ON FOUNDATION OF ERGODIC METHOD for calculation of invariant measures. *Conference devoted to 70-th anniversary of V.V.Kozlov, MIAN, 20-24/01/2020*

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20 января 2020 г.

INTRODUCTION

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$$(X, G) : \forall g \in G : g_*\mu = \mu.$$

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2. Quasi-invariant measures — Cocycle – Gibbs measures; measures with maximal entropy (stationary case);
Main problem — to describe all invariant (ergodic) probability measures on the given Borel (or topological) spaces for given ER.
It remains the problem of the description of integrals of classical dynamical systems.

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Horizontal and Vertical (only difference)

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Models-1

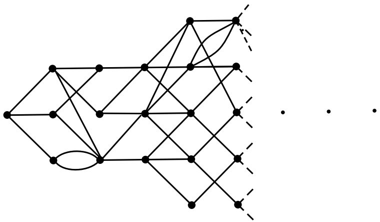

Рис.: Markov compact

Models-2

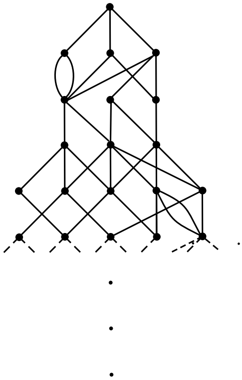

Рис.: Bratteli diagram

Continuation

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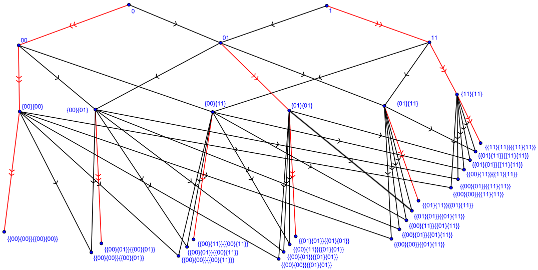
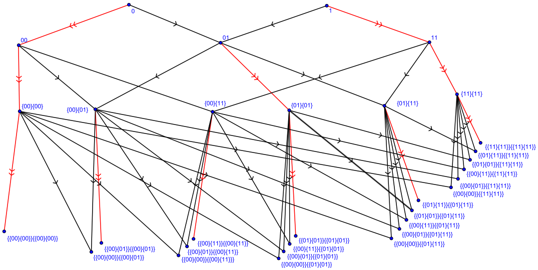
b) Natural technique: transition and CO-transition probabilities.
C) Existence of Universal Markov chain (uniadic universal graph).

Universal "uniadic graph":



Universal "uniadic graph":



Theorem

(A.V.-P.Zatitski,2018) Each ergodic automorphism can be realized as adic map of this graph with some central measure.

Continuation

The Problem of description of invariant measures is equivalent to the more concrete one:
TO FIND ALL MARKOV PROCESSES (e.g. its TRANSITION PROBABILITIES) WITH GIVEN COTRANSITION PROBABILITIES.
This problem unites many questions: the method of defining of Gibbs measures, Martin boundary etc.
(Kolmogorov, Dynkin, Dobrshin.) The notion of centrality of measure is a generalisation of the notion of independence of the sequence of random variables, which reflects the structure of graphs, space etc.

5.Example A -Young Graph

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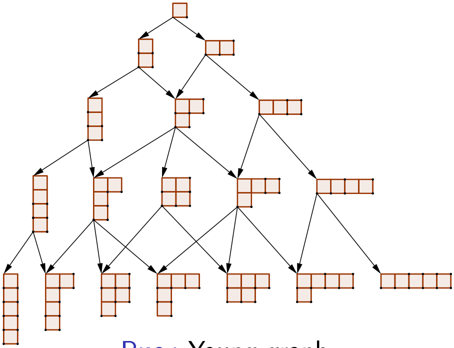

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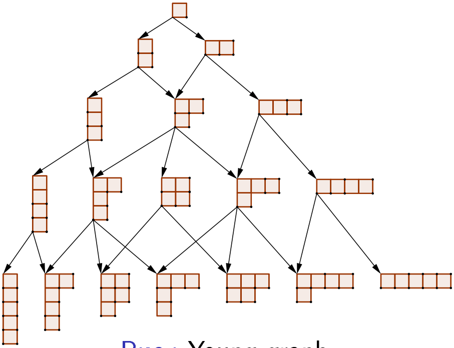
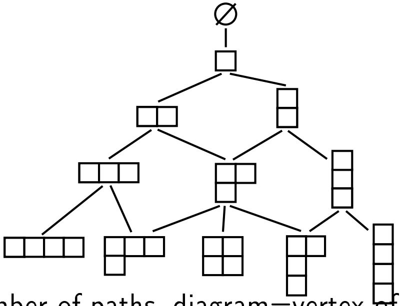


Рис.: Young graph

A.Young graph (Thoma, AV-Kerov, etc): List of ergodic invariant measures: 1)Plancherel measure on the space of infinite Young tableuax,
2) quasi-independent measures on the space of minimal infinite ideals of the $\mathbb{Z}_+^2$
3) mixture of 1)and 2).

