Nonholonomic systems in a magnetic field

A.V. Borisov, A.V. Tsiganov

- When a magnetic field is applied to a charged dielectric rigid body it starts to spin
- When a magnetic field is applied to a ferromagnetic rigid body it starts to spin
- When a magnetic field is applied to a superconducting rigid body it starts to spin

V.V. Kozlov studied this topic in
Problem of the Rotation of a Solid Body in a Magnetic Field, 1985
simultaneously studied
The Theory of Integration of Equations of Nonholonomic Mechanics, 1985.
We tried to continue his research in
On the Chaplygin Sphere in a Magnetic Field, 2019;
On nonholonomic Routh sphere in a Magnetic Field, 2020.

Table: Experiments on rotation and magnetization

	Rotation by magnetization	Magnetization by rotation
ferromagnets	Einstein-de Haas effect (1915)	Barnett effect (1915)
super conductors	gyromagnetic effect Kikoin and Gubar (1940)	London moment (1933)

These experimental results follow from conservation laws.

The question of how angular momentum causes magnetization or magnetization causes angular momentum is a separate question that needs to be addressed and understood.

Recently appear some new theories of rotation/magnetization:

- R. Jaafar, E.M. Chudnovsky, D.A. Garanin, Dynamics of the Einstein-de Haas effect: Application to a magnetic cantilever, 2009;
- D.A. Garanin, E. M. Chudnovsky, Angular momentum in spin-phonon processes, 2015.
- Hirsch, J. E., Moment of inertia of superconductors, 2019;
- Hirsch, J. E., Spinning superconductors and ferromagnets, 2019;
- J. H. Mentink, M. I. Katsnelson, M. Lemeshko, Quantum many-body dynamics of the Einstein-de Haas effect, 2019;
- Recent Advances in Topological Ferroics and their Dynamics, 2019, etc.

We have to also take account of theories including static crystal fields, magnetic ordering, etc.

As a result, we have a few different phenomenological theories of rigid body dynamics in a magnetic field.

Let us take standard equations describing rotation of the rigid body around a fixed point in a potential field

$$\dot{\gamma} = \gamma \times \omega\,, \qquad \dot{M} = M \times \omega - \frac{\partial V}{\partial \gamma} \times \gamma\,,$$

Here

- M is the angular momentum vector;
- γ is the Poisson vector;
- $\omega = AM$ is the angular velocity vector;
- $A = I^{-1} = \mathrm{diag}(a_1, a_2, a_3)$ is an inverse matrix to the tensor of inertia I.

Vector field associated with equations of motion is Hamiltonian.

Imposing magnetic field we can transfer from original Hamiltonian equations to

- new Hamiltonian equations;
- new non-Hamiltonian equations.

At first case we continue to study standard Kirchhoff equations with new Hamiltonian involving generalized potential which is a linear function on velocities

$$\tilde{H} = \frac{1}{2}(\omega, I\omega) + U\,, \qquad U = f_1(\gamma)\omega_1 + f_2(\gamma)\omega_2 + f_3(\gamma)\omega_3 + f_4(\gamma).$$

At second case we have to study non Hamiltonian equations

$$\dot{M} = M \times \omega - \frac{\partial V}{\partial \gamma} \times \gamma + M_{ext}\,,$$

where M_{ext} is a gyromagnetic torque associated with Barnett-London, Einstein-de Haas and so on effects.

Examples of external magnetic moments

- for dielectric rigid body we have so-called Grioli problem (Hamiltonian case)

  $$M_{ext} = B\gamma \times \omega\,,$$

  was studied Grioli, Goldstein, Bogoyavlensky, etc,
- for Barnett-London and Einstein-de Haas effects (non Hamiltonian case)

  $$M_{ext} = C\omega \times \gamma$$

  was studied by Samsonov, Kozlov, etc;
- for Barnett-London and Einstein-de Haas effects (Hamiltonian case)

  $$M_{ext} = D\gamma \times \omega + D\omega \times \gamma + D(\gamma \times \omega)\,,$$

  was studied by Burov and Subkhankulov, etc.

