Remarks on unstable equilibria of conservative systems

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Introduction

- A conservative mechanical system

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial x_i'}K\left(x,x'\right)-\frac{\partial}{\partial x_i}K\left(x,x'\right)+\frac{\partial}{\partial x_i}U\left(x\right)=0,\ \ i=1,...,n$$

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- J.-L. Lagrange 1788 used positivity of second derivatives of U to conclude that a equilibrium point is stable.
- The final form this result was given by G. Lejeune-Dirichlet 1846: *equilibrium point x_0 is stable if this point is strict local minimum of potential function U*. (due conservation of the energy).

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- The positive answer to this problem seems obvious: "*if at an equilibrium position potential energy is not a minimum, then the equilibrium is unstable"* (several references).
- P.Painlevé 1904 gave an example of a C^∞ potential function U whose stable equilibrium point **is not a minimum**, see also A. Wintner 1941.

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- P. Hagedorn 1971, V.Palamodov 1977, S.Taliaferro 1980, E.Liubushin 1980, M.Laloy-K.Peiffer 1982, V.Kozlov-V.Palamodov 1982, M.Sofer 1983, V.Kozlov 1981,1986,1987,1989, V.Moauro - P.Negrini 1991, M.Paternain 2019 and other.

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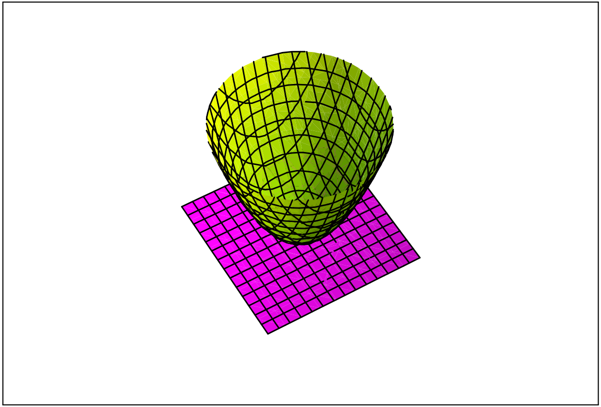
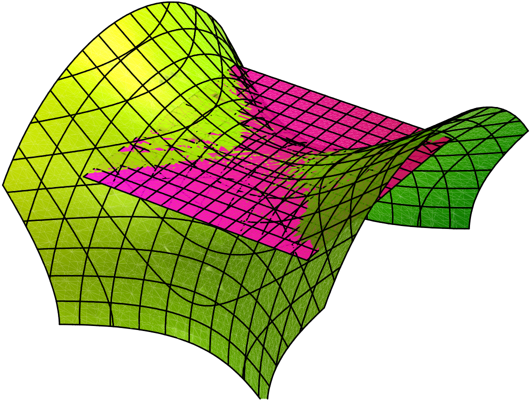
- This problem was discussed by J.Hadamard 1897, E.Routh 1898, A.Wintner 1941, P.Appel 1955, V.Arnold 1976.

