Dynamics of Akhiezer polynomials, elliptical billiards, and Painlevé VI equations

Vladimir Dragović

The University of Texas at Dallas
Mathematical Institute SASA, Belgrade

Classical mechanics, dynamical systems and mathematical physics on the occasion of Academician Valery V. Kozlov's 70th birthday

References

- V. Dragović, V. Shramchenko, *Algebro-geometric approach to an Okamoto transformation, the Painlevé VI and Schlesinger equations* Annales Henri Poincaré, (2019)
- V. Dragović, V. Shramchenko, *Algebro-geometric solutions of the Schlesinger systems and the Poncelet-type polygons in higher dimensions*, Internat. Math. Res. Notes, (2018).
- V. Dragović, V. Shramchenko, *Note on algebro-geometric solutions to triangular Schlesinger systems*, J. Nonlin. Math. Phys. (2017).
- V. Dragović, R. Gontsov, V. Shramchenko, *Triangular Schlesinger systems and superelliptic curves*, ArXiv: 1812.09795.
- V. Dragović, V. Shramchenko, *Dynamics of the Akhiezer polynomials and Painlevé VI equation*, in preparation

Six Painlevé equations

- Paul Painlevé (1863-1933) classified all second order ODEs of the form $\frac{d^2y}{dx^2} = F(\frac{dy}{dx}, y, x)$ with F rational in the first two arguments, meromorphic in x whose solutions have no movable critical points.
- Six new equations which cannot be solved in terms of known special functions.
- The sixth Painlevé equation, PVI, is the most general of them: PVI$(\alpha, \beta, \gamma, \delta)$.

$$\frac{d^2y}{dx^2}=\frac{1}{2}\left(\frac{1}{y}+\frac{1}{y-1}+\frac{1}{y-x}\right)\left(\frac{dy}{dx}\right)^2-\left(\frac{1}{x}+\frac{1}{x-1}+\frac{1}{y-x}\right)\frac{dy}{dx}$$

$$+\frac{y(y-1)(y-x)}{x^2(x-1)^2}\left(\alpha+\beta\frac{x}{y^2}+\gamma\frac{x-1}{(y-1)^2}+\delta\frac{x(x-1)}{(y-x)^2}\right).$$

Manin 1998

$$\frac{d^2 z}{d\tau^2} = \frac{1}{2\pi i} \sum_{j=0}^{3} \alpha_j \wp_z \left(z + \frac{T_j}{2}, \tau \right)$$

T_j - periods $(0, 1, \tau, 1+\tau)$. Relationship between parameters:

$$(\alpha_0, \alpha_1, \alpha_2, \alpha_3) = (\alpha, -\beta, \gamma, 1/2 - \delta).$$

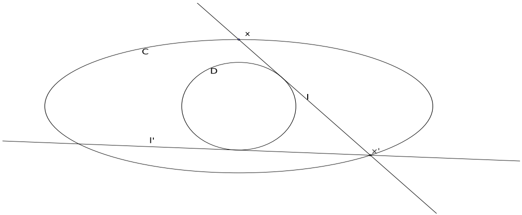
Painlevé 1906. Example: $(\alpha_0, \alpha_1, \alpha_2, \alpha_3) = (0, 0, 0, 0)$,
$(\alpha, \beta, \gamma, \delta) = (0, 0, 0, 1/2)$ – Picard solution (1889):

$$y_0(x) = \wp(z_0(x)),$$

where transformed $\wp$ satisfies:
$(\wp'(z))^2 = \wp(z)\,(\wp(z) - 1)\,(\wp(z) - x)$.

$$z_0(x) := c_1 + \tau(x) c_2.$$

Poncelet problem



- C and D are two smooth conics in $\mathbb{CP}^2$
- Question: Is there a closed trajectory inscribed in C and circumscribed about D?
- Poncelet Theorem: Let $x \in C$ be a starting point. The Poncelet trajectory originating at x closes up after n steps iff so does a Poncelet trajectory originating at any other point of C.

Solution of Poncelet problem

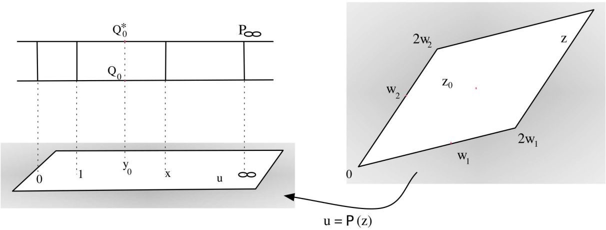
Griffiths, P., Harris, J., On Cayley's explicit solution to Poncelet's porism (1978)

- Let C and D be symmetric 3×3 matrices defining the conics C and D in $\mathbb{CP}^2$.
- $E = \{(x, y) \in \mathbb{CP}^1 \times \mathbb{CP}^1 : x \in C, y \in D^*, x \in y\}$ is an elliptic curve of the equation $v^2 = \det(D + uC)$.
- A closed Poncelet trajectory of length k exists for two conics C and D iff the point $(u, v) = (0, \sqrt{\det D})$ is of order k on E.
- $k\mathscr{A}_\infty(Q_0) \equiv 0 <=> \exists f \in L(-kP_\infty)$ with zero of order k at Q_0.

