} + a^{0}_{Y_j} + a^{-}_{Y_j} \quad ; \quad j \in D \quad (18)$$

Corollary 1 The following are equivalent:
(i) To give a triple

$$(\mathcal{H}, (P_{n]})_{n\in\mathbb{N}}, (Y_j))_{j\in D})$$

satisfying the conditions of Theorem (2).
(ii) To give:
(ii.1) an **orthogonal Hilbert space gradation**

$$(\mathcal{H}, \langle \,\cdot\, , \,\cdot\, \rangle) = \bigoplus_{n\in\mathbb{N}} (\mathcal{H}_n, \langle \,\cdot\, , \,\cdot\, \rangle_n) \tag{19}$$

(ii.2) for each $j \in D$ an **adjointable operator on $\mathcal{H}$ gradation increasing** of degree $+1$, i.e. such that

$$a_j^+ \ : \ \mathcal{H}_n \to \mathcal{H}_{n+1} \tag{20}$$

(ii.3) for each $j \in D$ an **Hermitean gradation preserving** operator on $\mathcal{H}$ (i.e. of degree 0):

$$(a_j^0)^* = a_j^0 \ : \ \mathcal{H}_n \to \mathcal{H}_n \tag{21}$$

Definition 2 Let D be a set. A **generalized Fock quantization with 1–particle space** $V \sim \mathbb{R}^{|D|}$ is given by;

$$(\mathcal{H}_n, \langle \cdot , \cdot \rangle_n)_{n\in\mathbb{N}} \quad ; \quad (a_j^+)_{j\in D} \quad ; \quad (a_j^0)_{j\in D}$$

Satisfying the conditions of Corollary 1.
– $\mathcal{H}_n$ is called **generalized** n**–particle space**;
– The $(a_j^+)_{j\in D}$ are called **generalized creators**;
– The $(a_j^0)_{j\in D}$ are called **generalized preservation operators**;
– The adjoints of the creators

$$(a_j^+)^* := a_j^- : \mathcal{H}_n) \to \mathcal{H}_{n-1} \tag{22}$$

are called **generalized annihilators**.
They are gradation decreasing of degree -1.
– The **CAP operators** is the union of the 3 sets above.

– The identity

$$Y_j = a_j^+ + a_j^0 + a_j^- \quad ; \quad j \in D \tag{23}$$

is called **the quantum decomposition of** Y_j associated to the triple $(\mathcal{H}, (P_{n]})_{n\in\mathbb{N}}, (Y_j))_{j\in D}$. The equivalent formulation of the identity (23) given by

$$Y_j P_n = P_{n+1} Y_j P_n + P_n Y_j P_n + P_{n-1} Y_j P_n \ ; \ n \in \mathbb{N} \tag{24}$$

is called the **symmetric** 3**–diagonal (Jacobi) relation** associated to the triple $(\mathcal{H}, (P_{n]})_{n\in\mathbb{N}}, (Y_j))_{j\in D}$.

Theorem 3 The quantum decomposition (23) **is unique** up to unitary isomorphism.

Type I commutation relations

The products of CAP operators

$$a_{Y_j}a^+_{Y_k} \quad , \quad a^+_{Y_k}a_{Y_j} \quad , \quad j,k \in D$$

are well defined on $\mathcal{D}$, **gradation preserving**. In particular satisfy the following **commutation relations**:

$$[a_{Y_j}, a^+_{Y_k}] = \partial\Omega_{jk} \quad , \quad j,k \in D \qquad (25)$$

where $(\partial\Omega_{jk})$ is a kernel with values into the gradation preserving adjointable linear operators $\mathcal{H} \to \mathcal{H}$.

Definition 3 The commutation relations (25) are called **type I commutation relations**. (The reasons of this terminology will be clear in the following.)

