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Rigorous theory of 1d turbulence and the stochastic Burgers equation

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BASED ON:

– my own work, starting late 1990's;

– works of my former PhD students Andrey Biriuk and Alexandre Boritchev;

– a MS of my book, written with Alex Boritchev (same title as the talk)

§1. K41 THEORY

The Kolmogorov theory of turbulence was created by A. N. Kolmogorov in three articles, published in 1941 (partially based on the previous work of Taylor and von Karman–Howard), and usually is called the K41 theory. It describes statistical properties of turbulent flows and is now the most popular theory of turbulence. I will briefly describe it for a fluid flow $u(t,x)$ of order 1, space–periodic of period $L \sim 1$. The Reynolds number R of the flow is

$$R = \frac{\langle\text{space-scale of the flow}\rangle\langle\text{characteristic speed of the flow}\rangle}{\nu} \sim \nu^{-1},$$

where ν is the viscosity. If R is large, then the velocity field $u(t,x)$ of the flow becomes very irregular, i.e. turbulent.

All constants in my talk are independent from ν (and R).

Kolmogorov: short scale in x features of a turbulent flow $u(t,x)$ display a universal behaviour which depends on the particularities of the system only through a few parameters. To make this point clearer, we will introduce **several definitions** which apply to a snapshot $u(t,\cdot) =: u(x)$ of the flow.

Let us decompose $u(x)$ in Fourier: $u(x) = \sum_{\xi \in L^{-1}\mathbb{Z}^d} \hat{u}(\xi) e^{2\pi i \xi \cdot x}$.

◇ The *dissipation scale* l_d of the flow u is $l_d = \nu^{-c_d}$, where $c_d \geq 0$ is the smallest number such that for any $N \geq 0$,

$$\langle |\hat{u}(\xi)|^2 \rangle \leq C_N |\xi|^{-N} \qquad \forall\, |\xi| \geq l_d,\ \forall\, \nu \in (0,1],$$

with suitable ν–independent constants C_N, where the brackets signify a suitable averaging. The K41 theory predicts that $l_d = \nu^{-3/4}$, and that for $|x_1 - x_2| \ll 1/l_d$ the increment of the velocity field $v(t,x_1) - v(t,x_2)$ is of order $|x_1 - x_2|$, while for $|x_1 - x_2|$ bigger then $1/l_d$ it behaves differently. The interval $I_{diss} = (l_d, \infty)$ is called the *dissipation range* (in the Fourier presentation).

◇ The complementary interval $I_{inert} = [1, l_d]$ is called the *inertial range*.

This is the most interesting zone, where the flow exhibits the non-trivial universal short-scale behaviour. That is, Fourier coefficients $\hat{u}(s)$ "behave interestingly" when $|s| \in I_{inert}$, and increments of velocity $u(t,x) - u(t,x+r)$ "behave interestingly" when $|r|^{-1} \in I_{inert}$.

Two the most important quantities of K41 are:

◇ in the x-presentation, the *structure function*

$$S_p^{||}(r,u) = \left\langle \int \left| \big(u(x+r) - u(x)\big) \cdot \tfrac{r}{|r|} \right|^p dx \right\rangle, \qquad p \geq 0;$$

This is the p-th moment of the norm of the speed's increment $\left|\big(u(x+r) - u(x)\big) \cdot \tfrac{r}{|r|}\right|$.

◇ in the Fourier presentation, the *energy spectrum*

$$E_k = |\Sigma_k|^{-1} \sum_{\xi \in \Sigma_k} \tfrac{1}{2} \left\langle |\hat{u}(\xi)|^2 \right\rangle, \qquad k > 0,$$

where $\Sigma_k \subset L^{-1}\mathbb{Z}^d$ is a suitable spherical layer around the sphere $|\xi| = k$ of a width, significantly bigger than L^{-1}.

The most interesting is the behaviour of $S_p^{||}(r,u)$ and of E_k in the inertial zone, i.e. when $|r|^{-1}, |k| \in I_{inert} = [1, \nu^{3/4}]$.

$\diamond$ The K41 theory predicts that if $\nu \ll 1$ (i.e. the Reynolds number of the flow is high), then for the increments r in the inertial zone, when $|r|^{-1} \in I_{inert}$, we have:

$$S_p^{||}(r,u) \sim |r|^{p/3}, \quad p \geq 0.$$

This is *Kolmogorov's 1/3–law.*

$\diamond$ For $k \in I_{inert}$ K41 states that

$$E_k \sim k^{-5/3}.$$

This is the *Kolmogorov–Obukhov law*.