Hitchin's work

Hitchin, N. Poncelet polygons and the Painlevé equations (1992)

For two conics and a Poncelet trajectory of length k there is an associated algebraic solution of $PVI(\frac{1}{8}, -\frac{1}{8}, \frac{1}{8}, \frac{3}{8})$.



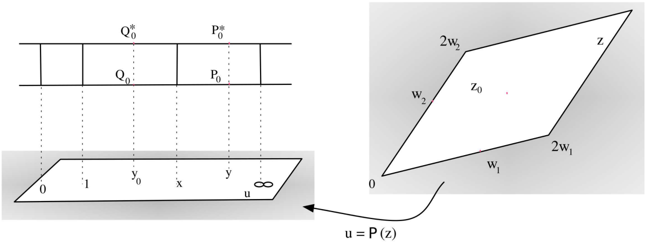
- Existence of the Poncelet trajectory of length k implies $kz_0 \equiv 0$. $\quad (z_0 := 2w_1\frac{m_1}{k} + 2w_2\frac{m_2}{k}.)$
- $z_0 = \mathscr{A}_\infty(Q_0)$, where $\mathscr{A}_\infty$ is the Abel map based at P_∞.
- A function $g(u, v)$ on the curve $v^2 = u(u-1)(u-x)$ having a zero of order k at Q_0 and a pole of order k at P_∞

Hitchin's work

Hitchin, N. Poncelet polygons and the Painlevé equations (1992)

- The function
$$s(u,v) = \frac{g(u,v)}{g(u,-v)}$$
has a zero of order k at Q_0 and a pole of order k at Q_0^* and no other zeros or poles.
- ds has exactly two zeros away from Q_0 and Q_0^*.
- These two zeros are paired by the elliptic involution.
- Their u-coordinate as a function of x solves $PVI(\frac{1}{8}, -\frac{1}{8}, \frac{1}{8}, \frac{3}{8})$.

Our construction



- $z_0 = 2w_1 c_1 + 2w_2 c_2, \quad z_0 = \mathscr{A}_\infty(Q_0), \quad y_0(x) = \wp(z_0(x)).$
- c_1, c_2 arbitrary.
- Differential of the third kind on the elliptic curve $\mathscr{C}$:

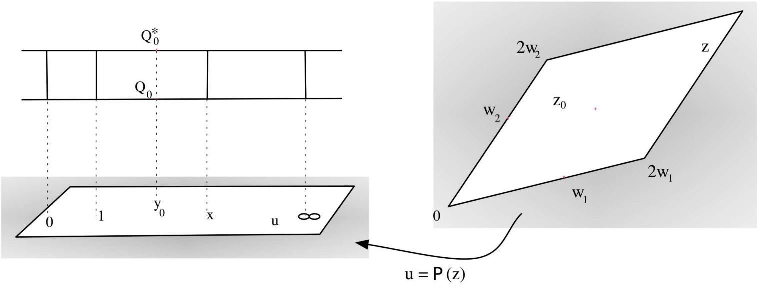
$$\Omega(P) = \Omega_{Q_0,Q_0^*}(P) - 4\pi\mathrm{i}c_2\omega(P).$$

- $\Omega_{Q_0,Q_0^*}(P)$ - meromorphic differential of the third kind with poles at Q_0, Q_0^*, with zero a - periods;
- $\omega(P)$ -holomorphic normalized differential on $\mathscr{C}$ in terms of z has the form: $\omega = \frac{dz}{2w_1}$.

Theorem (V.D., V. Shramchenko)

The differential Ω has two simple poles at Q_0 et Q_0^*, related by the elliptic involution, which project to y_0, the general solution of PVI $(0, 0, 0, \frac{1}{2})$.
The differential Ω has two simple zeros at P_0 et P_0^*, related by the elliptic involution, which project to y, the general solution of PVI$(\frac{1}{8}, -\frac{1}{8}, \frac{1}{8}, \frac{3}{8})$.