Remark. Type I commutation relations only arise from the fact that both $a_{Y_j}a^+_{Y_k}$ and $a^+_{Y_k}a_{Y_j}$ are **gradation preserving**:

Product states on polynomial algebras

Let V be a real topological vector space and let $e \equiv (e_n)_{n\in D}$ be a linear basis of V (D an at most countable set). Then

$$V = \dot{\sum}_{n\in D} \mathbb{C} \cdot e_n$$

where $\dot{\sum}_{n\in D}$ denotes algebraic direct sum of vector spaces.

Moreover for each $n \in D$,

$$\mathcal{P}_{\mathbb{C}\cdot e_n} \subseteq \mathcal{P} \equiv \mathcal{P}_V \quad ; \quad m \neq n \Rightarrow \mathcal{P}_{\mathbb{C}\cdot e_m} \cap \mathcal{P}_{\mathbb{C}\cdot e_n} = 1_{\mathcal{P}}$$

$$\mathcal{P} = \bigvee_{n\in D} \mathcal{P}_{\mathbb{C}\cdot e_n} = \text{lin. span}\left\{ \prod_{n\in D} q_{\mathbb{C}\cdot e_n} \ : \right.$$

$$\left. q_{\mathbb{C}\cdot e_n} \in \mathcal{P}_{\mathbb{C}\cdot e_n} \ , \ q_{\mathbb{C}\cdot e_n} = 1_{\mathcal{P}_{\mathbb{C}\cdot e_n}} \quad \text{almost all } n \in D \right\}$$

Definition 4 A state φ on $\mathcal{P}_V$, is called a **product state** if there exists a linear basis $e \equiv (e_n)_{n\in D}$ of V such that, denoting

$$\varphi_n := \varphi\Big|_{\mathcal{P}_{\mathbb{C}\cdot e_n}} \text{ (restriction)} \ , \ n \in D$$

one has

$$\varphi\left(\prod_{n\in F} p_{\mathbb{C}\cdot e_n}\right) = \prod_{n\in F} \varphi_n(p_{\mathbb{C}\cdot e_n})$$

for any finite sub–set $F \subseteq I$ and any family of polynomials $p_{\mathbb{C}\cdot e_n} \in \mathcal{P}_{\mathbb{C}\cdot e_n}$ $(n \in F)$.

The Boson canonical commutation relations

Recall the type I commutation relations

$$[a_v, a_u^+] = \partial\Omega(v,u) \quad , \quad j,k \in D$$

The simplest gradation preserving operators are the **multiples of identity**:

$$\partial\Omega(v,u) = \partial\omega(v,u) \cdot 1$$

with $\partial\omega(v,u) \in \mathbb{C}$.
In this case **the Type I** CCR become:

$$\partial\Omega(v,u) = [a_v^-, a_u^+] = \partial\omega(v,u) \cdot 1 \qquad (26)$$

Hence their vacuum expectation value is

$$0 \leq \|a_v^+ \Phi_0\|^2 = \langle \Phi_0, \partial\omega(v,v)\Phi_0 \rangle = \partial\omega(v,v)$$

Therefore the sesqui–linear form

$$\langle v,u \rangle_{V_{\mathbb{C}}} := \partial\omega(v,u)$$

is positive definte and therefore it defines a **semi–scalar product** $\langle \cdot , \cdot \rangle_{V_{\mathbb{C}}}$) **on** $V_{\mathbb{C}}$.

Theorem 4 Let V be a real vector space and let $(v,u) \in V_{\mathbb{C}} \times V_{\mathbb{C}} \mapsto \partial\omega(v,u)$ be a non–identically zero semi–scalar product on $V_{\mathbb{C}} \equiv V \dot{+} iV$ (the complexification of V) taking real values on $V \times V$.
For any classical random field X on V the following statements are equivalent.
(i) The CAP operators of X satisfy
$a_v^0 = 0$, $\forall v \in V$ (i.e. X is symmetric) and