The two K41 predictions are in reasonably good agreement with experimental and numerical data. But how to prove them? Firstly, which equations describe correctly the turbulence?

It is widely believed that the right set of equations to describe turbulence is the Navier–Stokes system, perturbed by a smooth and homogeneous in x random force:

$$\dot{u}(t,x) + (u\cdot\nabla)u - \nu\Delta u + \nabla p = \eta^\omega(t,x), \qquad \text{div}\, u = 0,$$

considered under the zero-meanvalue periodic boundary condition

$$x \in \mathbb{T}^3_L = \mathbb{R}^3/L\mathbb{Z}^3, \qquad \int_{\mathbb{T}^3_L} u(t,x)\,dx = \int_{\mathbb{T}^3_L} \eta(t,x)\,dx = 0,$$

(since we talk about space-periodic flows of period L). But this system is extremely complicated, and it is hard to tell anything about its solutions.

So starting 1940's physicists examine various models, where the Navier–Stokes system is replaced by easier equations. One of the most popular models is given by the Burgers equation. It was suggested by Jan Burgers as a model for turbulence in 1940's and has been intensively studied afterwords.

§2. THE EQUATION.

The stochastic Burgers equation is:

$$
\text{(B)}\qquad
\begin{aligned}
&u_t(t,x) + uu_x - \nu u_{xx} = \eta(t,x), \qquad u(0,x) = u_0(x),\\
&t \geq 0,\ x \in S^1, \quad \int u\,dx = 0,\ \ 0 < \nu \leq 1,
\end{aligned}
$$

where $\eta = \partial_t \xi(t,x)$ and ξ is a Wiener process in the space of functions of x of the form

$$
\xi^\omega(t,x) = \sum_{s=\pm 1, \pm 2, \dots} b_s \beta_s^\omega(t) e_s(x), \quad B_0 := \sum_s b_s^2 < \infty.
$$

Here $\{e_s, s = \pm 1, \pm 2, \dots\}$ is the trigonometric basis in the space of periodic function with zero mean, $\{\beta_s^\omega(t)\}$ are standard independent Brownian processes and $\{b_s\}$ are real numbers, fast converging to zero. So $\xi^\omega(t,x)$ is a smooth function of x.

I decompose solutions $u(t,x)$ to Fourier series:

$$
u(t,x) = \sum\nolimits_{s=\pm 1, \pm 2, \dots} \hat{u}_s(t) e_s(x).
$$

Eq. (B) has a unique strong solution $u^{\nu\omega}(t,x) =: u(t,x)$ which defines a Markov process in a suitable Hilbert function space H. I will usually write u as a random process in H, $u(t) = u^{\nu\omega}(t) \in H$.

I will soon explain that $u(t,x)$ is of order one. That is,

For any u_0, $\mathbf{E}\|u^\nu(t)\|^2_{L_2} \sim 1$ uniformly in $t \geq 0$ and $\nu \in (0,1]$.

Since the order of magnitude of a solution u^ν equals $\sqrt{\mathbf{E}\|u^\nu(t)\|^2_{L_2}} \sim 1$, then its Reynolds number is R$(u^\nu) \sim \nu^{-1}$. So (B) with small ν describes the 1d turbulence, called by Uriel Frisch *burgulence*.

THE GOALS are: 1) to study the equation (B) and the Markov process which it defines for small ν and for $0 \leq t \leq \infty$, and

2) to relate the obtained results with the K41 theory (regarding the Burgers equation (B) as a 1d model for the real turbulence).

Inspired by the heuristic work on the stochastic Burgers equation by U. Frisch with collaborators, Sinai in 1990's and later Sinai and others in the well known paper

E, Khanin, Mazel, Sinai *Invariant measures for Burgers equation with stochastic forcing*, Ann. Math. **151**, 877-960 (2000)

used the Lax-Oleinik formula to write down the limiting dynamics of (B) as $\nu \to 0$, and next studied the obtained limiting random field $u(t, x)$. In this way they arrived at a beautiful theory of stochastic Lagrangians which allows to obtain a representation for the limiting dynamics in (B) as $\nu \to 0$, for all $t \leq \infty$. They solved the problem 1) above, but their solution is not efficient in the sense that it does not allow to get the asymptotic relations, claimed by the theory of turbulence. So they did not resolve the problem 2).