Picard solution to PVI $(0, 0, 0, \frac{1}{2})$



- Transformed $\wp$ satisfies:
  $(\wp'(z))^2 = \wp(z)\,(\wp(z) - 1)\,(\wp(z) - x)$.
- Define

$$z_0 := 2w_1 c_1 + 2w_2 c_2.$$

- $z_0 = \mathscr{A}_\infty(Q_0)$.
- Picard's solution to PVI $(0, 0, 0, \frac{1}{2})$:

$$y_0(x) = \wp(z_0(x)).$$

Okamoto transformations $\sim$ 1987

- a group of symmetries of PVI$(\alpha, \beta, \gamma, \delta)$.

- Lemma (V. D., V. Shramchenko): Okamoto transformation from PVI$(0, 0, 0, \frac{1}{2})$ to PVI$(\frac{1}{8}, -\frac{1}{8}, \frac{1}{8}, \frac{3}{8})$:

  y_0 - Picard's solution of PVI$(0, 0, 0, \frac{1}{2})$

  y - the general solution of PVI$(\frac{1}{8}, -\frac{1}{8}, \frac{1}{8}, \frac{3}{8})$

$$y(x) = y_0 + \frac{y_0(y_0 - 1)(y_0 - x)}{x(x-1)y_0' - y_0(y_0 - 1)}.$$

Ω_{Q_0,Q_0*} as the Okamoto transformation

- Write the differential Ω in terms of the coordinate u:

$$\Omega(P) = \frac{\omega(P)}{\omega(Q_0)}\left[\frac{1}{u(P)-y_0} - \frac{I}{2w_1}\right] - 4\pi\mathrm{i}c_2\omega(P).$$

where $I = \oint_a \frac{du}{(u-y_0)\sqrt{u(u-1)(u-x)}}$.

$y = u(P)$ is projection of zeros of Ω iff

$$\frac{1}{y-y_0} = \frac{I}{2w_1} + 4\pi\mathrm{i}c_2\omega(Q_0).$$

- By differentiating the relation $\int_{P_\infty}^{Q_0}\omega = c_1 + c_2\tau$ with respect to x we find the derivative $\frac{dy_0}{dx}$:

$$\frac{dy_0}{dx} = -\frac{1}{4}\Omega(P_x)\frac{\omega(P_x)}{\omega(Q_0)}$$
$$= \frac{1}{4}\frac{\omega^2(P_x)}{\omega^2(Q_0)}\left[4\pi\mathrm{i}\,c_2\,\omega(Q_0) - \frac{1}{x-y_0} + \frac{I}{2w_1}\right].$$

Ω_{Q_0,Q_0*} as the Okamoto transformation

- Thus we get for the relationship between y and y_0 :

$$\frac{1}{y-y_0} = 4\frac{\omega^2(Q_0)}{\omega^2(P_x)}\frac{dy_0}{dx} + \frac{1}{x-y_0}.$$

- The holomorphic normalized differential in terms of the u-coordinate has the form

$$\omega(P) = \frac{du}{2w_1\sqrt{u(u-1)(u-x)}}.$$

- Therefore

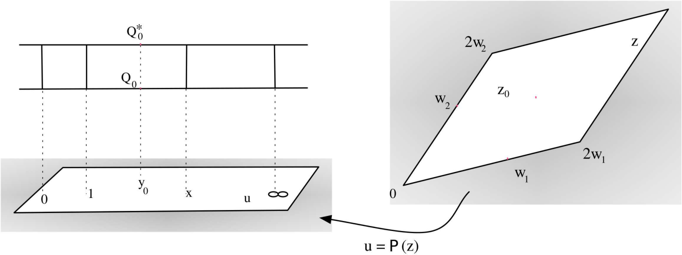
$$\omega(P_x) = \frac{2}{2w_1\sqrt{x(x-1)}} \quad \text{and} \quad \omega(Q_0) = \frac{1}{2w_1\sqrt{y_0(y_0-1)(y_0-x)}}.$$

- Okamoto transformation:

$$y(x) = y_0 + \frac{y_0(y_0-1)(y_0-x)}{x(x-1)y_0' - y_0(y_0-1)}. \qquad (1)$$

Remark on $\frac{dy_0}{dx}$

$y_0(x) = \wp(z_0(x))$ - the Picard solution to PVI $(0, 0, 0, \frac{1}{2})$



$$\frac{dy_0}{dx} = -\frac{1}{4}\Omega(P_x)\frac{\omega(P_x)}{\omega(Q_0)} \qquad (2)$$

$$(z_0 = 2w_1c_1 + 2w_2c_2 \qquad \Omega(P) = \Omega_{Q_0,Q_0*}(P) - 4\pi i c_2\omega(P))$$

Normalization of the differential Ω

- $z_0 = 2w_1 c_1 + 2w_2 c_2$.
- $\Omega(P) = \Omega_{Q_0, Q_0*}(P) - 4\pi \mathrm{i} c_2 \omega(P)$.
- The constants c_1 and c_2 determine the periods of Ω :

$$\oint_a \Omega = -4\pi \mathrm{i} c_2 \qquad \oint_b \Omega = 4\pi \mathrm{i} c_1.$$

- Ω does not depend on the choice of a- and b-cycles.
- Therefore our construction is global on the space of elliptic two-fold coverings of $\mathbb{C}P^1$ ramified above the point at infinity.