$$\partial\Omega(v,u) := [a_v^-, a_u^+] = \partial\omega(v,u) \cdot 1 \tag{27}$$

(ii) The quantum field canonically associated to X is the Fock Boson field over the Hilbert space $(V_{\mathbb{C},\partial\omega}\ ,\ \langle\ \cdot\ ,\ \cdot\ \rangle_{\partial\omega})$, obtained by completing $V_{\mathbb{C}}$ with the semi–scalar product $\partial\omega$.
(iii) X is the standard classical Gaussian field on the real Hilbert space $(V_{\partial\omega}\ ,\ \langle\ \cdot\ ,\ \cdot\ \rangle_{\partial\omega_V})$, obtained by completing V with the semi–scalar product $\partial\omega_V$ where $\partial\omega_V$ is the restriction of $\partial\omega$ on V.

Remark. Note that the semi–scalar product $\langle\ \cdot\ ,\ \cdot\ \rangle_{\partial\omega_V}$, on V, is **real valued** because φ is real on real polynomials.

The theorem above follows from the following more general result.

Product measures:
Statistics implies algebra
The canonical commutation relations of Boson quantum field theory (CCR)

$$[a_v^-, a_u^+] = \langle v, u\rangle_{V_{\mathbb{C}}} \cdot 1$$

imply in particular that

$$v \perp u \ \Rightarrow \ [a_v^-, a_u^+] = 0$$

Therefore, the canonical commutation relations of Boson quantum field theory enjoy the following property:
Property (C)
If $e \equiv (e_j)_{j\in D}$ is an orthogonal basis of $\Big(V_{\mathbb{C}}\ ,\ \langle\ \cdot\ ,\ \cdot\ \rangle_{V_{\mathbb{C}}}\Big)$, one has:

$$[a_{e_j}^-, a_{e_k}^+] = 0 \qquad ; \qquad \forall j \neq k \tag{28}$$

The following question then arises naturally:
For which classical $V-$valued random variables X does there exist a linear basis $e \equiv (e_j)_{j\in D}$ of $V_{\mathbb{C}}$ for which condition (28) is satisfied?

Theorem 5 (Accardi, Kuo, Stan)
For a state φ on $\mathcal{P}$ associated to a classical V–valued random variable X and a linear basis $e \equiv (e_j)_{j \in D}$ of $V_{\mathbb{C}}$, **the following statements are equivalent**:
(i) The classical real–valued random variables

$$X_{e_j} = a^{+}_{e_j} + a^{0}_{e_j} + a^{-}_{e_j} \qquad ; \qquad j \in D \tag{29}$$

are **mutually independent**.
(ii) φ is **a product state** with respect to the linear basis $e \equiv (e_j)_{j \in D}$ of $V_{\mathbb{C}}$.
(iii) The CAP operators in the quantum decomposition of the X_{e_j} **mutually commute**, i.e. for all $j \neq k \in D$, and all $\varepsilon, \eta \in \{+1, 0, -1\}$, one has

$$[a^{\varepsilon}_{e_j}, a^{\eta}_{e_k}] =: [a^{\varepsilon}_{j}, a^{\eta}_{k}] = 0 \tag{30}$$

Properties of product states

From Definition 4 it follows a state is a product state if and only if its restriction to an increasing sequence of finite dimensional sub–spaces is a product state.
This reduces the problem to the finite dimensional case.
Therefore from now on we suppose that for $d \in \mathbb{N}^*$,

$$D := \{1, \cdots, d\}$$

Thus V is a real d–dimensional vector space, $e \equiv \{e_j\}_{j \in D}$ a linear basis of V

$$D_n := \left\{ \bar{m} := (m_1, \cdots, m_d) \in \mathbb{N}^d \ : \ \sum_{j \in D} m_j = n \right\}$$

and φ a state on $\mathcal{P}_V$.