On the contrary, we study (B) for small but POSITIVE ν, i.e. when

not $\nu \to 0$, but $0 < \nu \ll 1$,

using basic tools of PDEs and stochastic processes. This approach allows to get the asymptotic, similar to those, claimed by the K41 theory. Our results rigorously justify the heuristic theory, built in the influential paper

Aurell, Frisch, Lutsko, Vergassola, J. of Fluid Mechanics, **238**, 467–486, 1992.

§3. APRIORY ESTIMATES. The key starting point is the Oleinik-Kruzkov inequality, which we apply to solutions of (B) with fixed ω. The inequality was proved by O–K for the free (B) equation, but with minimal effort their argument applies to the stochastic equation and implies:

THEOREM O-K. For ANY initial data u_0, any $p \geq 1$ and any $\nu, \theta \in (0, 1]$, uniformly in $t \geq \theta$ we have:

$$\mathbf{E}\Big(|u^\nu(t,\cdot)|_\infty^p + |u_x^\nu(t,\cdot)|_1^p\Big) \leq C\theta^{-p}.$$

The constant C depends only on the random force.

This very powerful estimate, jointly with usual PDE tricks, allows to bound from above moments of all Sobolev norms of solutions:

THEOREM 1. For any u_0, every $m \in \mathbb{N}$, $0 < \nu \leq 1$ and every $\theta > 0$,

$$\mathbf{E}\|u^\nu(t)\|_m^2 \leq C(m,\theta)\nu^{-(2m-1)} \quad \text{if} \quad t \geq \theta.$$

Remark. For $m = 0$ this is wrong, and instead then we have $\mathbf{E}\|u^\nu(t)\|_0^2 \sim 1.$

§4. LOWER BOUNDS ON MOMENTS OF SOBOLEV NORMS OF SOLUTIONS.

Applying the Ito formula to $\frac{1}{2}\|u(t)\|_0^2$ we get the Balance of Energy Relation for solutions of (B). Namely, for any $\sigma > 0$,

$$\mathbb{E}\int \tfrac{1}{2}|u(T+\sigma,x)|^2dx - \mathbb{E}\int \tfrac{1}{2}|u(T,x)|^2dx + \nu\mathbb{E}\int_T^{T+\sigma}\int |u_x(s,x)|^2dxds = \sigma B_0,$$

where $B_0 = \sum b_s^2 > 0$.

Let $T \geq 1$. Then by Oleinik-Kruzkov, the first two terms are bounded by a constant C_*, which depends only on the random force. Choosing σ so big that $C_* \leq \frac{1}{4}\sigma B_0$, we get that

$$\boxed{\nu\mathbb{E}\,\frac{1}{\sigma}\int_T^{T+\sigma}\int |u_x^\nu(s,x)|^2dxds \geq \tfrac{1}{2}B_0.}$$

For any $m \in \mathbb{N}$ we will denote $\langle\langle \|u\|_m^2 \rangle\rangle = \mathbb{E}\, \frac{1}{\sigma} \int_T^{T+\sigma} \|u(s)\|_m^2 \, ds.$ – This is the squared m–th Sobolev norm, averaged in ensemble and locally averaged in time. We have seen that

$$\langle\langle \|u^\nu\|_1^2 \rangle\rangle \geq \nu^{-1} C_1 .$$

But by Theorem 1 it is $\leq \nu^{-1} C_2$! So

$$\langle\langle \|u^\nu\|_1^2 \rangle\rangle \sim \nu^{-1}$$

§5. LOWER BOUNDS FOR HIGHER NORMS. The Gagliardo-Nirenberg interpolation inequality + Oleinik-Kruzkov imply:

$$\langle\langle |u_x^\nu|_{L_2}^2\rangle\rangle \overset{G-N}{\leq} C'_m \langle\langle \|u^\nu\|_m^2\rangle\rangle^{\frac{1}{2m-1}} \langle\langle |u_x^\nu|_{L_1}^2\rangle\rangle^{\frac{2m-2}{2m-1}} \overset{O-K}{\leq} C_m \langle\langle \|u^\nu\|_m^2\rangle\rangle^{\frac{1}{2m-1}}$$

Using the already obtained lower bound for the averaged first Sobolev norm $\langle\langle \|u^\nu\|_1^2\rangle\rangle \geq \nu^{-1}C$ we get from here a lower bound for $\|u^\nu\|_m$:

$$\langle\langle \|u^\nu\|_m^2\rangle\rangle \geq C''_m \nu^{-(2m-1)} \qquad \forall m \geq 1.$$

Combining this with the upper bound in Theorem 1, we get:

THEOREM 2. For any u_0, any $0 < \nu \leq 1$ and every $m \in \mathbb{N}$,

$$C_m^{-1}\nu^{-(2m-1)} \leq \langle\langle \|u^\nu\|_m^2\rangle\rangle \leq C_m \nu^{-(2m-1)}$$

This theorem and the Oleinik–Kruzkov result turns out to be an efficient tool to study the turbulence in the 1d Burgers equation (B) (the burgulence).