Schlesinger system (four points)

- Linear matrix system

$$\frac{d\Phi}{du} = A(u)\Phi, \qquad A(u) = \frac{A^{(1)}}{u} + \frac{A^{(2)}}{u-1} + \frac{A^{(3)}}{u-x}$$

$u \in \mathbb{C}, \Phi \in \mathrm{M}(2,\mathbb{C}), A \in sl(2,\mathbb{C})$

- Isomonodromy condition (Schlesinger system)

$$\begin{aligned}
\frac{dA^{(1)}}{dx} &= \frac{[A^{(3)}, A^{(1)}]}{x}; \\
\frac{dA^{(2)}}{dx} &= \frac{[A^{(3)}, A^{(2)}]}{x-1}; \\
\frac{dA^{(3)}}{dx} &= -\frac{[A^{(3)}, A^{(1)}]}{x} - \frac{[A^{(3)}, A^{(2)}]}{x-1}.
\end{aligned}$$

$A^{(1)} + A^{(2)} + A^{(3)} = const.$

Solution to the Schlesinger system (four points)

- By conjugating, assume $A^{(1)} + A^{(2)} + A^{(3)} = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}$.
- Then the term A_{12} is of the form:

$$A_{12}(u) = \kappa \frac{(u-y)}{u(u-1)(u-x)}$$

- The zero y as a function of x satisfies the

$$\mathrm{PVI}\left(\frac{(2\lambda-1)^2}{2},\ -\mathrm{tr}(A^{(1)})^2,\ \mathrm{tr}(A^{(2)})^2,\ \frac{1-2\mathrm{tr}(A^{(3)})^2}{2} \right)$$

- For PVI$(\frac{1}{8}, -\frac{1}{8}, \frac{1}{8}, \frac{3}{8})$ $\lambda = -1/4$. Our construction implies

$$A_{12}(u) = \frac{\Omega(P)}{\omega(P)} \frac{(u-y_0)}{u(u-1)(u-x)}, \qquad P \in \mathscr{L}, \ \ u = u(P).$$

Solution to the Schlesinger system (four points)

- Let $\phi(P) = \frac{du}{\sqrt{u(u-1)(u-x)}}$ - a non-normalized holom. diff.

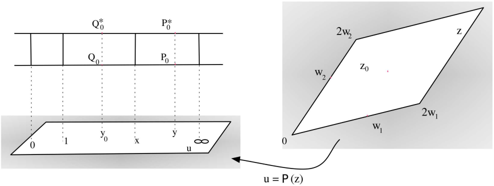
$$A_{12}^{(1)} = -\frac{1}{4} y_0 \Omega(P_0)\phi(P_0), \qquad \beta_1 := -\frac{y_0}{4}\left(\Omega(P_0)\right)^2,$$
$$A_{12}^{(2)} = \frac{1}{4}(1-y_0)\Omega(P_1)\phi(P_1), \qquad \beta_2 := \frac{1-y_0}{4}\left(\Omega(P_1)\right)^2,$$
$$A_{12}^{(3)} = \frac{1}{4}(x-y_0)\Omega(P_x)\phi(P_x), \qquad \beta_3 := \frac{x-y_0}{4}\left(\Omega(P_x)\right)^2.$$

- Then the following matrices solve the Schlesinger system

$$A^{(i)} := \begin{pmatrix} -\frac{1}{4} - \frac{\beta_i}{2} & A_{12}^{(i)} \\ -\frac{1}{4}\frac{\beta_i + \beta_i^2}{A_{12}^{(i)}} & \frac{1}{4} + \frac{\beta_i}{2} \end{pmatrix}, \qquad i = 1, 2, 3.$$

- Eigenvalues of matrices $A^{(i)}$ are $\pm 1/4$.
- cf. Kitaev, A., Korotkin, D. (1998); Deift, P., Its, A., Kapaev, A., Zhou, X. (1999)

