Two model problems about charged particles motion with remarkable dynamics

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Based on joint works with A.Artemyev, D.Turaev

Plan.

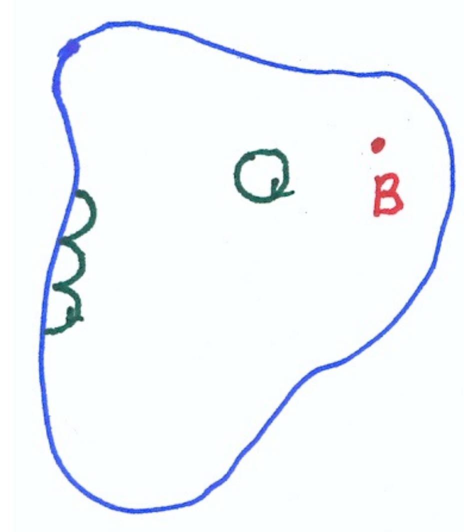
I. Destruction of adiabatic invariance in dynamics in a magnetic billiard (with A. Artemyev)

II. Dynamics near magnetic field null lines (with A. Artemyev, D. Turaev)

I. Billiards in a strong magnetic field

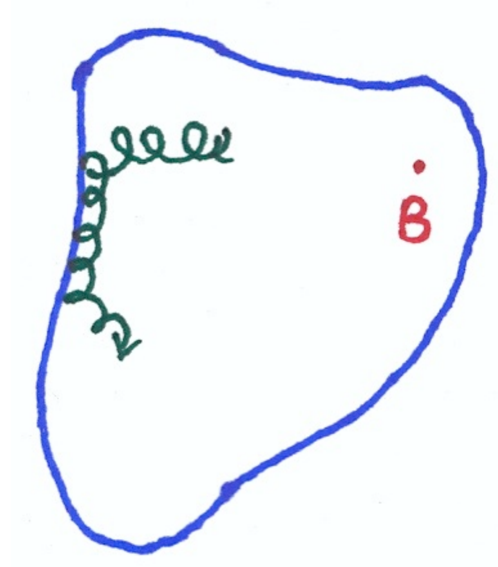
a) Uniform magnetic field

(M.Berry, M. Robnik (1985),
N.Berglund, H.Kunz(1996),
V.Zharnitsky (1998))



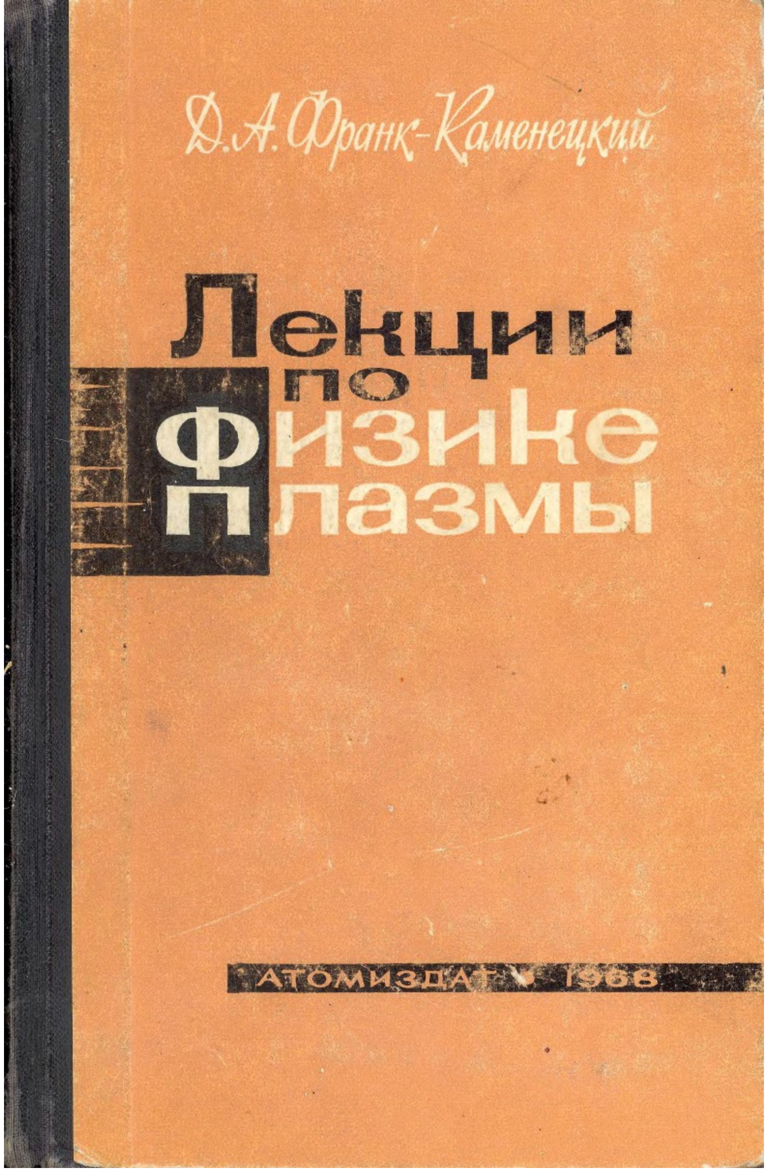
Distance of the centre of the Larmor cirle from the boundary is a perpetual adiabatic invariant.

b) Nonuniform magnetic field



Adiabatic invariant? Adiabatic description of dynamics?

Destruction of adiabatic invariance due to change of the mode of motion?



Если плазма находится в полном термодинамическом равновесии, то она не должна отдавать энергию наружу, т. е. частицы должны упруго отражаться от границы плазмы. Но в результате упругого отражения возникает краевой ток, циркулирующий в направлении, обратном току циклотронного вращения (рис. 9). Пусть магнитное поле направлено к нам, циклотронные токи текут по часовой стрелке,

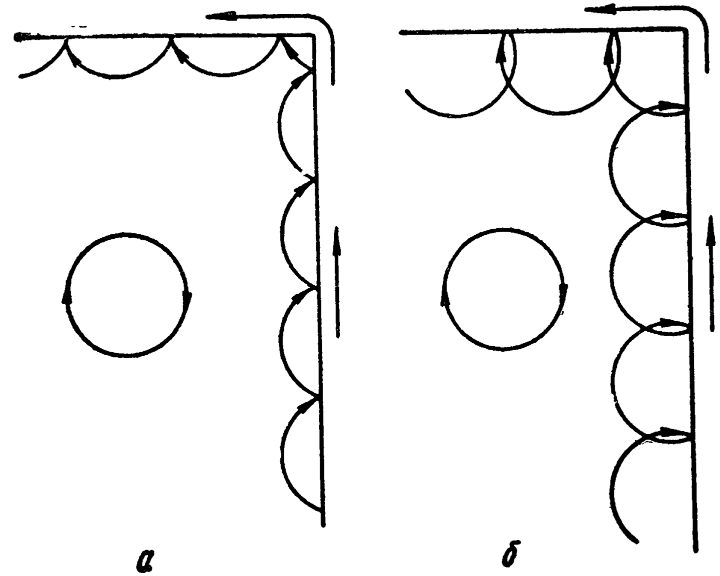


Рис. 9. Краевой парамагнитный ток при упругом отражении частиц:

а — отражение под острым углом; *б* — отражение под тупым углом.

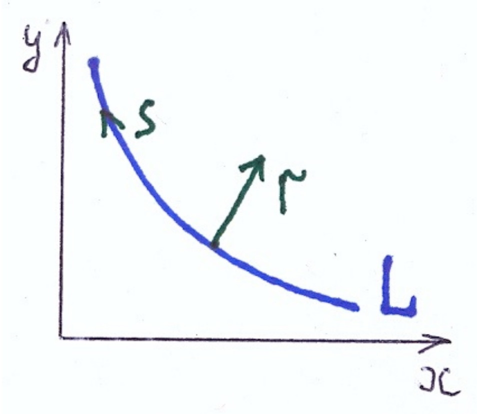
тогда краевой ток течет против часовой стрелки. Этот ток возникает по чисто геометрической причине, из-за того что при упругом отражении угол отражения равен углу падения. Краевой ток создает парамагнитный момент, направленный по исходному полю. Величина магнитного мо-

V.I. Arnold: “As a general phenomenon, it is more convenient to think about mappings, but it is easier to calculate with flows” (Mathematical methods of classical mechanics, 1988)

We consider a billiard in strong magnetic field as a slow-fast Hamiltonian system and use an adiabatic perturbation theory for description of motion with collisions.

Such an approach was justified (in a similar problem) in: I.Gorelyshev, A.N., Jump in adiabatic invariant at a transition between modes of motion for systems with impacts, Nonlinearity, **21**, 661 (2008).