There are also other expressions for M_{ext} associated with other phenomenological theories.

Below we put

$$M_{ext} = \alpha + B\gamma \times \omega + C\omega \times \gamma + D(\gamma \times \omega)\,,$$

where α is a constant gyroscopic moment and B, C, D are symmetric matrices (non-diagonal in generic case). This will allow us to cover most existing phenomenological theories.

Equations of motion

$$\dot\gamma = \gamma \times \omega\,, \qquad \dot M = M \times \omega - \frac{\partial V}{\partial \gamma} \times \gamma + M_{ext}\,,$$

preserve first integrals

$$(\gamma, \gamma) = 1\,, \qquad \text{and} \qquad H_1 = (\gamma, M)\,.$$

which are independent of the form of the external moment M_{ext}.

Samsonov case

Original Hamilton equations have standard invariant measure $\mu = d\gamma dM$ which remains unchanged after imposing the magnetic field only if

$$C - D = \Lambda \equiv \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \qquad \lambda_k \in \mathbb{R}.$$

If Λ is proportional to the unit matrix we have Samsonov case

$$\Lambda = \lambda \mathrm{Id},$$

when equations of motion preserve original mechanical energy

$$H = \frac{1}{2}(M,\omega) + V(\gamma_1,\gamma_2,\gamma_3),$$

Reduction to the Kirchhoff equations proposed V.V. Kozlov.

New case at $I_1 \neq I_2 \neq I_3$

If

$$\alpha = 0, \quad D = \mathrm{M} - \Lambda - B\,, \qquad \mathrm{M} = \begin{pmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu_3 \end{pmatrix}.$$

elements of the Λ satisfy

$$(I_2 - I_3)(I_1 - I_2 - I_3)\lambda_1 + (I_3 - I_1)(I_2 - I_1 - I_3)\lambda_2 + (I_1 - I_2)(I_3 - I_1 - I_2)\lambda_3 = 0$$

elements of matrix M satisfy the Clebsch type relation

$$(I_2 - I_3)\mu_1 + (I_3 - I_1)\mu_2 + (I_1 - I_2)\mu_3 = 0\,,$$

then equations of motion are Hamiltonian w.r.t. Hamiltonian

$$H = \frac{(M_1 + \sigma I_1^2\gamma_1)^2}{2I_1} + \frac{(M_2 + \sigma I_2^2\gamma_2)^2}{2I_2} + \frac{(M_3 + \sigma I_3^2\gamma_3)^2}{2I_3} + U + V.$$

New case at $I_1 \neq I_2 \neq I_3$

Here

$$\sigma = (\mu_3 - \mu_2 + \lambda_2 - \lambda_3) I_1 + (\mu_1 - \mu_3 + \lambda_3 - \lambda_1) I_2 + (\mu_2 - \mu_1 + \lambda_1 - \lambda_2) I_3 \,,$$

and we suppose that potential $V(\gamma_1, \gamma_2, \gamma_3)$ satisfies differential equation

$$\gamma_1\gamma_2(I_2 - I_3)\frac{dV}{d\gamma_1} + \gamma_1\gamma_2(I_3 + I_1)\frac{dV}{d\gamma_2} + \gamma_1\gamma_2(I_1 - I_2)\frac{dV}{d\gamma_3} = 0 \,.$$

This new Hamiltonian system integrable at $V = 0$ if

$$\frac{\lambda_i - \lambda_j}{\mu_i - \mu_j} = 1 - \frac{2I_k}{I_1 + I_2 + I_3} \,, \quad (i, j, k) \sim (1, 2, 3).$$

Second integral of motion is equal to $M^2 = (M, M)$.