Generalization to hyperelliptic curves



Let $z_0 \in Jac(\mathscr{L})$, $z_0 = c_1 + c_2^t \mathbb{B}$, and $\sum_{j=1}^{g} \mathscr{A}_\infty(Q_j) = z_0$.
Define the differential

$$\Omega(P) = \sum_{j=1}^{g} \Omega_{Q_j Q_j^*}(P) - 4\pi \mathrm{i}\, c_2^t \omega(P).$$

Let $q_j = u(Q_j)$. Then

$$\frac{\partial q_j}{\partial u_k} = -\frac{1}{4}\Omega(P_k) v_j(P_k), \qquad (3)$$

where

$$v_j(P) = \frac{\phi(P)\prod_{\alpha=1,\alpha\neq j}^{g}(u - q_\alpha)}{\phi(Q_j)\prod_{\alpha=1,\alpha\neq j}^{g}(q_j - q_\alpha)}, \qquad j = 1, \ldots, g$$

Normalization of the differential Ω

$$\Omega(P) = \sum_{j=1}^{g} \Omega_{Q_j Q_j^{\tau}}(P) - 4\pi\mathrm{i}\, c_2^t \omega(P)$$

where $z_0 = c_1 + c_2^t \mathbb{B}$ and $\sum_{j=1}^{g} \mathscr{A}_\infty(Q_j) = z_0$;

$c_1, c_2 \in \mathbb{R}^g$.

- The constant vectors $c_1 = (c_{11}, \dots c_{1g})^t$ and $c_2 = (c_{21}, \dots, c_{2g})^t$ determine the periods of Ω :

$$\oint_{a_k} \Omega = -4\pi\mathrm{i} c_{2k} \qquad \oint_{b_k} \Omega = 4\pi\mathrm{i} c_{1k}.$$

- Ω does not depend on the choice of a- and b-cycles.

Schlesinger system (n points)

$$\frac{d\Phi}{du} = A(u)\Phi, \qquad A(u) = \sum_{j=1}^{2g+1} \frac{A^{(j)}}{u - u_j},$$

where $u \in \mathbb{C}$, $\Phi(u) \in \mathrm{M}(2, \mathbb{C})$, $A^{(j)} \in sl(2, \mathbb{C})$.

- Schlesinger system for residue-matrices $A^{(i)} \in sl(2, \mathbb{C})$:

$$\frac{\partial A^{(j)}}{\partial u_k} = \frac{[A^{(k)}, A^{(j)}]}{u_k - u_j}; \qquad A^{(1)} + \cdots + A^{(2g+1)} = -A^{(\infty)} = const$$

- by removing the conjugation freedom assume

$$A^{(\infty)} = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}.$$

Solution to the Schlesinger system (n points)

- Let $\phi(P) = \frac{du}{\sqrt{\prod_{i=1}^{2g+1}(u-u_i)}}$ - a non-normalized holom. diff.
- Use the differential Ω to construct an analogue of A_{12} in the hyperelliptic case

$$A_{12}(u) = \frac{\Omega(P)}{\phi(P)} \frac{\prod_{\alpha=1}^{g}(u - q_\alpha)}{\prod_{j=1}^{2g+1}(u - u_j)},$$

- Its residues at the simple poles:

$$A_{12}^{(n)} = \frac{\kappa}{4}\Omega(P_n)\phi(P_n)\prod_{\alpha=1}^{g}(u_n - q_\alpha). \tag{4}$$

- Introduce the following quantities:

$$\beta_n := \frac{1}{4}\,\Omega(P_n)\sum_{j=1}^{g} v_j(P_n) - \frac{1}{2}\Omega(\infty)A_{12}^{(n)}.$$

- The following matrices $A^{(i)}$ with $i = 1, \ldots, 2g+1$ solve the Schlesinger system

$$A^{(i)} := \begin{pmatrix} -\frac{1}{4} - \frac{\beta_i}{2} & A_{12}^{(i)} \\ -\frac{1}{4}\frac{\beta_i + \beta_i^2}{A_{12}^{(i)}} & \frac{1}{4} + \frac{\beta_i}{2} \end{pmatrix};$$

- $$A^{(1)} + \cdots + A^{(2g+1)} = -A^{(\infty)} = \begin{pmatrix} -1/4 & 0 \\ 0 & 1/4 \end{pmatrix}.$$

- cf. Kitaev, A., Korotkin, D. (1998); Deift, P., Its, A., Kapaev, A., Zhou, X. (1999)

- Zeros of Ω are zeros of $A_{12}(u)$ and are solutions of the multidimensional Garnier system.