We denote

$$\langle a \cdot \Phi_0, b \cdot \Phi_0 \rangle := \varphi(a^* b)$$

the semi–scalar product defined by the state

$$\varphi := \langle \Phi_0, (\ \cdot\ ) \Phi_0 \rangle$$

and $a_j^\varepsilon := a_{e_j}^\varepsilon$ ($\varepsilon \in \{+, 0, -\}$) the CAP operators associated with φ.
The n–particle vectors (in the e–basis) are then defined by

$$\Phi_{n,\bar{m}} := a_1^{+m_1} \cdots a_d^{+m_d} \Phi_0 \ ; \ \bar{m} := (m_1, \cdots, m_d) \in D_n \tag{31}$$

where

$$\Phi_0 \equiv \Phi_{0,(0,\cdots,0)}$$

is the vacuum vector.

Theorem 6 Let $e \equiv \{e_j\}_{j\in D}$ be a linear basis of $V_{\mathbb{C}}$. Let φ be a product state on $\mathcal{P}$ and let $a^+_{e_j}, a_{e_j}, a^0_{e_j}$ be its CAP operators in the e–basis. Then for any $d \in \mathbb{N}$ with $d \le |D|$,

$$\langle a^{+m_1}_{e_1} \cdots a^{+m_d}_{e_d} \Phi_0 \ , \ a^{+n_1}_{e_1} \cdots a^{+n_d}_{e_d} \Phi_0 \rangle = 0 \quad (32)$$

whenever $\bar{m} := (m_1, \cdots, m_d) \neq \bar{n} := (n_1, \cdots, n_d) \in \mathbb{N}^d$, while if $\bar{m} \neq \bar{n}$,

$$\|a^{+n_1}_{e_1} \cdots a^{+n_d}_{e_d} \Phi_0\|^2 = \prod_{k\in D} \|a^{+n_k}_{e_k} \Phi_0\|^2 \quad (33)$$

It is natural to ask oneself if the converse of Theorem 6 is true, namely:

given a state φ on $\mathcal{P}_V$ such that the associated semi–scalar product satisfies the orthogonality condition: $\bar{m} := (m_1, \cdots, m_d) \neq \bar{n} := (n_1, \cdots, n_d) \quad \Rightarrow$

$$\langle a_{e_1}^{+m_1} \cdots a_{e_d}^{+m_d} \Phi_0 \ , \ a_{e_1}^{+n_1} \cdots a_{e_d}^{+n_d} \Phi_0 \rangle = 0 \quad (34)$$

then is a product state.

Theorem 7 Let $\left(\mathcal{H}, V_{\mathbb{C}}, a^{+}, \Phi_0\right)$ be a **symmetric** interacting Fock space over $V_{\mathbb{C}}$. For any linear basis $e \equiv (e_j)_{j\in D}$ of V, $n \in \mathbb{N}$ and $\bar{m} = (m_1, \ldots, m_d) \in D_n$, denote

$$\Phi_{n,\bar{m}} := a_{e_1}^{+m_1} \cdots a_{e_d}^{+m_d} \Phi_0 := a_e^{\bar{m}} \Phi_0 \qquad (35)$$

Then the following statements are equivalent.
(i) There exists a linear basis $e \equiv (e_j)_{j\in D}$ of V such that for all $n \in \mathbb{N}$ and $\bar{m} \neq \bar{n} \in D_n$,

$$\langle \Phi_{n,\bar{m}}, \Phi_{n,\bar{n}} \rangle$$

$$= \langle a_{e_1}^{+m_1} \cdots a_{e_d}^{+m_d} \Phi_0 \ , \ a_{e_1}^{+n_1} \cdots a_{e_d}^{+n_d} \Phi_0 \rangle = 0 \qquad (36)$$

(ii) There exists a linear basis $e \equiv (e_j)_{j\in D}$ of V such that, with the convention that $0/0 := 0$, for all $j \in D$, $n \in \mathbb{N}$ and $\bar{m} \neq \bar{n} \in D_n$,

$$a_j \Phi_{n,\bar{m}} =$$

$$= \begin{cases} 0, \quad \text{if } m_j = 0 \\ \dfrac{\|\Phi_{n,\bar{m}}\|^2}{\|\Phi_{n,(m_1,\cdots,m_j-1,\cdots,m_d)}\|^2}\Phi_{n,(m_1,\cdots,m_j-1,\cdots,m_d)}, \\ \text{if } m_j > 0 \end{cases} \tag{37}$$