Kozlov case at $I_1 = I_2 = I_3 = a$

If $\alpha = D = 0$ there are also two integrable cases with Hamiltonian

$$H = \lambda_1 M_1^2 + \lambda_2 M_2^2 + \lambda_3 M_3^2 = (M, \Lambda M), \qquad \Lambda = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

at

$$B = \begin{pmatrix} \frac{\lambda_2 c_3 - \lambda_3 c_2}{\lambda_2 - \lambda_3} & 0 & 0 \\ 0 & \frac{\lambda_1 c_3 - \lambda_3 c_1}{\lambda_1 - \lambda_3} & 0 \\ 0 & 0 & \frac{\lambda_1 c_2 - \lambda_2 c_1}{\lambda_1 - \lambda_2} \end{pmatrix}.$$

V.V. Kozlov found integrable case for Clebsch type conditions

$$(\lambda_2 - \lambda_3)c_1 + (\lambda_3 - \lambda_1)c_2 + (\lambda_1 - \lambda_2)c_3 = 0\,,$$

and later E. Veselova reduced this system to the standard Clebsch system without magnetic field.

New case at $I_1 = I_2 = I_3 = a$

We found second case at

$$(\lambda_2 - \lambda_3)\lambda_1 c_1 + (\lambda_3 - \lambda_1)\lambda_2 c_2 + (\lambda_1 - \lambda_2)\lambda_3 c_3 = 0.$$

when integrals of motion are equal to

$$H_1 = \lambda_1 M_1^2 + \lambda_2 M_2^2 + \lambda_3 M_3^2 = (M, \Lambda M)$$

$$H_2 = (M, M) - 2(M, C\gamma) + \frac{\mathrm{tr}\,(C - B)}{\mathrm{tr}\Lambda^{-1}}\,(C\gamma, \Lambda^{-1}\gamma).$$

We suppose that there is a counterpart of the Veselova transformation which reduces this system of equations

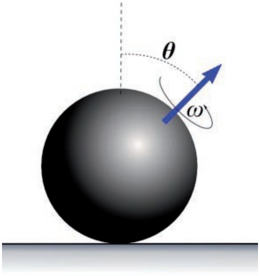
$$a\dot{M} = B\gamma \times M + C\,M \times \gamma, \qquad a\dot{\gamma} = \gamma \times M$$

to the well-known integrable Steklov – Lyapunov case of rigid body motion.

Nonholonomic Chaplygin sphere

The nonholonomic Chaplygin sphere is that of a dynamically balanced 3-dimensional ball that rolls on a horizontal table without slipping or sliding.

‘Dynamically balanced’ means that the geometric center coincides with the center of mass but the mass distribution is not assumed to be homogeneous.



Nonholonomic Chaplygin sphere

Equations of motion

$$\dot{\gamma} = \gamma \times \omega, \qquad \dot{M} = M \times \omega - \frac{\partial V}{\partial \gamma} \times \gamma$$

looks like equations of motion describing the rotation of the rigid body around a fixed point in a potential field, but angular velocity vector now depends on $\boldsymbol{\gamma}$

$$\omega = A_\gamma \, M, \qquad A_\gamma = A + \frac{dA\,\gamma \otimes \gamma\,A}{g^2}, \qquad a_k = (I_k + d)^{-1}$$

with

$$g = \sqrt{1 - d(\gamma, A\gamma)}.$$

This vector field is conformally Hamiltonian

$$X = g^{-1} P_g \mathrm{d}H, \quad H = \frac{1}{2}(M, \omega) + V(\gamma).$$

Nonholonomic Chaplygin sphere

The corresponding Poisson bivector

$$P_g = g\,P - \frac{d}{g}\,(M, A\gamma)\begin{pmatrix} 0 & 0 \\ 0 & \Gamma \end{pmatrix},$$

is a trivial deformations of the Lie-Poisson bivector on $e^*(3)$

$$P = \begin{pmatrix} 0 & 0 & 0 & 0 & \gamma_3 & -\gamma_2 \\ 0 & 0 & 0 & -\gamma_3 & 0 & \gamma_1 \\ 0 & 0 & 0 & \gamma_3 & -\gamma_1 & 0 \\ 0 & \gamma_3 & -\gamma_2 & 0 & M_3 & -M_2 \\ -\gamma_3 & 0 & \gamma_1 & -M_3 & 0 & M_1 \\ \gamma_2 & -\gamma_1 & 0 & M_2 & -M_1 & 0 \end{pmatrix},$$

which generates Kirchhoff equations in rigid body dynamics in potential and magnetic fields.