Back to Poncelet and to billiards

$$n = 2g + 2$$

Consider the case of a point z_0 with rational coordinates $c_1, c_2 \in \mathbb{Q}^g$ with respect to the Jacobian of the hyperelliptic curve of genus g. It corresponds to a periodic trajectory of a billiard ordered game associated to g quadrics from a confocal family in $d = g + 1$ dimensional space.
For billiard ordered games see V. Dragović, M. Radnović, JMPA 2006.

Reference:

Algebro-geometric solutions of the Schlesinger systems and the Poncelet-type polygons in higher dimensions, by V. D., Vasilisa Shramchenko, IMRN, 2017

Akhiezer polynomails

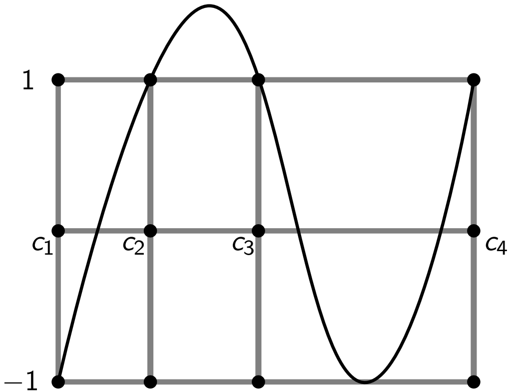
Problem: find monic polynomial of degree n minimizing the uniform norm at the union of two (or more) intervals. Denote the solution as $\hat{P}_n$ and its norm as L_n.

The polynomial $\hat{p}_n$ is the solution of the Pell equation on $[c_1, c_2] \cup [c_3, c_4]$ if and only if:

(i) $\hat{p}_n = \hat{P}_n / \pm L_n$

(ii) the set $[c_1, c_2] \cup [c_3, c_4]$ is the maximal subset of $\mathbf{R}$ for which $\hat{P}_n$ is the minimal polynomial in the above sense.

Akhiezer polynomials



An elliptic curve

An elliptic curve $\mathscr{L}_\lambda$ defined by the equation

$$\mu^2 = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)(\lambda - \lambda_4)\,. \tag{5}$$

It has two points at infinity ∞^- and ∞^+, where around ∞^+ we have $\mu \sim \lambda^2$, while around ∞^- we have $\mu \sim -\lambda^2$.
Suppose there exists a polynomial $P_n(\lambda)$ of degree n which solves the Pell equation:

$$P_n^2(\lambda) - \mu^2 Q_{n-2}(\lambda)^2 = 1\,, \tag{6}$$

where $Q_{n-2}(\lambda)$ is of degree $n-2$.

The meromorphic function

On the curve (5) there is a meromorphic function:

$$P^{(n)}(\lambda, \mu) = P_n(\lambda) + \sqrt{P_n^2(\lambda) - 1}\,.$$

From Pell's equation (6), the function has the form:

$$P^{(n)}(\lambda, \mu) = P_n(\lambda) + \mu Q_{n-2}(\lambda).$$

The elliptic involution transforms it:

$$P^{(n)}(\lambda, -\mu) = \frac{1}{P^{(n)}(\lambda, \mu)}\,.$$

Because of that and since $P^{(n)}(\lambda, \mu)$ has a pole of order n in ∞^+, we get $P^{(n)}(\infty^-) = 0\,.$

The differential

We construct the differential

$$\hat{\Omega}(p) = \frac{1}{n} \frac{dP^{(n)}(p)}{P^{(n)}(p)} \tag{7}$$

having a simple pole at ∞^+ with residuum -1, and a simple pole at ∞^- with residuum $+1$ without other poles.
Lemma. If the curve $\mathscr{L}_\lambda$ given by the equation (5) allows the solution of the Pell equation (6) of degree n then $+\infty$ is the point of order n or $2n$.

The differential – the second time

Consider the differential

$$\hat{\Omega}_1 = \frac{1}{n}\frac{P_n'(\lambda)d\lambda}{Q_{n-2}(\lambda)\mu} = \frac{(\lambda - \hat{\gamma})d\lambda}{\mu} \tag{8}$$

Lemma. Two differentials defined with (7) and (8) coincide:

$$\hat{\Omega} = \hat{\Omega}_1 \, .$$

The Möbius transformation

Let the curve $\mathscr{L}_\lambda$ (5) maps to the curve $\mathscr{L}$ (9),

$$v^2 = u(u-1)(u-x)\,, \tag{9}$$

with the Möbius transformation

$$u = \phi(\lambda) = \frac{\lambda - \lambda_1}{\lambda - \lambda_4}\,\frac{\lambda_2 - \lambda_4}{\lambda_2 - \lambda_1}\,,$$

and let the point ∞^+ maps to $q_0 \in \mathscr{L}$ with the projection to the u-plane denoted as y_0 . We have

$$y_0 = \frac{\lambda_2 - \lambda_4}{\lambda_2 - \lambda_1} \qquad \text{and} \qquad x = \frac{\lambda_3 - \lambda_1}{\lambda_3 - \lambda_4}\, y_0\,.$$

The point q_0 remains of order n or $2n$ on $\mathscr{L}$ under the variation of the variable branch point x.