1. Hamiltonian and symplectic structure



$$m\ddot{x} = \frac{e}{c}\frac{B(x,y)}{\varepsilon}\dot{y}, \quad m\ddot{y} = -\frac{e}{c}\frac{B(x,y)}{\varepsilon}\dot{x}$$

Ideal reflection at the boundary of the billiard *L={x,y : x=X(s), y=Y(s)}*

$$H = \frac{1}{2m}\left(p_x^2 + p_y^2\right) - \text{Hamiltonian},$$

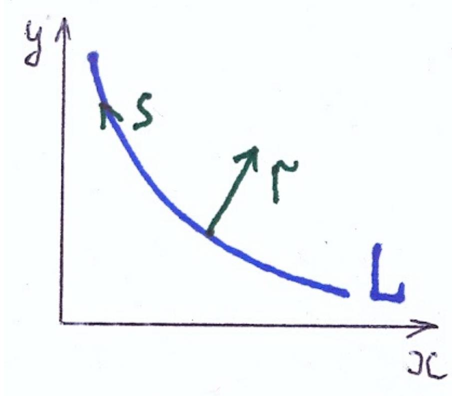
$$\omega^2 = dp_x \wedge dx + dp_y \wedge dy + \varepsilon^{-1}\frac{e}{c}B(x,y)dx \wedge dy - \text{symplectic structure},$$

x, y, p_x, p_y – phase variables,

m, e - mass and charge of the particle, c - the speed of light,

$\varepsilon^{-1}B(x,y)$ – strength of the magnetic field, $0 < \varepsilon << 1.$

2. Transformation of Hamiltonian and symplectic structure



Canonical transformation:

$$(x, y, p_x, p_y) \mapsto (r, s, p_r, p_s)$$

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_s^2}{(1+k(s)r)^2}\right), \; k(s) \text{ - the curvature of the boundary}$$

$$\omega^2 = dp_r \wedge dr + dp_s \wedge ds + \varepsilon^{-1}(1+k(s)r)\frac{e}{c}B(r,s)dr \wedge ds,$$

Canonical momentum:

$$P_s = p_s + \varepsilon^{-1}A(r,s),$$

$$A(r,s) = \int_0^r (1+k(s)\xi)\frac{e}{c}B(\xi,s)d\xi = \frac{e}{c}\left(B_0(s)r + O(r^2)\right)$$

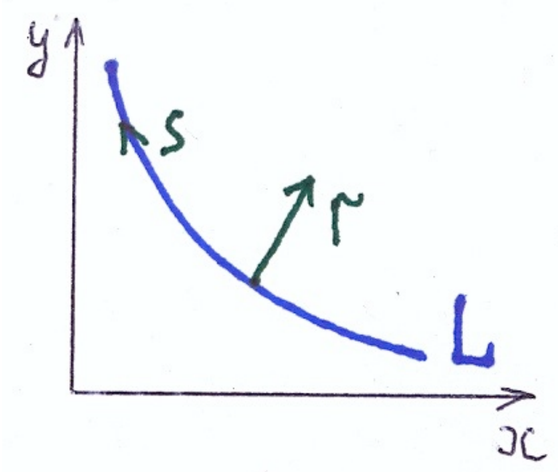
$$H = \frac{1}{2m}\left(p_r^2 + \left(\frac{P_s - \varepsilon^{-1}A(r,s)}{1+k(s)r}\right)^2\right), \quad \omega^2 = dp_r \wedge dr + dP_s \wedge ds$$

3.Rescaling

$$\bar{r} = r/\varepsilon,\ \bar{s} = s/\varepsilon,\ \bar{t} = t/\varepsilon,$$

$$H = H_0(\bar{r}, p_r, \varepsilon\bar{s}, P_s) + O(\varepsilon\bar{r}),$$

$$H_0 = \frac{1}{2m}\left[p_r^2 + \left(P_s - \frac{e}{c}B_0(\varepsilon\bar{s})\bar{r}\right)^2\right]$$



Equations of motion between collisions with boundary:

$$\frac{d\bar{r}}{d\bar{t}} = \frac{\partial H}{\partial p_r},\quad \frac{dp_r}{d\bar{t}} = -\frac{\partial H}{\partial \bar{r}}$$

$\bar{r}, p_r$ -fast variables

$$\frac{ds}{d\bar{t}} = \varepsilon\frac{\partial H}{\partial P_s},\quad \frac{dP_s}{d\bar{t}} = -\varepsilon\frac{\partial H}{\partial s}$$

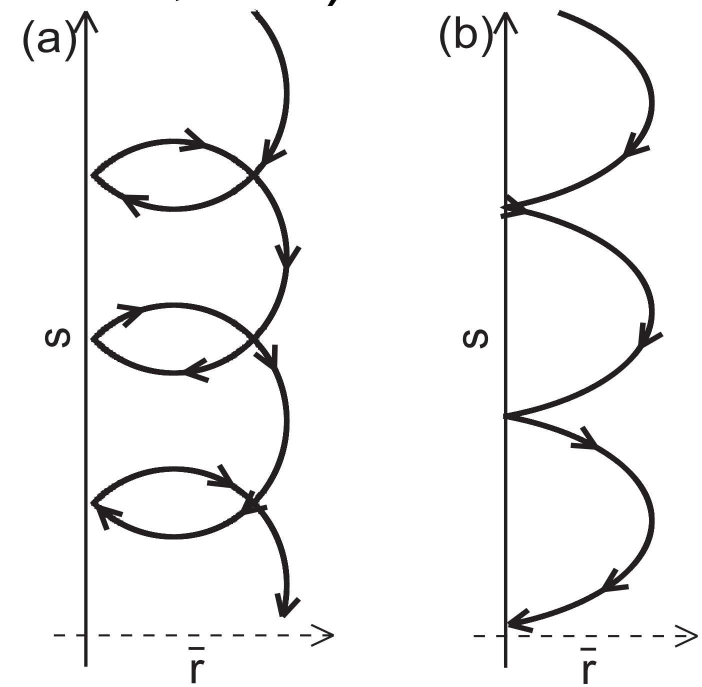
s, P_s -slow variables

At collisions ($r = 0$) s and P_s are continuous, while p_r changes the sign.

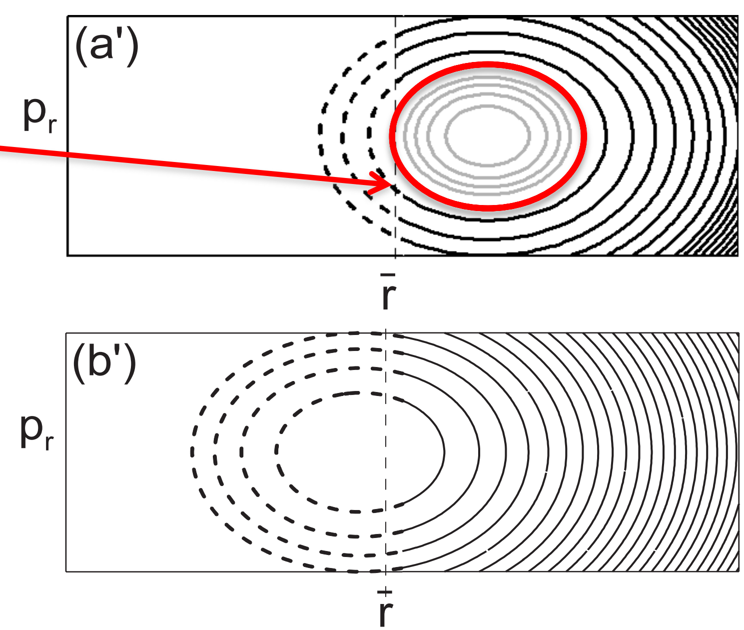
4. Unperturbed dynamics (s, P_s=const, $r \geq 0$)

$$H_0 = \frac{1}{2m}\left[p_r^2 + \left(P_s - \frac{e}{c}B_0(s)\bar{r}\right)^2\right]$$

$$\frac{d\bar{r}}{d\bar{t}} = \frac{\partial H_0}{\partial p_r}, \quad \frac{dp_r}{d\bar{t}} = -\frac{\partial H_0}{\partial \bar{r}}$$



Linear oscillations between collisions



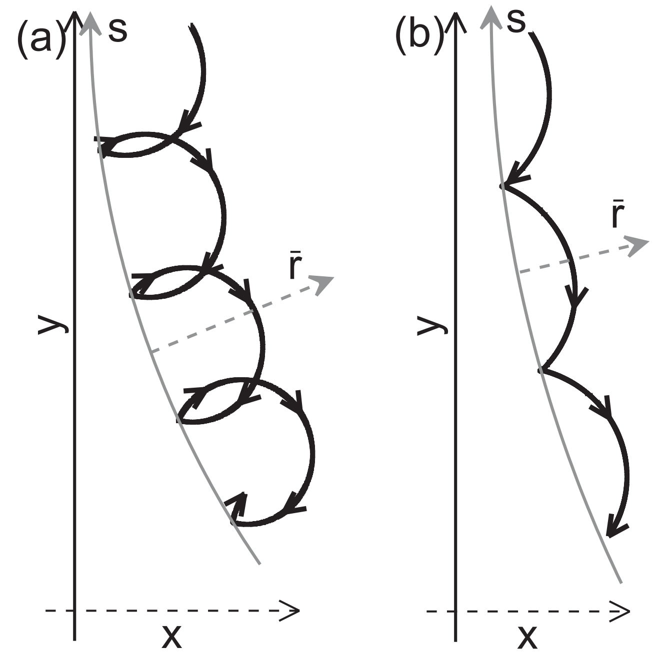
Action $I = I(h, s, P_s) =$ "$area$"$/(2\pi)$

5. Adiabatic approximation for skipping motion

$$H_0 = h(I, s, P_s) = \mathrm{const}, \; I = \mathrm{const}$$

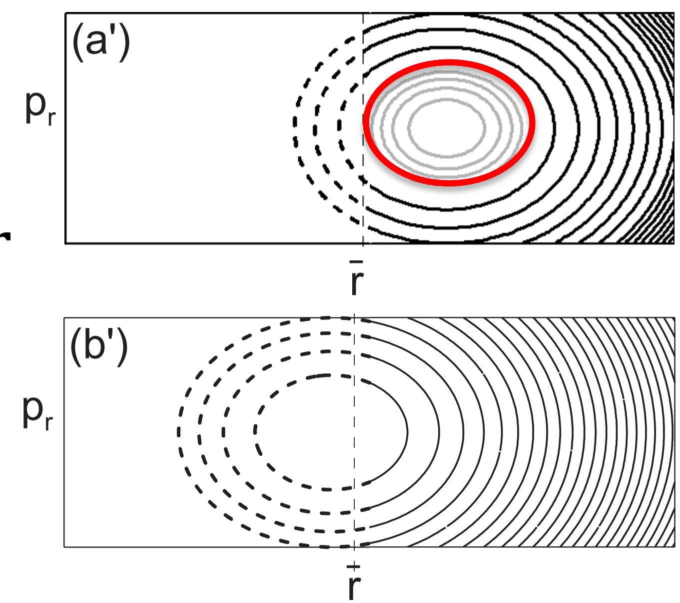
$$\frac{ds}{dt} = \frac{\partial h}{\partial P_s}, \quad \frac{dP_s}{dt} = -\frac{\partial h}{\partial s}$$

$$I = I(h, s, P_s) = \text{"}area\text{"}/(2\pi)$$



Meaning of *I* :