Solenoidal field

Let us consider Chaplygin sphere in a solenoidal field

$$\dot{\gamma} = \gamma \times \omega, \qquad \dot{M} = \Big(M + b(\gamma)\Big) \times \omega - \frac{\partial V}{\partial \gamma} \times \gamma.$$

In partial case $b(\gamma) = B\gamma$ it is nonholonomic Grioly problem, i.e. ball consists of dielectric material only.

- work done by external forces is equal to zero;
- total mechanical energy H is altered

$$H = \frac{1}{2}(M, \omega) + V(\gamma);$$

- form of invariant measure is altered

$$\mu = g^{-1}\,\mathrm{d}\gamma\,\mathrm{d}M;$$

- vector field remains conformally Hamiltonian

$$X = g^{-1} P_{bg}\,\mathrm{d}H.$$

Magnetic Poisson bivector

Here instead of

$$P_g = g\,P - \frac{d}{g}\,(M, A\gamma)\begin{pmatrix} 0 & 0 \\ 0 & \Gamma \end{pmatrix}$$

we have to use other deformation

$$P_{gb} = g\,P_b - \frac{d}{g}\,(M, A\gamma)\begin{pmatrix} 0 & 0 \\ 0 & \Gamma \end{pmatrix}.$$

Here P_b is a well-known magnetic Poisson bivector obtained by Poisson map

$$\varphi: \quad M_i \to M_i + c_i(\gamma), \qquad i = 1, 2, 3,$$

from the Lie-Poisson bivector P on $e^*(3)$.

Magnetic Poisson bivector

Functions $b(\gamma)$ and $c(\gamma)$ are related to each other

$$b_1 = \left(\frac{\partial c_2}{\partial \gamma_2} + \frac{\partial c_3}{\partial \gamma_3}\right)\gamma_1 - \frac{\partial\left(\gamma_2 c_2 + \gamma_3 c_3\right)}{\gamma_1} - c_1\,,$$
$$b_2 = \left(\frac{\partial c_1}{\partial \gamma_1} + \frac{\partial c_3}{\partial \gamma_3}\right)\gamma_2 - \frac{\partial\left(\gamma_1 c_1 + \gamma_3 c_3\right)}{\partial\gamma_2} - c_2\,,$$
$$b_3 = \left(\frac{\partial c_1}{\partial \gamma_1} + \frac{\partial c_2}{\partial \gamma_2}\right)\gamma_3 - \frac{\partial\left(\gamma_1 c_1 + \gamma_2 c_2\right)}{\partial\gamma_3} - c_3\,,$$

so that

$$(\mathrm{rot}\, b, \gamma) = (\nabla \times b, \gamma) = 0 \quad \text{or} \quad \mathrm{div}\, \gamma \times b = 0\,.$$

The corresponding deformations of the Casimir functions read as

$$C_1 = (\gamma, \gamma), \qquad C_2 = (\gamma, M - c), \quad P_b \mathrm{d} C_1 = 0\,, \quad P_b \mathrm{d} C_2 = 0.$$

Barnett-London and Einstein-de Haas effects

Equations of motion generates non Hamiltonian vector field

$$\dot{M} = (M + B\gamma + \alpha) \times \omega + \left(C\omega - \frac{\partial V}{\partial \gamma} \right) \times \gamma, \qquad \dot{\gamma} = \gamma \times \omega.$$