Theorem

Theorem. The function $y(x)$ given with (10),

$$y = \phi(\hat{\gamma}) = \frac{\hat{\gamma} - \lambda_1}{\hat{\gamma} - \lambda_4} \frac{\lambda_2 - \lambda_4}{\lambda_2 - \lambda_1} = \frac{\hat{\gamma} - \lambda_1}{\hat{\gamma} - \lambda_4} y_0 \, , \tag{10}$$

the zero of the differential Ω is the solution of the Painlevé-VI$(1/8, -1/8, 1/8, 3/8)$ equation, where

$$y_0 = \frac{\lambda_2 - \lambda_4}{\lambda_2 - \lambda_1} \, , \qquad x = \frac{\lambda_3 - \lambda_1}{\lambda_3 - \lambda_4} y_0 \, , \qquad \text{and thus} \qquad \lambda_3 = \frac{x\lambda_4 - y_0\lambda_1}{x - y_0} \, , \tag{11}$$

Conversly, if y, y_0 are solutions of the Painlevé VI equations connected through the Okamoto transformation, then:

$$\hat{\gamma} = \frac{y\lambda_4 - y_0\lambda_1}{y - y_0}. \tag{12}$$

The Akhiezer parametrization

The union of intervals $[-1, \alpha_{n,m}] \cup [\beta_{n,m}, 1]$, where

$$\alpha_{n,m} = 1 - 2\,\mathrm{sn}^2(\frac{m}{n}K), \quad \beta_{n,m} = 2\,\mathrm{sn}^2\left(\frac{n-m}{n}K\right) - 1. \qquad (13)$$

Define:

$$TA_n(x, m, \kappa) = L\left(v^n(u) + \frac{1}{v^n(u)}\right), \qquad (14)$$

where

$$v_{n,m}(u) = \frac{\theta_1\left(u - \frac{m}{n}K\right)}{\theta_1\left(u + \frac{m}{n}K\right)}, \quad x_{n,m} = \frac{\mathrm{sn}^2(u)\,\mathrm{cn}^2(\frac{m}{n}K) + \mathrm{cn}^2(u)\,\mathrm{sn}^2(\frac{m}{n}K)}{\mathrm{sn}^2(u) - \mathrm{sn}^2(\frac{m}{n}K)},$$

and

$$L_{n,m} = \frac{1}{2^{n-1}}\left(\frac{\theta_0(0)\theta_3(0)}{\theta_0(\frac{m}{n}K)\theta_3(\frac{m}{n}K)}\right), \quad \kappa^2_{n,m} = \frac{2(\beta_{n,m} - \alpha_{n,m})}{(1-\alpha_{n,m})(1+\beta_{n,m})}.$$

The Akhiezer theorem

(a) $TA_n(x,m,\kappa)$ is a monic polynomials of degree n in x where the second term is equal to $-n\tau_1^{(n,m)}$, where

$$\tau_1^{(n,m)} = -1 + 2\frac{\mathrm{sn}(\frac{m}{n}K)\,\mathrm{cn}(\frac{m}{n}K)}{\mathrm{dn}(\frac{m}{n}K)}\left(\frac{1}{sn(\frac{2m}{n}K)} - \frac{\theta'(\frac{m}{n}K)}{\theta(\frac{m}{n}K)}\right).$$

(b) The maximum of modulus of TA_n on the union of the intervals $[-1,\alpha_{n,m}]\cup[\beta_{n,m},1]$ is $L_{n,m}$.

(c) The polynomials $TA_n(x,m,\kappa_{n,m})$ are the generalized Chebyshev polynomials on the unions of two intervals $E_{n,m}=[-1,\alpha_{n,m}]\cup[\beta_{n,m},1]$ with the norm $L_{n,m}=||TA_n(x,m,\kappa_{n,m})||_{E_{n,m}}$ and

$$E_{n,m} = TA_n^{-1}[-L_{n,m},L_{n,m}].$$

(d) Outside $E_{n,m}$ the derivative of $TA_n(x,m,\kappa_{n,m})$ with respect to x has a unique zero $c_{n,m}$. It belongs to $[\alpha_{n,m},\beta_{n,m}]$ and