$I = e\Phi/(2\pi c\varepsilon)$ where Φ is the flux of the magnetic field through the area bounded by the particle Larmor trajectory and the billiard boundary



Condition for take-off:

$$B_0(s) = \frac{mch}{eI}$$

(skipping motion while $B_0(s) \le \frac{mch}{eI}$)

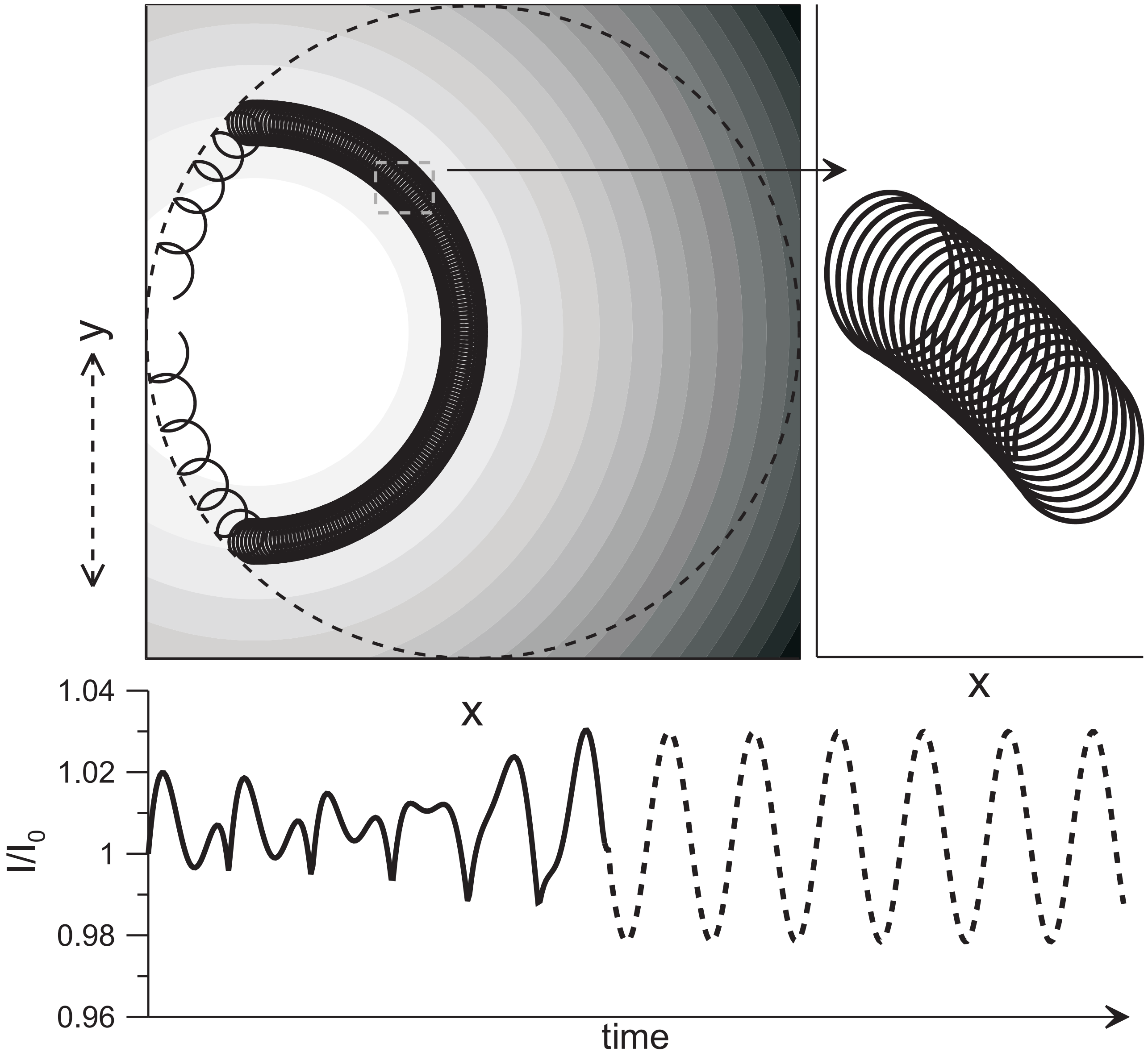
6. Adiabatic approximation for gradient drift: guiding centre theory

Centre of Larmor circle moves along level line of magnetic field strength: $B(x,y) = \text{const}:$

$$\frac{dy}{dt} = -\frac{\varepsilon}{mB(x,y)}\frac{\partial B(x,y)}{\partial x} I, \quad \frac{dx}{dt} = \frac{\varepsilon}{mB(x,y)}\frac{\partial B(x,y)}{\partial y} I$$

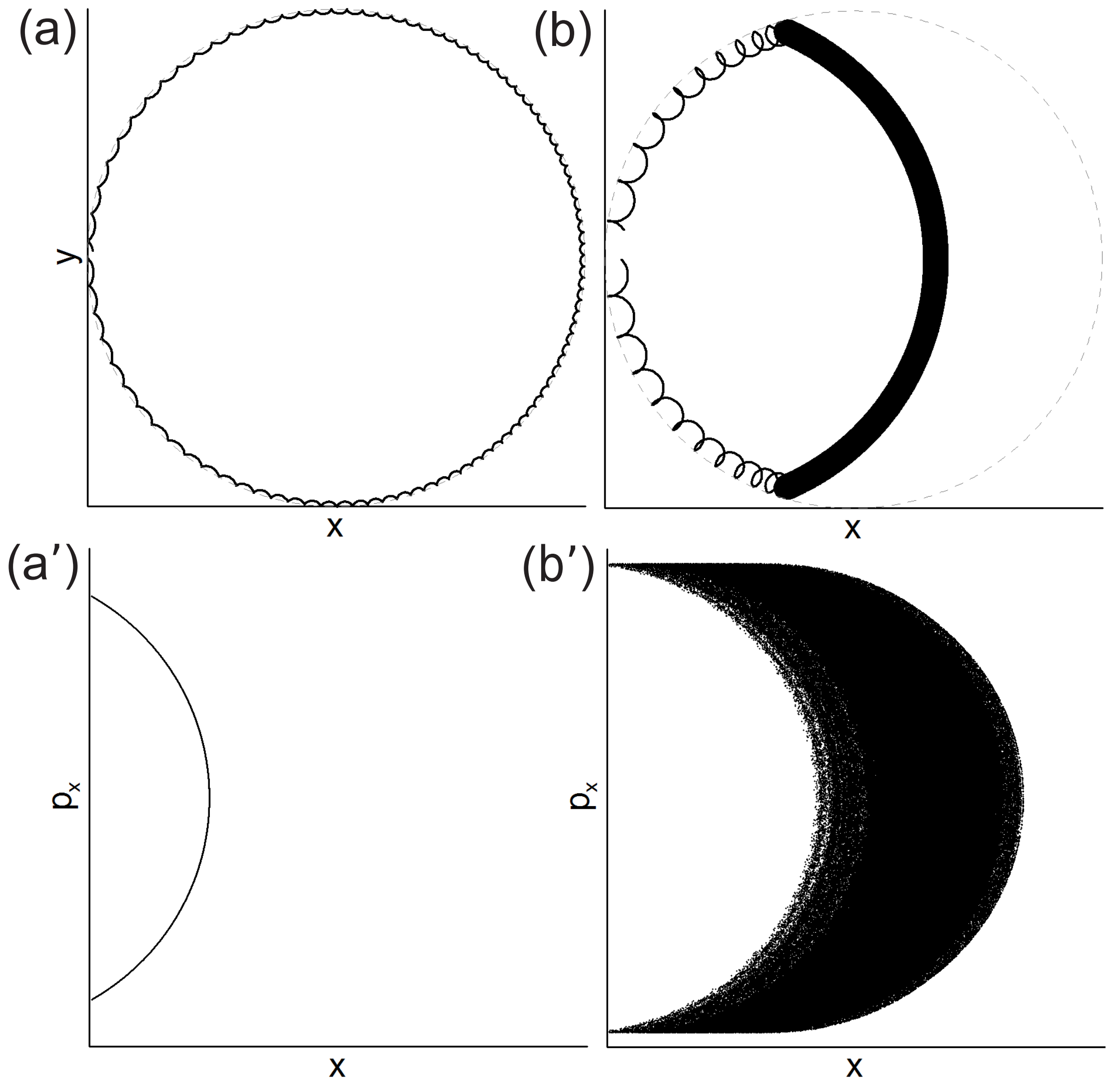
Thus, speed of the gradient drift, $\sim \varepsilon$, is much smaller than speed of skipping along billiard boundary, ~ 1, which is much smaller than angular speed of Larmor rotation, $\sim \varepsilon^{-1}$.

Here $I = (mc/e)(\mu/\varepsilon)$, where μ is the magnetic moment of the particle, the ratio of of the kinetic energy to to the value of the magnetic field.



y
x
x
1.04
1.02
1
0.98
0.96
I/I0
time

7. Destruction of adiabatic invariance due to multiple changes of modes of motion (right figures)

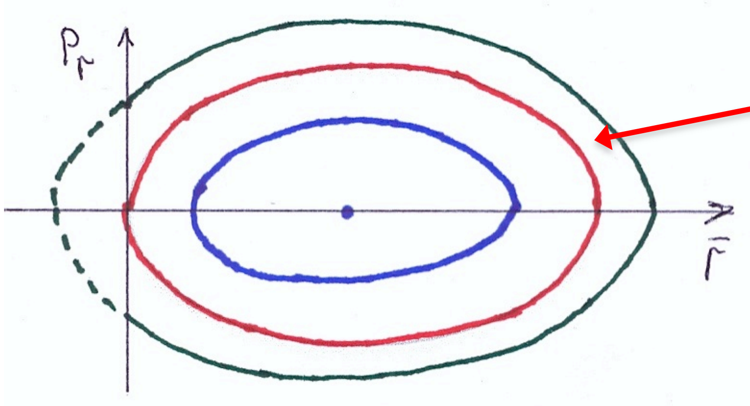


Billiard trajectories

Poincaré sections

$$y = 0,\ p_y < 0$$

8. An estimate of jump of adiabatic invariant at take-off



"separatrix"

Denote $K = H - H_{sep}(P_s)$.