At $\omega = A_g M$ the vector field X has an invariant measure

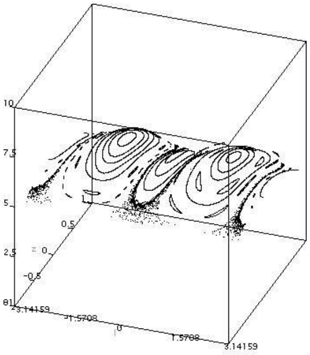
$$\mu = g^{-1} \mathrm{d}\gamma \mathrm{d}M, \qquad g = \sqrt{1 - d(\gamma, A\gamma)},$$

only if the matrix C is a diagonal matrix with entries satisfying the Clebsch type condition

$$\frac{c_2 - c_3}{a_1} + \frac{c_3 - c_1}{a_2} + \frac{c_1 - c_2}{a_3} = 0.$$

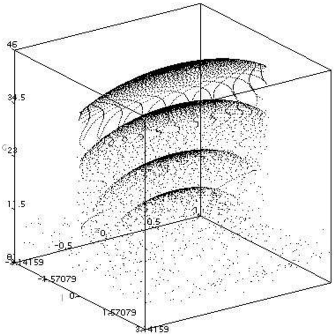
In holonomic case at $\omega = AM$ invariant measure exists if C is diagonal matrix.

System with invariant measure, holonomic case



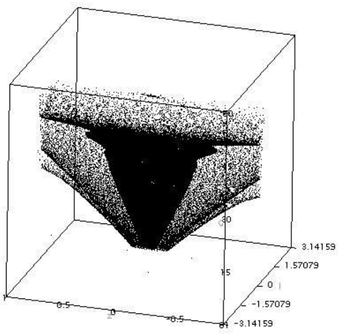
Regular trajectories and small sections of only conservative chaos, since the presence of a measure prevents the occurrence of significant dissipation.

System without invariant measure, holonomic case



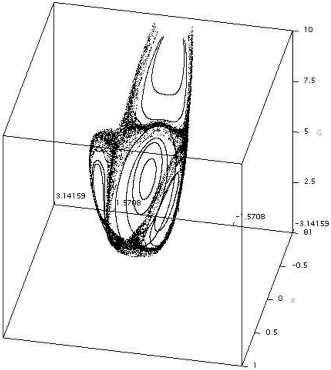
An increase in dissipation is observed, but it is still not enough to fully form a strange attractor. The dissipation resulting from the appearance of a magnetic field is rather weak.

System without invariant measure, holonomic case



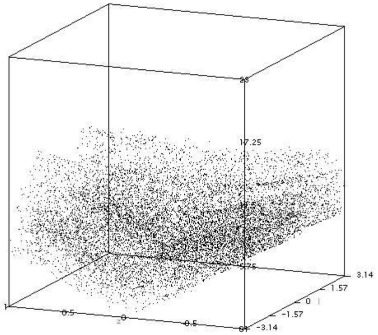
Thick attractor stratified into many coexisting cycles, may be exist a quasi-integral. The sum of Lyapunov indicators is small ≥ -0.08.

System with invariant measure, nonholonomic case



Dissipation is somewhat stronger than in the holonomic case. Under the Clebsch condition, regular trajectories and sections of conservative chaos are also observed.

System without invariant measure, nonholonomic case



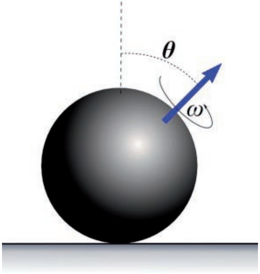
An increase in dissipation is observed, but it is still not enough to fully form a strange attractor, since the dissipation resulting from the appearance of a magnetic field is rather weak.

There is no additional integral or even quasi-integral.

Nonholonomic Routh sphere

For nonholonomic Routh sphere geometric center differs on center of mass and two momenta of inertia coincide to each other, for instance $I_1 = I_2$.

The line joining the center of mass and the geometric center is an axis of inertial symmetry.