In particular, with the notation

$$(\bar{m})_{j,\pm} = (m_1, \ldots, m_j, \ldots m_d)_{j,\pm}$$

$$:= (m_1, \ldots, m_j \pm 1, \ldots m_d) \in \mathbb{N}^d \tag{38}$$

(36) becomes

$$a_j a^{+\bar{m}}\Phi_0 = \begin{cases} 0, & \text{if } m_j = 0 \\ \dfrac{\|a^{+\bar{m}}\Phi_0\|^2}{\|a^{+(\bar{m})_{j,-}}\Phi_0\|^2} a^{+(\bar{m})_{j,-}}\Phi_0, & \text{if } m_j > 0 \end{cases} \tag{39}$$

Corollary 2 In the notations and assumptions of Theorem 7, the orthogonality condition (36) is equivalent to the following condition.

(iii) Introducing for $\bar{m} = (m_1, \ldots, m_j, \ldots m_d) \in D_n$ and $j, k \in D$, the notation,

$$\pi^{j,k}(\bar{m}) := ((\bar{m})_{k,+})_{j,-} = (m_1, \ldots, m_j - 1, \ldots, m_k + 1, \ldots \tag{40}$$

where, if $j > k$, the order of $m_j - 1$ and $m_k + 1$ in (40) is inverted, one has

$$a_j a_k^+ \Phi_{n,\bar{m}} = \begin{cases} 0, \qquad \text{if } j \neq k \text{ and } m_j = 0 \\ \frac{\|\Phi_{n+1,(\bar{m})_{k,+}}\|^2}{\|\Phi_{n,\pi^{j,k}(\bar{m})}\|^2} \Phi_{n,\pi^{j,k}(\bar{m})}, \\ \text{if either } j = k \text{ or } m_j > 0 \end{cases} \tag{41}$$

A characterization of product states

Theorem 8 In the above notations, suppose that φ is a state on $\mathcal{P}_V$. Then φ is a product state on $\mathcal{P}$ if and only if either condition (i) or (ii) of Theorem 7 or condition (iii) of Corollary 2 holds and for all $j \in D$, $n \in \mathbb{N}$ and $\bar{m} \neq \bar{n} \in D_n$, one has

$$\frac{\|\Phi_{n+1,(m_1,\cdots,m_j,\cdots,m_k+1,\cdots,m_d)}\|^2}{\|\Phi_{n,(m_1,\cdots,m_j-1,\cdots,m_k+1,\cdots,m_d)}\|^2}$$

$$= \frac{\|\Phi_{n,(m_1,\cdots,m_j,\cdots,m_k,\cdots,m_d)}\|^2}{\|\Phi_{n-1,(m_1,\cdots,m_j-1\cdots,m_k,\cdots,m_d)}\|^2} \qquad (42)$$

Remark. Denoting, for each $j \in D$, $\{\omega_{j,k}\}_{k\in\mathbb{N}}$ the Jacobi sequence of the restriction of the state φ on the $*-$ algebra $\mathcal{P}(a^{\pm}_{e_j})$ generated by the CA operators $a^{\pm}_{e_j}$, **if φ is a product state** then one has the factorization condition:

$$\|\Phi_{n,\bar{m}}\|^2 = \|\Phi_{(m_1,0,\cdots,0)}\|^2 \cdots \|\Phi_{(0,\cdots,0,m_d)}\|^2$$
$$= \prod_{j\in D} \omega_{j,m_j}! \qquad (43)$$

which implies that (42) is satisfied.

Proposition 1 Condition (42) is equivalent to the existence of a family $\{\omega_k^{(j)}\}_{k\in\mathbb{N}}$ $(j \in D)$ of sequences of strictly positive numbers satisfying (43).