Decay of K for one round $\cong \varepsilon$

Oscillation of I inside one round $\cong \varepsilon$

Change of I for the whole round $\cong \quad \varepsilon^2 / K^{1/2}$

Improved adiabatic invariant $J = I + \varepsilon u(\bar{r}, p_r, s, P_s)$

Oscillation of J inside one round $\cong \varepsilon^2 / K^{1/2}$

Change of J for the whole round $\cong \quad \varepsilon^3 / K^{3/2}$

Total change of $J \cong \dfrac{1}{\varepsilon}\displaystyle\int_{\varepsilon}^{1} \frac{\varepsilon^3 dk}{k^{3/2}} \cong -\left.\frac{\varepsilon^2}{k^{1/2}}\right|_{\varepsilon}^{1} \cong \varepsilon^{3/2}$

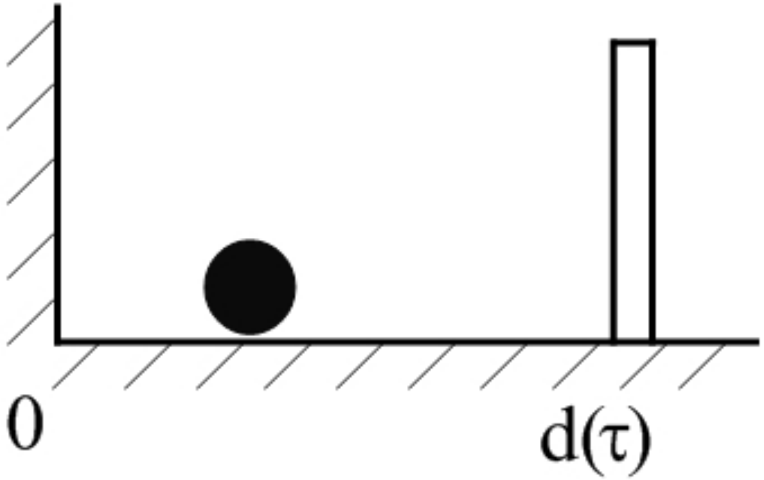
9. Asymptotic formula for jump of adiabatic invariant at take-off

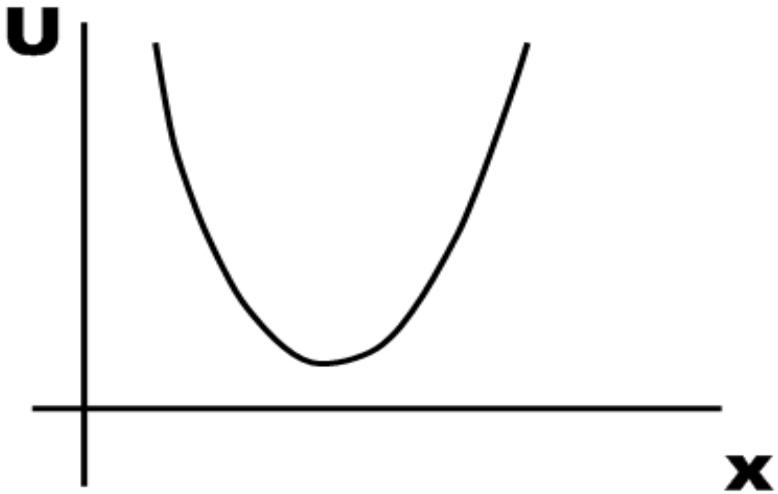
$$\Delta J = -A(\varepsilon\Theta)^{\frac{3}{2}} f(\xi) + o\left(\varepsilon^{\frac{3}{2}}\right), \qquad A, \Theta\text{=const}$$

$$f(\xi) = \lim_{N\to\infty}\left[\frac{3}{2}\sum_{m=0}^{N}(\xi+m)^{1/2} - \frac{3}{4}(\xi+N)^{1/2} - (\xi+N)^{3/2}\right]$$

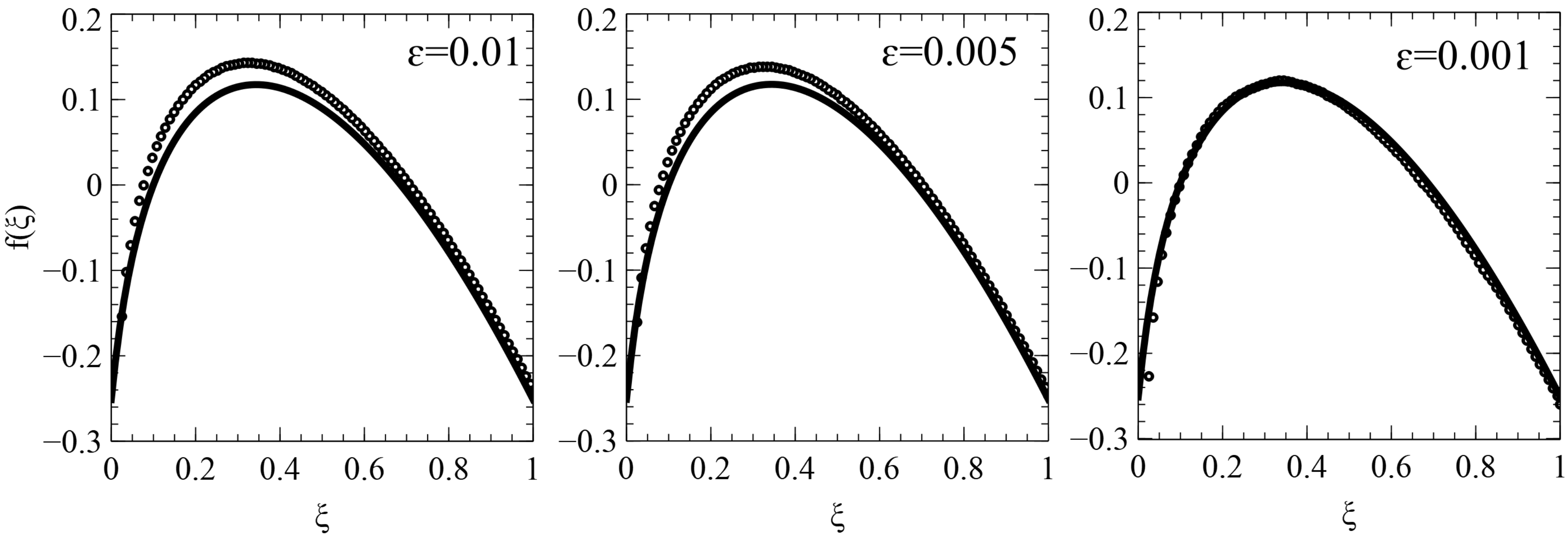
Here value $\xi \in (0,1)$ characterices the phase of the particle on the Larmor circle at the take-off. It is interpreted as a random value with uniform distribution on the interval $(0,1)$. Thus the obtained formula provides distribution of ΔJ as a random value.

This formula for jump of adiabatic invariant has the same form as in Gorelyshev, N., 2008, where a one-dimensional motion between slowly moving walls in slowly changing potential is considered.





Numerical check of formula:



10. Mechanism of destruction of adiabatic invariance

- jump of (improved) adiabatic invariant due to change of mode of motion (take-off) $\sim \varepsilon^{3/2}$. This jump depends on the phase of the particle on Larmor circle at the moment of take-off.
- this jump changes the time of motion from take-off to landing by a value $\sim\varepsilon^{3/2}\,\varepsilon^{-1}\sim\varepsilon^{1/2}$.
- this will change phase of the particle on Larmor circle at the next take-off by a value $\sim\varepsilon^{1/2}\varepsilon^{-1}\sim\varepsilon^{-1/2}$
- thus for phases φ and φ' on Larmor circle (equivlently, for variables ξ and ξ') at two consecutive take-offs we have a relation of the form $d\varphi'/d\varphi \sim\varepsilon^{-1/2} >> 1$. This stretching of phase prevents existence of regular dynamics and conservation of adiabatic invariance.

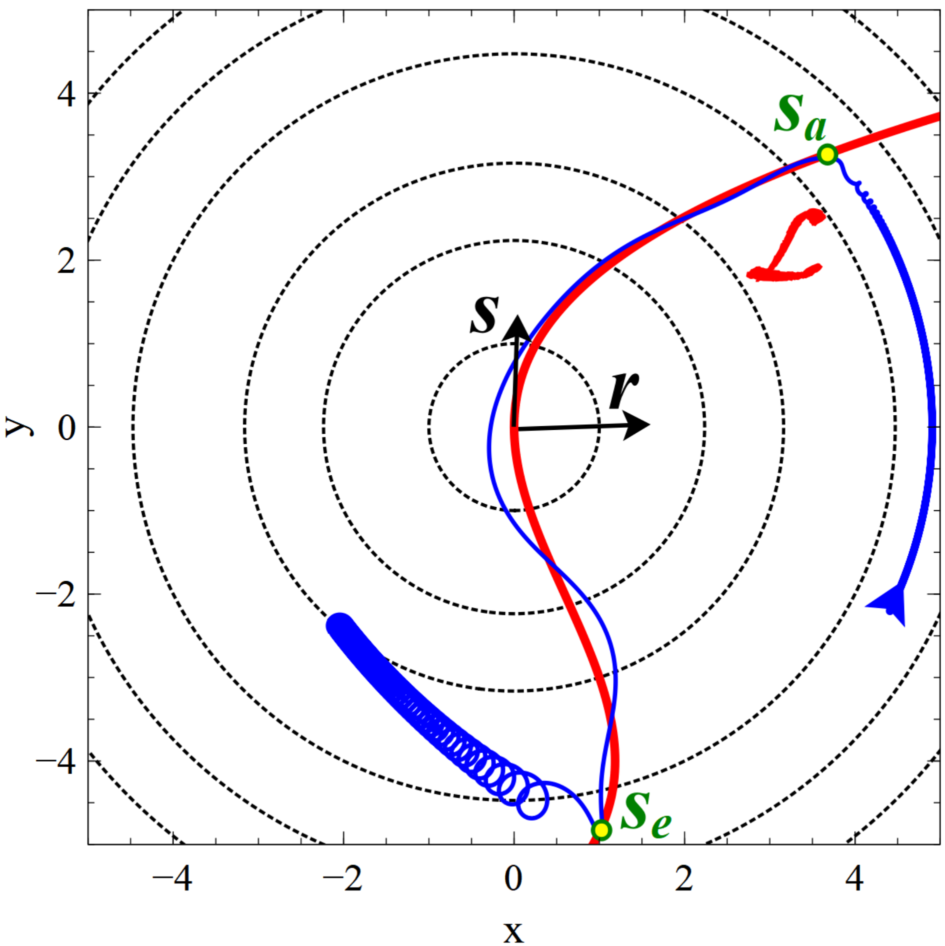
II. Dynamics near magnetic field null line

We consider planar motion of a charged particle in strong stationary magnetic field with a null line and a strong electrostatic field.