§6. SPACE–INCREMENTS OF THE VELOCITY FIELD $u(t,x)$.

I recall that I write $u(t,x)$ as the Fourier series $\sum_{s=\pm1,\pm2,\dots} \hat u_s(t) e_s(x)$.

Theorem 2 and a Tauberian argument imply:

THEOREM 3. The dissipation space-scale l_d of u equals ν^{-1}. That is,

$$|\hat u_s(t)| \le C_m |s|^{-m} \qquad \forall\, |s| \ge \nu^{-\gamma} \ \text{ if } \ \gamma > 1,$$

for each m and all $\nu > 0$. If $\gamma < 1$, then the relation above is wrong for some $|s| \ge \nu^{-\gamma}$ and some m.

So the dissipation range is $I_{diss} = \{k \ge C\nu^{-1}\}$, and the inertial range is $I_{iner} = \{1 \le k \le C\nu^{-1}\}$

The short-scale space-increments of a velocity field $u(t,x)$ are measured by the Structure Function $S_p(l,u)$, which for solutions u of the 1d Burgers equation (B) is defined as

$$S_p(l,u) = \langle\langle \int |u(x+l) - u(x)|^p dx \rangle\rangle, \quad 0 < l \le 1.$$

Theorem 2 and Oleinik–Kruzkov imply the behaviour of $S_p(l,u)$ in the inertial range $c > |l| \geq C\nu$:

THEOREM 4. For suitable constants $C, c > 0$ and for l in the inertial range $c > |l| \geq C\nu$ we have

$$S_p(l,u^\nu) \sim l^p \quad \text{if} \quad 0 < p \leq 1,\ \forall\, \nu > 0;$$

$$S_p(l,u^\nu) \sim l \quad \text{if} \quad p \geq 1,\ \forall\, \nu > 0. \tag{$*$}$$

While in the dissipation range $\{|l|^{-1} \geq C_0\nu^{-1}\}$,

$$S_p(l,u^\nu) \sim l^p \qquad \forall\, p > 0;\ \forall\, \nu > 0.$$

For the water turbulence (governed by the 3d Navier-Stokes equation) the K41 theory predicts that in the inertial range, $S_p(l) \sim l^{p/3}$. So $(*)$ is an "abnormal scaling". For (B) this scaling was predicted by Frisch and others.

§7. DISTRIBUTION OF ENERGY ALONG THE SPECTRUM. Write $u^\nu(t,x)$ in Fourier: $\sum_{s=\pm1,\pm2,\dots}\hat u_s(t)e_s(x)$.

The energy spectrum of $u^\nu(t,x)$ at time t is

$$E_k(u^\nu(t)) = \text{averaging of } \tfrac12\langle\langle|\widehat{u}_n^\nu(t)|^2\rangle\rangle \text{ in all } n, \text{ close to } k \text{ or } -k.$$

Theorem 3 (about the space-scale) tells that if $k \gg \nu^{-1}$, then

$$E_k(u^\nu(t)) \le C_N k^{-N} \qquad \forall N,\ \forall\,\nu,\ t\ge 1.$$

But what happens in the inertial zone, when $k \lesssim \nu^{-1}$? – Something very different:

Theorem 4 and a Tauberian argument imply:

THEOREM 5. For k in the inertial range, $1 \le k \le C_1 \nu^{-1}$, we have:

$$C^{-1} k^{-2} \le E_k(u^\nu) \le C k^{-2}, \quad \forall\, \nu > 0, \qquad (*)$$

with suitable constants C, C_1, and for all moments of time $T \ge 1$.