- **Example 1.** Cup equilibrium point is stable

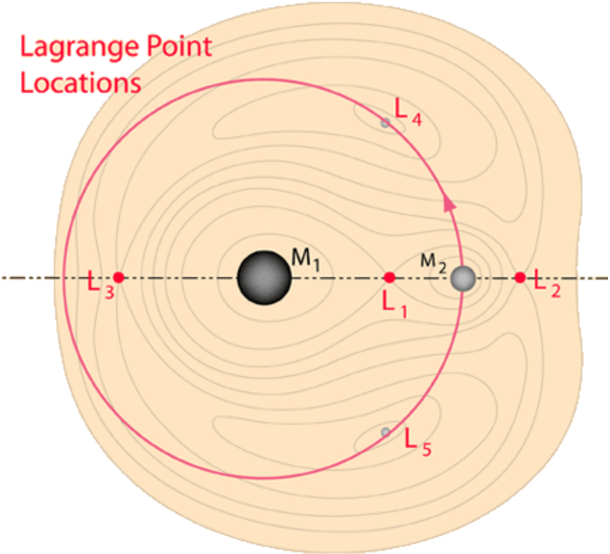


- **Example 2.** Saddle point make is unstable equilibrium



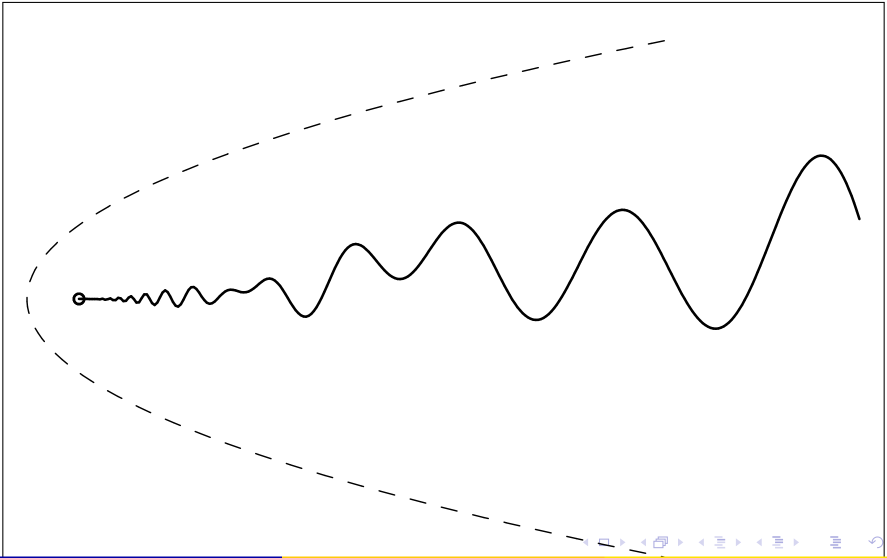
Saddle equilibrium point

- **Example 3.** Lagrange points



Asymptotic solutions

- Method of asymptotic solutions was first applied by Liapounov



Fatal instability

- **Definition.** We call equilibrium point x_0 of a function U on $\Omega \subset R^n$ *fatally* unstable if any trajectory in Ω with subcritical energy $E \leq U(x_0)$ started from a non-equilibrium point reaches boundary $\partial\Omega$ in finite time.

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- Definition has a sense only if point x_0 belongs to the boundary of set

$$X := \{x \in \Omega; U(x) < U(x_0)\}.$$

In this case fatal instability implies Liapounov's instability (not vice versa). If x_0 is a minimum (may be not strict) point of U in a neighborhood of x_0, then there is no motion started from a non-equilibrium position, hence this equilibrium is fatally unstable in this sense.

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- **Conjecture** *Any equilibrium point of an analytic potential function is fatally unstable.*
- Proved for $n = 2$ by P. 1977.

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$$\left(\xi \frac{\partial}{\partial x} \mathrm{A}(x_0), \xi \mathrm{d}x\right) |_{x=x_0} \geq 0.$$
- *Then any solution of the Lagrange system started in* X *reaches* ∂X *in finite time.*

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$$\mathrm{e}_w = \sum w_k x_k \frac{\partial}{\partial x_k}$$

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- **Proposition** Any quasihomogeneous potential function U, $\deg U > 0$ fulfils conditions (I) and (II) for $x_0 = 0$.
- *Proof.* Condition I is obvious and inequality

$$\left\langle \xi \frac{\partial}{\partial x} \mathrm{e}_w, \xi \mathrm{d}x \right\rangle = \sum w_k \xi_k^2 \geq 0$$

implies II.

Method of geodesic curves

- **Example 3.** (M. Paternain 2019) Function

$$U_1(x, y, z) = \pm\left(\left(y - x^2\right)^5 + \left(y - 2x^2\right)^5 + z^2\right)^2$$

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- **Example 4.** (M. Paternain 2019) Function

$$U_2 = \pm\left(\left(y-x^2\right)^5\left(y-2x^2\right)^5 + \left(z-x^3\right)^2\right)^2$$

  has no geodesic curve through the origin.

- This function is not quasihomogeneous but field

$$\mathrm{A} = \frac{1}{20} x \frac{\partial}{\partial x} + \frac{1}{10} y \frac{\partial}{\partial y} + \frac{1}{2} \left(z - \frac{7}{10} x^3 \right) \frac{\partial}{\partial z}$$

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- Quadratic form

$$\left(\xi \frac{\partial}{\partial x} \mathrm{A}, \mathrm{d}\xi \right) = \frac{1}{20} \xi_1^2 + \frac{1}{10} \xi_2^2 + \frac{1}{2} \xi_3^2 - \frac{21}{10} x^2 \xi_1 \xi_3$$

depending on x is positive defined if $|x| < \sqrt{2}/21$.

Results of eighties

- **Theorem** (V.Kozlov - V.Palamodov, 1982) *Let* $m \geq 2$ *and*

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where u_k *is for any* k, *a homogeneous polynomial of degree* k *and series converges in a neighborhood of the origin.*

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- **Theorem** (V.Kozlov, 1986) *The same conclusion holds for*

$$U(x) = u_2 + u_m + \sum_{k>m} u_k,$$

if $u_2 + u_m$ *has no local minimum at equilibrium point* $x = 0$.

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- **Perspective:** *to prove the Conjecture!*

Thank you for listening and reading!

Wishing to Valeriy Vasilievich Kozlov

further achievements

in science and beyond

for many years to come!