Red: null line of the magnetic field

Dashed: level lines of the electrostatic potential

Blue: particle trajectory



Schematic of the problem

1. Equations of motion.

(x, y) − Cartesian coordinates,

$\frac{1}{\varepsilon}B(x, y)$ − magnetic field,

$\frac{1}{\varepsilon}V(x, y)$ − electrostatic potential

$0 < \varepsilon \ll 1$

$$\boxed{m\ddot{x} = \frac{e}{c}\frac{B}{\varepsilon}\dot{y} - \frac{1}{\varepsilon}\frac{\partial V}{\partial x}, \quad m\ddot{y} = -\frac{e}{c}\frac{B}{\varepsilon}\dot{x} - \frac{1}{\varepsilon}\frac{\partial V}{\partial y}}$$

m, e − mass and charge of the particle

c − speed of light

We choose units such that $e = m$=c=1.

$$\boxed{\varepsilon\ddot{x} = B\dot{y} - \frac{\partial V}{\partial x}, \quad \varepsilon\ddot{y} = -B\dot{x} - \frac{\partial V}{\partial y}}$$

This is a Hamiltonian system with the following Hamiltonian and symplectic structure

$$H = \frac{1}{2}\left(p_x^2 + p_y^2\right) + \frac{1}{\varepsilon}V(x, y),$$

$$dp_x \wedge dx + dp_y \wedge dy + \frac{1}{\varepsilon}B(x, y)\,dx \wedge dy$$

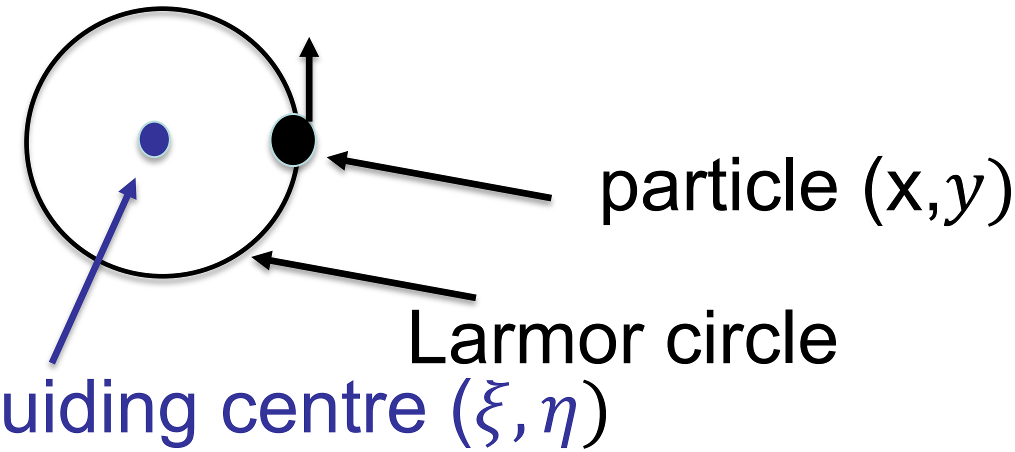
μ – magnetic moment, the ratio of of the kinetic energy of the particle to the value of the magnetic field:

$$\mu = \left(H - \frac{1}{\varepsilon}V(x, y)\right) / \left(\frac{1}{\varepsilon}B(x, y)\right) = \left(\varepsilon H - V(x, y)\right) / B(x, y)$$

μ is *an* adiabatic invariant for motion away from the null line

2. Guiding centre equation.

We consider $\mu \sim 1$



$$\boxed{\dot{\xi} = -\frac{\mu}{B}\frac{\partial B}{\partial \eta} - \frac{1}{B}\frac{\partial V}{\partial \eta}, \quad \dot{\eta} = \frac{\mu}{B}\frac{\partial B}{\partial \xi} + \frac{1}{B}\frac{\partial V}{\partial \xi}}$$

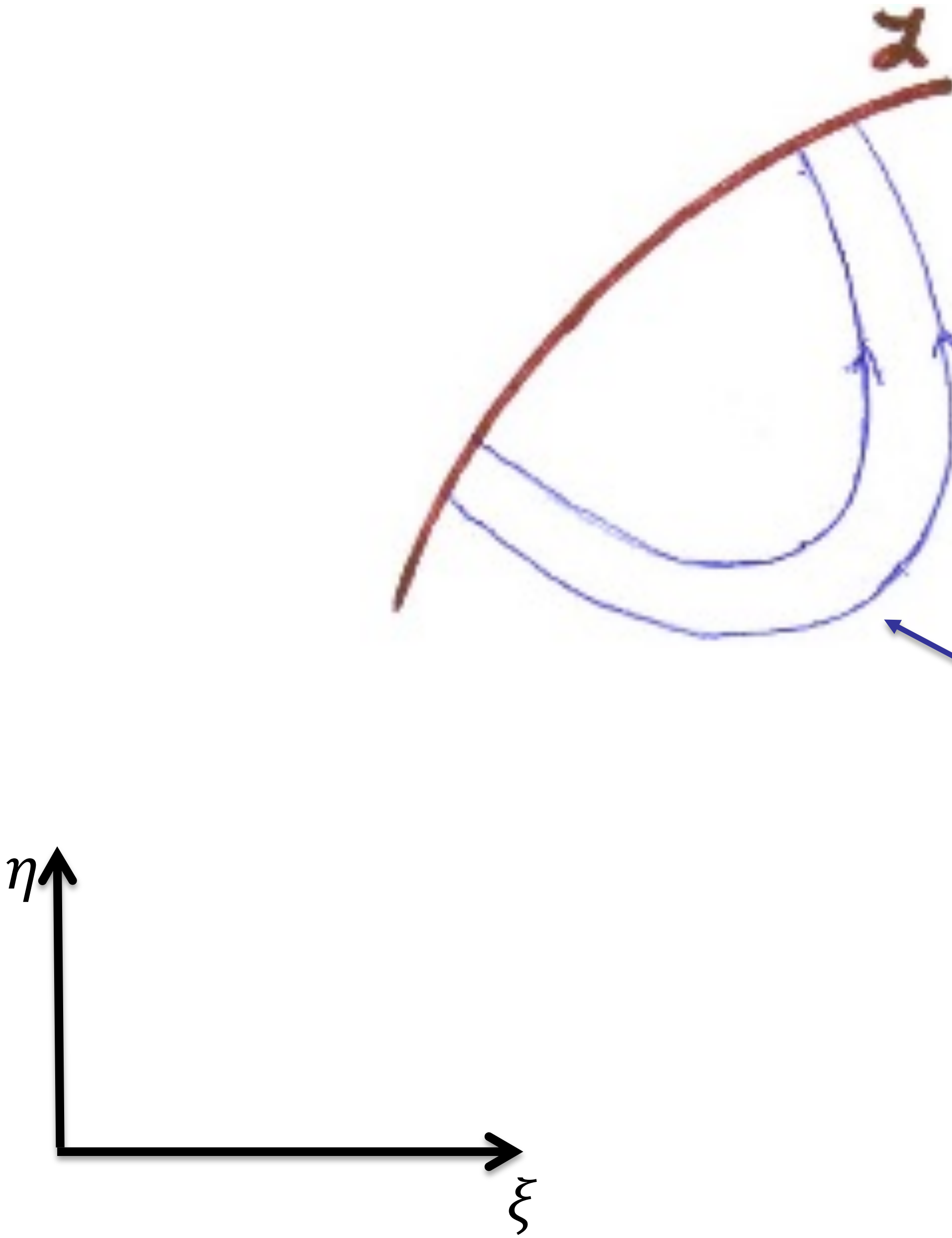
$$B = B(\xi, \eta), \quad V = V(\xi, \eta)$$

This is a Hamiltonian system:

Hamiltonian: $\varepsilon H = \mu B(\xi, \eta) + V(\xi, \eta)$,

Symplectic structure: $B(\xi, \eta) d\xi \wedge d\eta$

Guiding centre trajectories



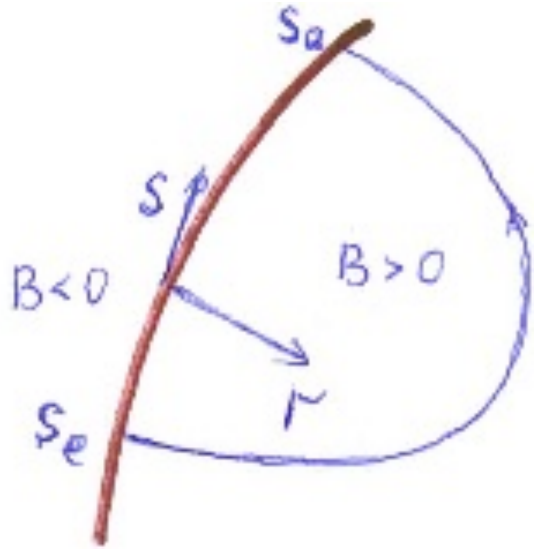
null line:

$B(\xi, \eta) = 0$

guiding centre trajectory

$\mu B(\xi, \eta) + V(\xi, \eta) = \text{const}$

3. New variables and expansion near the null line.



$$B = b(s)r + O(r^2), b(s) > 0$$

$$V = v_0(s) + v_1(s)r + O(r^2)$$

For the guiding centre equation $$\frac{d}{dt} r^2 \sim -\frac{2}{b(s)} v_0'(s)$$