Continuation



Dimension=number of paths, diagram=vertex of graph, table=path=numeration.

pause

Plancherel measure as central Markov measure

$$Prob\{\Lambda|\lambda\} \sim dim\Lambda : \Lambda \succ \lambda,$$

(transition probability)

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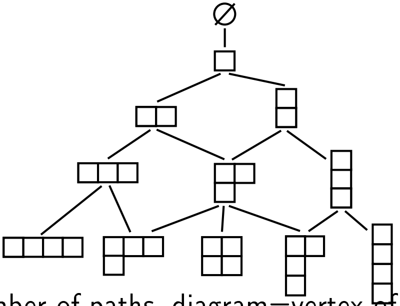

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(cotransition probability)

Planchrel measure and random numerations

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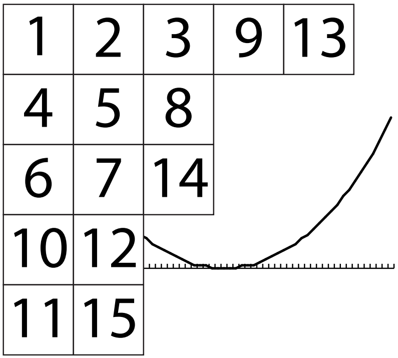
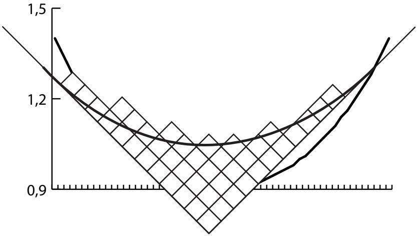
Monotonic Numeration of the lattice $\mathbb{Z}_+^2$

Theorem

There is only one nondegentrated central measure on $\mathbb{Z}_+^2$ *— Plancherel measure.*



Limit-shape



Each central measure possesses with limit shape — nonstandard Law of Large Number. This is limit shape Ω (Vershik-Kerov, Logan-Shepp — 1977)

Example B — Hermitian matrices

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Introduce GOE and GUE (Infinite Random Hermitian Matrices) in terms of Markov compact (V-F.Petrov), (Olshanski-V, Pickrel, etc); list of ergodoc invariant (up to conjugation) measures— (Wigner-Dyson) measure, and, so called, — Wishart measures. New Graph— Kurant, Gelfand-Zetlin,etc.

Absolute

6. The Absolute by definition —the space of all ergodic central measures: Chouqet boundary of the simplex of all invariant measures.
Projective limit of finite dimensional simplices

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It is a correct procedure for description of invariant measures but it is too abstract.

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7.Classical Ergodic Method (Gibbs, Boltzmann, Poincare, von Neumann, Birkhof -Ergodic theorem.
It is a correct procedure for description of invariant measures but it is too abstract. We define notion of "Satisfactory system of functions"which helps to make more concrete proceedure.
It will help to calculate lit of invariant measure in come cases.
END of INTRODUCTION

Probabilistic aspects: new kind of Laws of Large Numbers

The space is the space of pathes of the $\mathbb{N}$-graded graph or sequence of the paths of Markov chain.
Needed measures are central measure on this space.

Ergodic method

Let M- Markov compact (Bratteli diagram) and μ ergodic central measure. Then for μ-almost all sequnce $\{t_i\} \in M$ define the n-fragment $\{t_i\}_1^n$ denote it as t^n. The following is true weak limit of some atomic measures (in the space of measure on M) coincides with μ:

$$w - \lim \mu_n^{t^n} = \mu,$$

where the atomic measure $\mu_n^{t^n}$ is uniform measure on the tableaux of the same n-tail as t^n.

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Weak convergence and Problem.

Satifactory system

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Let $\{F_m\}$ countable system of Borel functions on M called satisfactory system for central measures in M (or in some Borel space with ER) if there is a unique central measure with given values of integrals of functions from system $\{F_m\}$. Or in other words: system $\{F_m\}$ can distinguish the set of ergodic central measures.

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The evident system of functions is cylindric ("linear"functions) but in many interesting case it is not satisfactory system in the sense above (see below).

Theorem and Conjectures

We can consider the polynomials of cylindric functions for which moment problem has positive solution.
The characteristic of the space of central measures is minimal degree of the polynomials which gives satisfactory system.

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1.The characteristic of the space of central measures could be arbitrary large and even infinite.
2.For the Young graph and for graph of Hermitian matrices the characteristic is eqaul 2.

3. The charactersitic of the graph of all Young diagrams of dimension d equal d.