Nonholonomic Routh sphere

Equations of motion

$$\dot{M} = M \times \omega + m\dot{r} \times (\omega \times r) - \frac{\partial V}{\partial \gamma} \times \gamma + M_{ext}\,, \quad \dot{\gamma} = \gamma \times \omega\,.$$

with generic magnetic moment

$$M_{ext} = \alpha + B\gamma \times \omega + C\omega \times \gamma + D(\gamma \times \omega)\,,$$

preserve geometric first integral $(\gamma, \gamma) = 1$ and have invariant measure

$$\mu = \mathrm{g}^{-1/2}\, \mathrm{d}\gamma\, \mathrm{d}M\,, \qquad \mathrm{g}(\gamma) = I_1 I_3 + I_1 mR^2(\gamma_1^2 + \gamma_2^2) + I_3 m(R\gamma_3 + a)^2$$

if

$$D - C = \Lambda = \mathrm{diag}(\lambda_1, \lambda_1, \lambda_3)\,.$$

Nonholonomic Routh sphere

If B, C, D diagonal then existence of measure guarantees an existence of counterpart of Jellet and Rauth-Chaplygin integrals of motion

$$\begin{aligned} H_J &= H_J + \upsilon(\gamma_3) \\ &= (M, r) - \tfrac{R(d_1 - d_3 + b_1 - b_3)}{2}\gamma_3^2 + \big(R\alpha_3 - a(b_1 - c_1 - d_3)\big)\gamma_3\,, \end{aligned}$$

$$\begin{aligned} H_R &= H_R + \nu(\gamma_3) \\ &= \sqrt{\mathrm{g}}\,\omega_3 + \int \left(m(R\gamma_3 + a)\frac{d\upsilon(\gamma_3)}{d\gamma_3} - I_1(b_1 - c_1 - d_3)\right)\frac{d\gamma_3}{\sqrt{\mathrm{g}}}\ ; \end{aligned}$$

Nonholonomic Routh sphere

There is also independent integral of motion which is polynomial of second order in momenta

$$H=\frac{1}{2}(M,\omega)+f_1(\gamma_3)M_3+f_2(\gamma_3)H_J+f_3(\gamma_3)+V(\gamma_3)\,,$$

where

$$\begin{aligned}
f_1(\gamma_3) &= \frac{I_1(c_1-c_3-d_1+d_3)}{\sqrt{\mathrm{g}}}\int\frac{d\gamma_3}{\sqrt{\mathrm{g}}}\,,\\
f_2(\gamma_3) &= m\int\Big(R\big(Rm(R+a\gamma_3)+I_3\big)f_1(\gamma_3)+(c_1-c_3-d_1+d_3)(R\gamma_3+a)\Big)\,\frac{d\gamma_3}{\mathrm{g}}\,,\\
f_3(\gamma_3) &= -\int\Big(\upsilon(\gamma_3)\frac{df_2(\gamma_3)}{d\gamma_3}+(b_1-c_1-d_3)f_1(\gamma_3)\Big)\,d\gamma_3.
\end{aligned}$$

Nonholonomic Routh sphere

On a common level surface of the linear integrals

$$H_J = \ell\,, \qquad H_R = k\,,$$

Hamiltonian depends only on θ and $\dot{\theta}$

$$H = \frac{m(R^2 + 2Ra\cos\theta + a^2) + I_1}{2}\left(\frac{d\theta}{dt}\right)^2 + V(\theta, k, \ell)$$

with well-defined derivative of effective potential

$$\frac{dV}{d\theta} = \frac{df_2}{d\theta}\,\ell + \frac{df_3}{d\theta} + \dots$$

It allows us estimate the stability of partial solutions by using standard tools.

Nonholonomic Routh sphere

Thus, in framework of this theory we can confirm the experimental results

- for strong magnetic field the Larmor precession stabilizes the system (matrix B)
- for low magnetic fields the magnetic anisotropy stabilizes the system via the Einstein-de Haas effect (matrices C and D)