The guiding centre velocity changes direction at the null line.
Thus, the guiding centre can't cross null line.

$$v_0'(s) > 0 \ - \text{absorption } (s = s_a)$$

$$v_0'(s) < 0 \ - \text{ejection } (s = s_e)$$

$$\mu B(\xi, \eta) + V(\xi, \eta) = \text{const} \implies \boxed{v_0(s_a) = v_0(s_e)}$$

Expanded original equation

Hamiltonian: $$E = \frac{1}{2}(p_r^2 + p_s^2) + \frac{1}{\varepsilon} v_0(s) + \frac{1}{\varepsilon} v_1(s) r$$

Symplectic structure: $$dp_r \wedge dr + dp_s \wedge ds + \frac{1}{\varepsilon} b(s) r\, dr \wedge ds$$

s-component of vector potential: $$\frac{1}{2\varepsilon} b(s) r^2$$

Canonical momentum: $$\mathcal{P}_s = p_s + \frac{1}{2\varepsilon} b(s) r^2$$

Hamiltonian:

$$E = \frac{1}{2}\left(p_r^2 + (\mathcal{P}_s - \frac{1}{2\varepsilon} b(s) r^2)^2\right) + \frac{1}{\varepsilon} v_0(s) + \frac{1}{\varepsilon} v_1(s) r$$

Canonical symplectic structure:

$$dp_r \wedge dr + d\mathcal{P}_s \wedge ds$$

Hamiltonian:

$$E = \frac{1}{2}\left(p_r^2 + (\mathcal{P}_s - \frac{1}{2\varepsilon} b(s) r^2)^2\right) + \frac{1}{\varepsilon} v_0(s) + \frac{1}{\varepsilon} v_1(s) r$$

For $|\mathcal{P}_s| \gg \varepsilon^{-1/3}$:

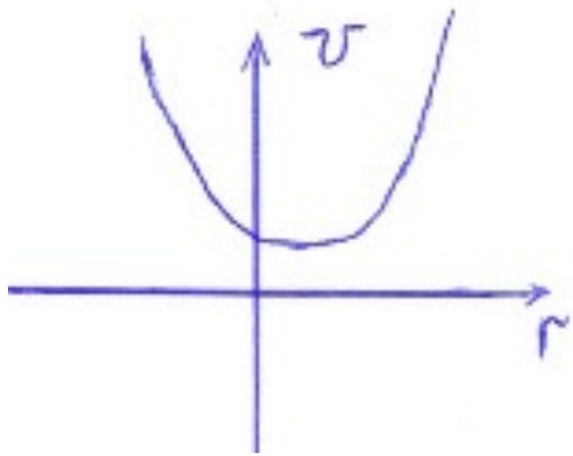
(r, p_r) – fast variables, $(s, \mathcal{P}_s)$ – slow variables

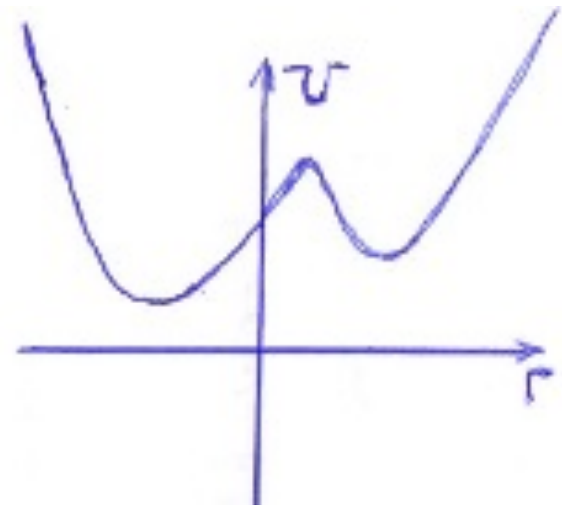
For frozen $(s, \mathcal{P}_s)$:

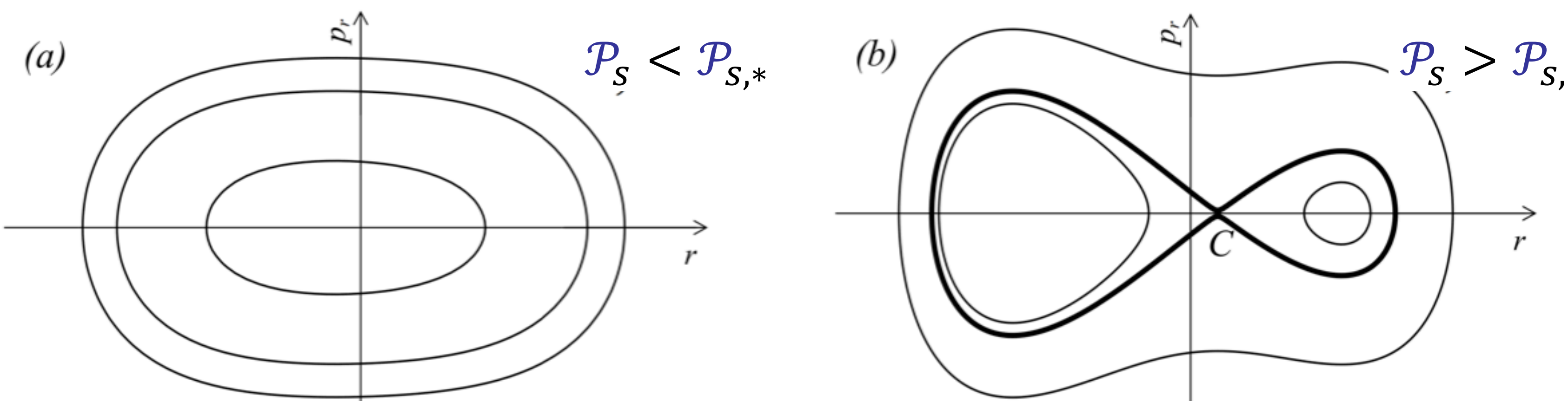
dynamics of (r, p_r) is a motion in the quartic potential

$$U = \frac{1}{2}(\mathcal{P}_s - \frac{1}{2\varepsilon} b(s) r^2)^2 + \frac{1}{\varepsilon} v_1(s) r$$

$$U = \frac{1}{2}\left(\mathcal{P}_s - \frac{1}{2\varepsilon}b(s)r^2\right)^2 + \frac{1}{\varepsilon}v_0(s) + \frac{1}{\varepsilon}v_1(s)r$$







Motion with slow evolution of $(s, \mathcal{P}_s)$ has adiabatic invariant, which coincides with $|\mu|$ for small oscillations near bottom of potential well.

4. Dynamics near point of absorption.

Rescaling near the point of absorption ($s \approx s_a, t \approx t_0$):

$$r = \varepsilon^{1/3}\,\hat{r},\;\; p_r = \varepsilon^{-1/3}\hat{p}_r,\, s - s_a = \varepsilon^{1/3}\,\hat{s}\,,\quad \mathcal{P}_s = \varepsilon^{-1/3}\hat{\mathcal{P}}_s$$

$$t - t_0 = \varepsilon^{2/3}\,\hat{t},\qquad E - \frac{1}{\varepsilon} v_0(s_a) = \varepsilon^{-2/3}\,\hat{E}$$

The system for the rescaled variables is given, in the limit as $\varepsilon \to 0$ by the rescaled Hamiltonian

$$\hat{E} = \frac{1}{2}\left[\hat{p}_r^2 + \left(\hat{\mathcal{P}}_s - b_a\frac{\hat{r}^2}{2}\right)^2\right] + v'_{0,a}\hat{s} + v_{1,a}\hat{r},$$

The corresponding equations of motion are

$$\frac{d}{d\hat{t}}\hat{r} = \hat{p}_r, \quad \frac{d}{d\hat{t}}\hat{p}_r = \left(\hat{\mathcal{P}}_s - b_a\frac{\hat{r}^2}{2}\right)b_a\hat{r} - v_{1,a},$$

$$\frac{d}{d\hat{t}}\hat{s} = \left(\hat{\mathcal{P}}_s - b_a\frac{\hat{r}^2}{2}\right), \quad \frac{d}{d\hat{t}}\hat{\mathcal{P}}_s = -v'_{0,a}.$$

Here $v'_{0,a} = v'_0(s_a), \;\; v_{1,a} = v_1(s_a), \;\; b_a = b(s_a)$

There is no small parameter in these equations,

$$\hat{\mathcal{P}}_s = -v'_{0,a}\hat{t}$$

Equation for $\hat{r}$ turnes out to be a nonhomogeneous Painlevé II equation:

$$\frac{d^2}{d\hat{t}^2}\hat{r} = \left(-v'_{0,a}\hat{t} - b_a\frac{\hat{r}^2}{2}\right) b_a\hat{r} - v_{1,a}$$

Dynamics at $\hat{t} \to -\infty$ and $\hat{t} \to \infty$ are oscillations at bottoms of potential wells. They have adiabatic invariants. Difference of values of these adiabatic invariants for one trajectory is a value of order 1.

Thus, passage through a neighbourhood of an absorption point leads to a change (jump) of the adiabatic invariant of order 1.

Connection formulas for solutions of Painlevé II equation at $\hat{t} \to -\infty$ and $\hat{t} \to \infty$ should allow to calculate this jump. (This is done for homogeneous Painlevé II, $v_{1,a} = 0$.)

5. Drift along the null line.

After passing through a neighbourhood of the absorption point the particle *drifts* along the null line with small *fast oscillations* across the null line.