$$c_{n,m} = \frac{\alpha_{n,m}+\beta_{n,m}}{2} - \tau_1^{(n,m)}. \qquad (15)$$

Theorem. The formulas give explicit solutions of the equation PVI$(1/8, -1/8, 3/8, 1/8)$:

$$y_{n,m}(x) = \frac{2 + \alpha_{n,m} + \beta_{n,m} - 2\tau_1^{(n,m)}}{2 - \alpha_{n,m} - \beta_{n,m} + 2\tau_1^{(n,m)}} \frac{1 - \alpha_{n,m}}{1 + \alpha_{n,m}}, \qquad (16)$$

where

$$x = \frac{\beta_{n,m} + 1}{\beta_{n,m} - 1} \frac{\alpha_{n,m} - 1}{\alpha_{n,m} + 1}.$$

Corollary. The evolution of the critical point $\hat{\gamma}$ is given with

$$\hat{\gamma} = \frac{y + y_0}{y - y_0},$$

where the variable x expresses as:

$$x = \frac{1}{k^2 - 1}.$$

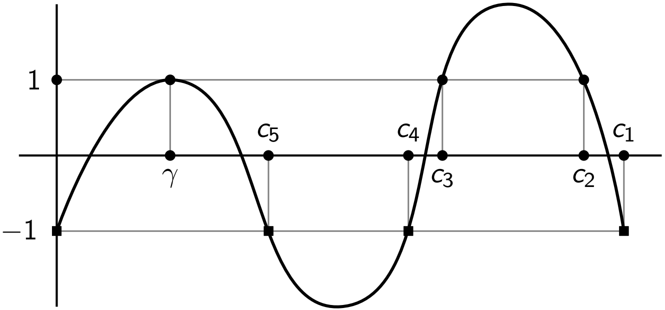
The case $g = 2$. The Schlesinger system with constraints

With Möbius transformations, we reduce the genus 2 curve:
$v^2 = u(u-1)(u-u_1)(u-x_1)(u-x_2)$ Let $a_j \in \{0, 1, u_1, x_1, x_2\}$
and $\phi = du/v$.
Theorem. The system

$$\begin{aligned}
\partial_{x_1} A_{a_j} &= \frac{[A_{x_1}, A_{a_j}]}{x_1 - a_j} + \frac{[A_{u_1}, A_{a_j}]}{u_1 - a_j}\frac{\partial u_1}{\partial x_1} \\
\partial_{x_1} A_{u_1} &= \frac{[A_{x_1}, A_{u_1}]}{x_1 - u_1} - \sum_{a_j \in \{0,1,x_1,x_2\}} \frac{[A_{u_1}, A_{a_j}]}{u_1 - a_j}\frac{\partial u_1}{\partial x_1} \\
\partial_{x_1} A_{x_1} &= -\sum_{a_j \in \{0,1,x_2,u_1\}} \frac{[A_{x_1}, A_{a_j}]}{x_1 - a_j} + \frac{[A_{u_1}, A_{x_1}]}{u_1 - x_1}\frac{\partial u_1}{\partial x_1},
\end{aligned} \tag{17}$$

(with similar equations w.r.t. x_2) describe the dynamics of the solutions of the Pell equation on three intervals ($d = 3$, $g = 2$), when the end-points of the intervals move.

Extremal polynomials for $g = 2$.



Solutions of the Schlesinger system with constraints

Let

$$A_{12} = \frac{t}{2}\frac{\Omega_A(P)}{\phi(P)}\frac{(u-y_0)^2 du}{v^2} \tag{18}$$

with t arbitrary, and

$$\beta_{a_j} = A^{12}_{a_j}\left(\frac{1}{\phi(q_0)(a_j-y_0)^2} + X_{a_j}\right) \text{, where} \tag{19}$$

$$X_{a_j} = -\frac{1}{2}\frac{\sum_k \frac{1}{a_k-y_0}}{(a_j-y_0)\phi(q_0)} - \Omega(P_\infty)\,.$$

Solutions of the Schlesinger system with constraints

Theorem. Matrices

$$A_{a_j}^{12} = \operatorname*{res}_{u=a_j} A_{12}(u) = \frac{t}{4}\Omega(a_j)\phi(a_j)(a_j - y_0)^2 \ ;$$

$$A_{a_j}^{11} = -\frac{1}{4} - \frac{1}{t}\beta_{a_j} \ ;$$

$$A_{a_j}^{21} = -\frac{t\beta_{a_j} + 2\beta_{a_j}^2}{2t^2 A_{a_j}^{12}}$$

solve the Schlesinger system with constraints (17) for $g = 2$.

Dear Valery Vasil'evich,

Congratulations with the jubilee! Many happy returns!