I recall that

$$E_k = \text{averaging of } \tfrac{1}{2}\langle\langle|\widehat{u}_n^\nu(t)|^2\rangle\rangle \text{ in all } n, \text{ close to } k \text{ or } -k,$$

where $\langle\langle f\rangle\rangle = \mathbf{E}\frac{1}{\sigma}\int_T^{T+\sigma} f\,dt$. In fact, the averaging $\mathbf{E}$ “almost is not needed”:

THEOREM $5'$. If in the definition of E_k we replace $\langle\langle\cdot\rangle\rangle$ by the local averaging in time $\frac{1}{\sigma}\int_T^{T+\sigma}\cdot\,dt$, then E_k becomes a r.v., and $(*)$ holds with high probability if $C \gg 1$ and $\sigma \gg 1$.

For the water turbulence the Kolmogorov–Obukhov law from the K41 theory predicts that $E_k \sim |k|^{-5/3}$ for k in the inertial range. But for solutions of (B), Jan Burgers in 1940 (!) predicted that $E_k \sim |k|^{-2}$, i.e. exactly $(*)$.

§8. THE MIXING. The mixing means that in the function space H of functions of x, where we study the equation, there exists a unique measure μ_ν, such that for any "reasonable" functional f on H and for any solution $u(t,x)$ of (B) we have

$$(1) \qquad \mathbb{E}f(u(t,\cdot)) \to \int_H f(u)\,\mu_\nu(du) \quad \text{as} \quad t \to \infty.$$

μ_ν is a *statistical equilibrium.*

This holds for (B), and may be derived from general theory. But then the rate of convergence would depend on ν. In the same time, in the theory of turbulence it should not depend on t, and for (B) it does not!

THEOREM 6. If the functional $f(u)$ is continuous in the L_1–norm, then the rate of convergence above does not depend on ν, and holds at least with the rate $t^{-1/6}$.

The theorem applies to the functional $u \mapsto \hat{u}_k$ and allows to specify the spectral law:

I recall that $E_k(u^\nu(t))$ is $\frac{1}{2}|\widehat{u}_n^\nu(t)|^2$, averaged in time, in ω and in n, close to $\pm k$:

$$E_k = \text{averaging of } \tfrac{1}{2}\langle\langle|\widehat{u}_n^\nu(t)|^2\rangle\rangle \text{ in all } n, \text{ close to } k \text{ or } -k,$$

and Theorem 4 states that

$$C^{-1}k^{-2} \le E_k(u^\nu) \le Ck^{-2} \quad \text{for} \quad k \le C_1\nu^{-1}. \qquad (*)$$

Consider the instant spectral density of energy for u^ν, without the time-average:

$$\mathcal{E}_k(t, u^\nu) = \frac{1}{2|I_k|}\sum_{n\in I_k\cup -I_k} \tfrac{1}{2}\mathbf{E}|\widehat{u}_n^\nu(t)|^2, \quad I_k = [M^{-1}k, Mk],$$

and the spectral density of the stationary measure μ_ν:

$$F_k(\mu_\nu) = \frac{1}{2|I_k|}\sum_{n\in I_k\cup -I_k} \tfrac{1}{2}\int_H |\widehat{u}_n|^2\, d\mu_\nu(u), \quad I_k = [M^{-1}k, Mk].$$

This F_k is a specific function of k which satisfies the estimates $(*)$.

For large t the spectral law holds without the local averaging in time:

THEOREM 6. The instant spectral density $\mathcal{E}_k(t, u^\nu)$ converges to $F_k(\mu_\nu)$, as $t \to \infty$, at least with the rate $t^{-1/6}$.

§9. THE INVISCID LIMIT. Another remarkable feature of (B) is that, as $\nu \to 0$, a solution u^ν of (B) converges to an inviscid limit:

$$u^\nu(t,\cdot) \to u^0(t,\cdot) \quad \text{in } L_p(S^1), \text{ a.s.},$$

for each $p < \infty$. This result is due to Lax–Oleinik (1957). The limit $u^0(t,x)$ is called an "inviscid solution ", or an "entropy solution " of (B) with $\nu = 0$. This function of (t,x) even is not continuous. But still the structure function and spectral energy of u^0 inherit the estimates for u^ν in the following form:

THEOREM 7. For all $\nu > 0$,

1) $E_k(u^0) \sim Ck^{-2}$ for all k;

2) $S_p(l,u^0) \sim l^p$ if $0 < p \le 1$, $l \le c$, and
$S_p(l,u^0) \sim l$ if $p \ge 1$, $l \le c$.

Now the energy law for E_k holds for ALL $k \ge 1$. So for u^0 the viscous space-scale l_v equals ∞, and the inertial range is $[1,\infty)$.