Oscillations have the adiabatic invariant

$$I_0 = \frac{\frac{1}{2}\left(p_r^2 + |\mathcal{P}_s| b(s) r^2/\varepsilon\right)}{(|\mathcal{P}_s| b(s)/\varepsilon)^{1/2}}.$$

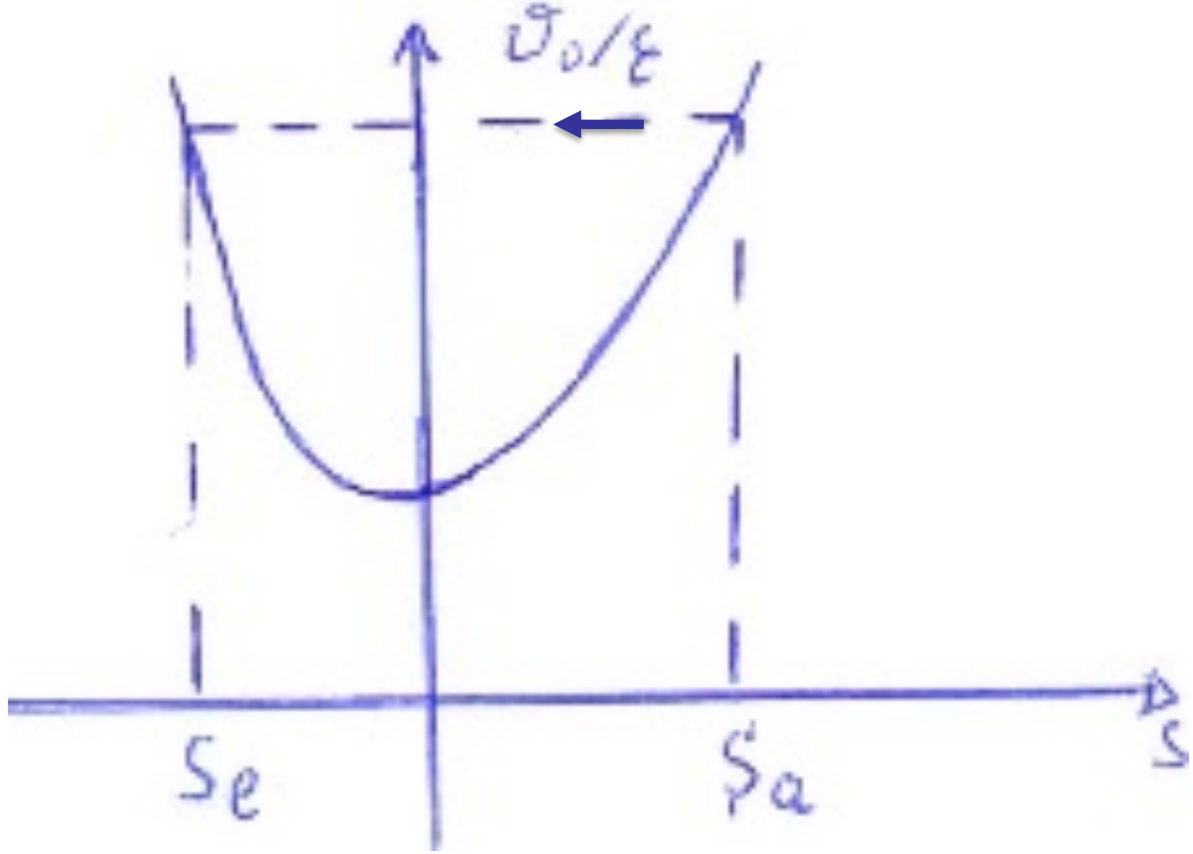
Drift is described by the Hamiltonian

$$\mathcal{E} = \frac{1}{2}\mathcal{P}_s^2 + v_0(s)/\varepsilon$$

Thus, drift is motion in the potential well of the potential $v_0\,(s)/\varepsilon$:

$$\varepsilon \ddot{s} + \frac{\partial v_0(s)}{\partial s} = 0$$

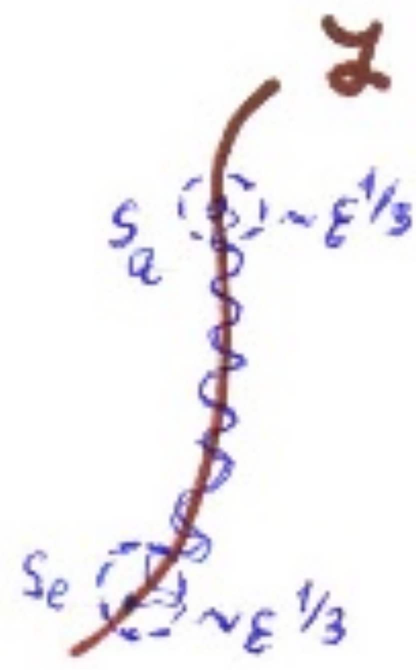
$$\varepsilon \ddot{s} + \frac{\partial v_0(s)}{\partial s} = 0$$



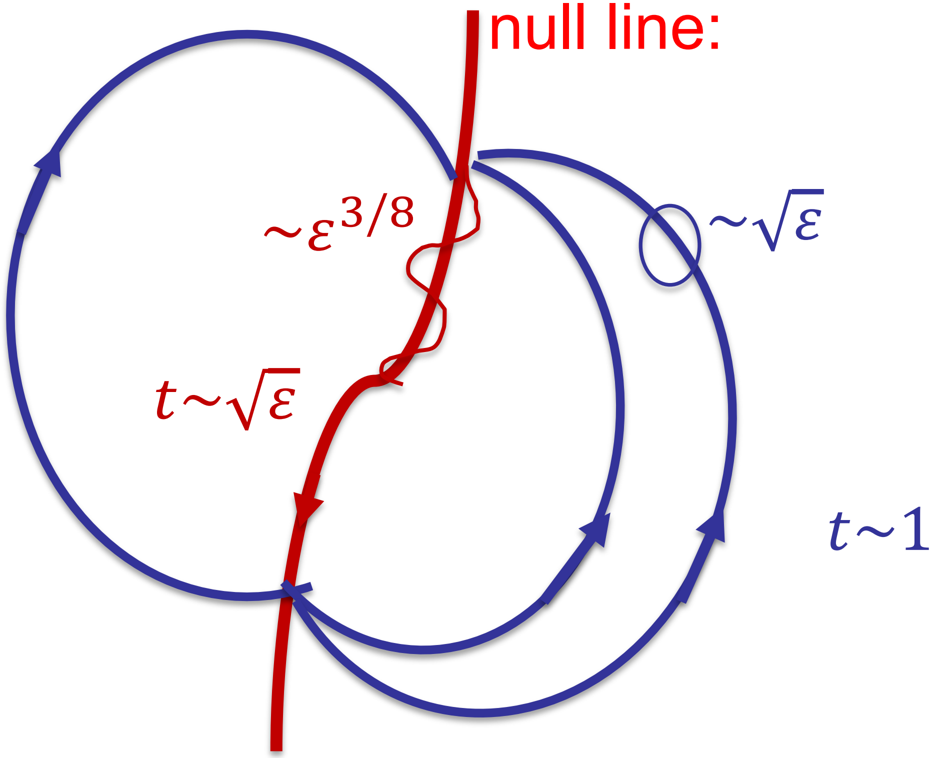
6. Ejection from the null line.

Ejection point is determined, in the principal approximation, by the relation $v_0\,(s_e) = v_0\,(s_a)$.

Dynamics near the ejection point is similar to that near the absorption point. It is described by Painlevé II equation and result in a jump ~ 1 in the adiabatic invariant invariant .



7. General description and asymptotic characteristics of motion.



8. Numerical example.

Magnetic field $\frac{B(x,y)}{\varepsilon}$, $B(x,y) = x$.

Electrostatic potential $\frac{V(x,y)}{\varepsilon}$, $V(x,y) = \frac{x^2+y^2}{2}$

Thus, null line: $x = 0$; $\quad r \equiv x, s \equiv y, v_0(s) = \frac{y^2}{2}$

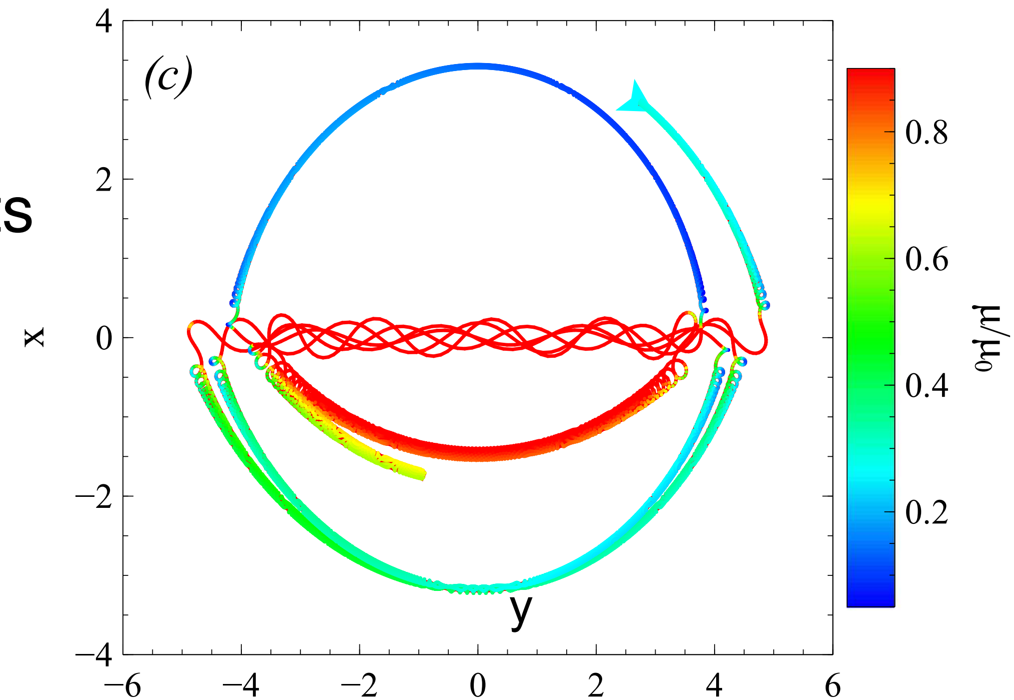
Guiding centre trajectories are arcs of circles $\mu x + \frac{x^2+y^2}{2} = h$

$h = \text{const}, \; \mu = \text{const}.$

Guiding centre trajectories cross the null line at the points

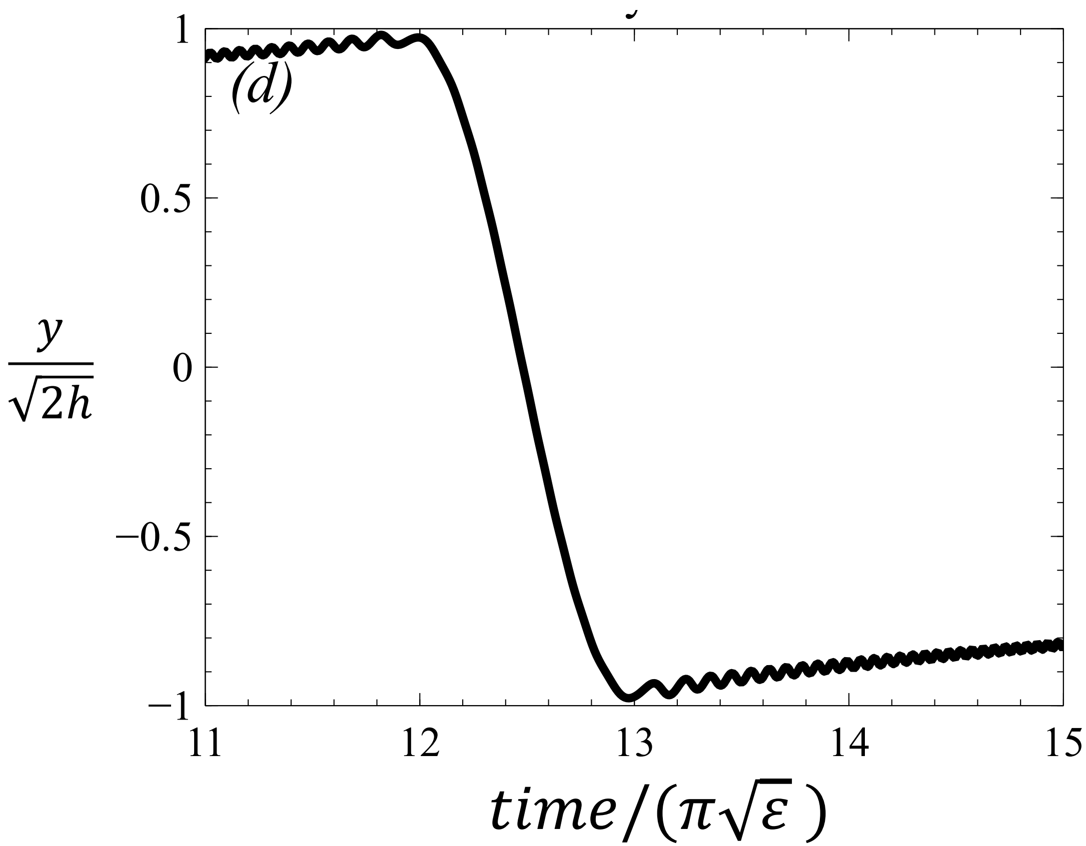
$$y = \pm\sqrt{2h}$$

$$2h = 25, \varepsilon = 10^{-3}$$



Drift along the null line is described by the equation $\varepsilon\ddot{y} + y = 0$ with the initial condition $y = \sqrt{2h}$, $\dot{y}$=0.

Thus $y(t) = \sqrt{2h}\,\cos(t/\sqrt{\varepsilon}\ )$.



9. Stationary distribution of the magnetic moment.

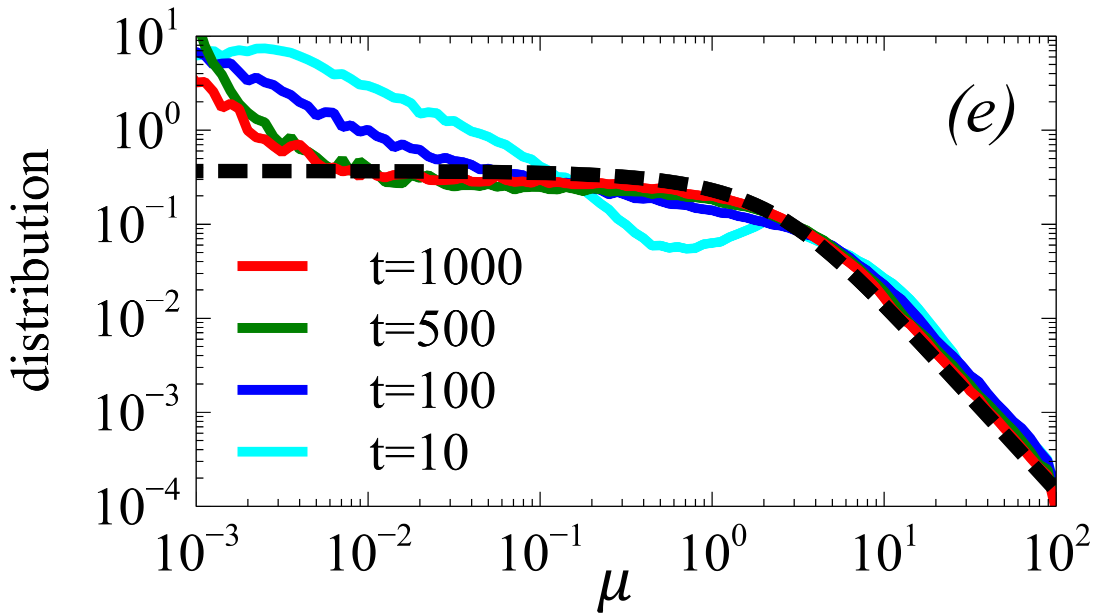
Heuristically, jumps of the magnetic moment can be considered as independent identically distributed random values. For an ensemble of particles, these jumps should lead to stationary distribution of the magnetic moment. A theoretical density of this distribution is

$$f(\mu) = 4/(\pi\sqrt{2h}) \cdot \left[1 - \chi \arcsin(1+\chi^2)^{-1/2}\right]$$

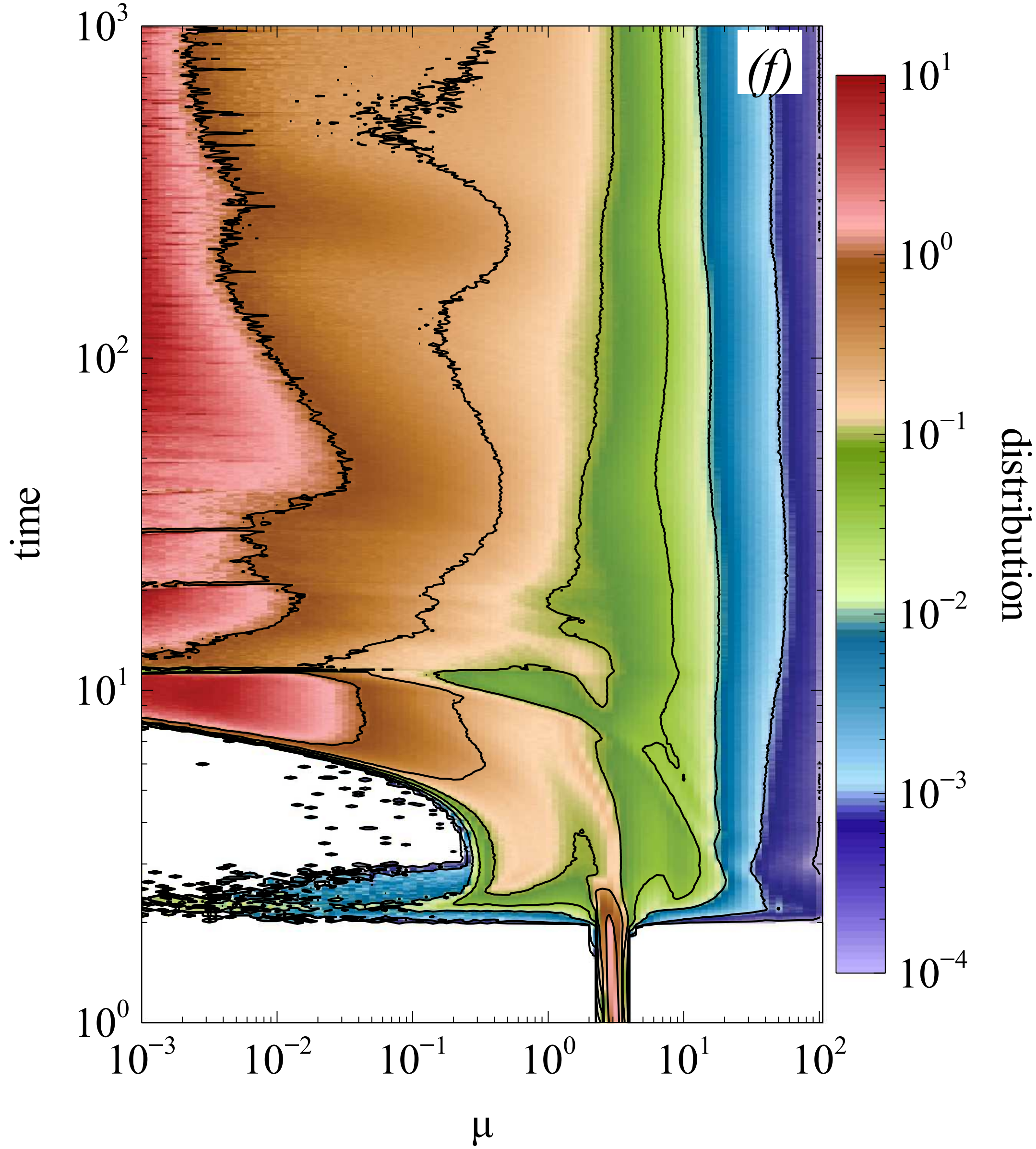
where $\chi = \frac{\mu}{\sqrt{2h}}$.

Numerical example of convergence to stationary distribution.

Dashed line – theoretical density



Evolution of distribution of the magnetic moment.



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3. A.Neishtadt, A. Artemyev, D. Turaev, Chaos, **29**, 051104 (2019)

С днем рождения!