New integrable two-centre problem on sphere with Dirac magnetic field

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Classical mechanics, dynamical systems and mathematical physics
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L. Euler *De motu corporis ad duo centra virium fixa attracti.* Novi Comm. Acad. Sci. Petrop. **10** (1766), 207-242.

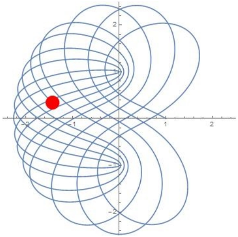


Рис.: Leonhard Euler (1707-1783) and orbits in his two-centre problem

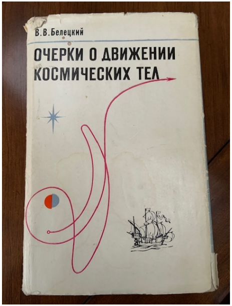
Celebrated Euler two-centre problem with

$$H = \frac{1}{2}(p_1^2 + p_2^2) - \frac{\mu}{r_1} - \frac{\mu}{r_2}, \quad r_{1,2} = \sqrt{q_1^2 + (q_2 \pm c)^2}$$

was the first non-trivial mechanical system integrated since Newton.

"Second birth of the old problem"

Aksenov, Grebennikov and Demin, 1961: Euler's system with imaginary distance between the centres as an approximation of the satellite motion in the gravitational field of the Earth spheroid



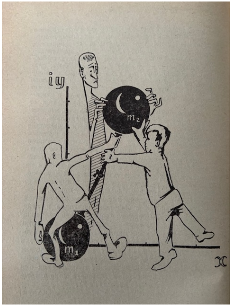


Рис.: Novizhilov's sketch of the authors in Beletski's book

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The corresponding spherical analogue of the Newton-Coulomb potential is

$$U = -\mu \cot \theta_1 - \mu \cot \theta_2,$$

where θ_1 and θ_2 are the spherical distances from the fixed centres.

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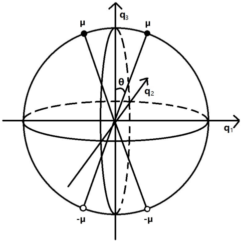

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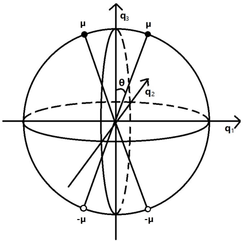

Adding the Dirac magnetic field seems to destroy the integrability...

Let $e(3)$ be the Lie algebra of the Euclidean motion group $E(3)$ of $\mathbb{R}^3$ and consider the canonical Lie-Poisson bracket on its dual space $e(3)^*$:

$$\{M_i, M_j\} = \epsilon_{ijk} M_k, \quad \{M_i, q_j\} = \epsilon_{ijk} q_k, \quad \{q_i, q_j\} = 0.$$

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$$C_1 = (p, p) = R^2, \qquad C_2 = (l, p) = \nu R.$$

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Novikov and Schmelzer, 1981: the variables

$$L_i = M_i - \frac{\nu}{R} q_i, \quad i = 1, 2, 3$$

satisfy $(q, q) = R^2$, $(L, q) = 0$. This identifies the coadjoint orbits with $T^* S^2$ with the symplectic form

$$\omega = \mathrm{d}P \wedge \mathrm{d}Q + \nu\, \mathrm{d}S,$$

where $\mathrm{d}P \wedge \mathrm{d}Q$ is the standard symplectic form on $T^* S^2$ and $\mathcal{H} = \nu\, \mathrm{d}S$ is the magnetic field of the Dirac monopole of charge ν.

Mamaev, 2003: In the coordinates M, q on $e(3)^*$ the Hamiltonian of the spherical analogue of the Euler two-centre problem is

$$H = \frac{1}{2}|M|^2 - \mu \frac{\beta q_3 - \alpha q_1}{\sqrt{q_2^2 + (\alpha q_3 + \beta q_1)^2}} - \mu \frac{\beta q_3 + \alpha q_1}{\sqrt{q_2^2 + (\alpha q_3 - \beta q_1)^2}},$$

where μ, α, β are parameters such that $\alpha^2 + \beta^2 = 1$.

Spherical Euler problem on $e(3)^*$

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At the special level $(M, q) = 0$ the system has an additional integral

$$F = \alpha^2 M_1^2 - \beta^2 M_3^2 - 2\alpha\beta\left(\mu\frac{\beta q_1 - \alpha q_3}{\sqrt{q_2^2 + (\beta q_1 - \alpha q_3)^2}} + \mu\frac{\alpha q_1 + \beta q_3}{\sqrt{q_2^2 + (\alpha q_3 + \beta q_1)^2}}\right).$$

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When $(M, q) \neq 0$ then the Poisson bracket $\{F, H\}$ is not vanishing and the system is believed to be non-integrable.

New integrable system on $e(3)^*$

Veselov and Ye, 2019: The system on $e(3)^*$ with the Hamiltonian

$$H = \frac{1}{2}|M|^2 - \mu \frac{|q|}{\sqrt{R(q)}},$$

$$R(q) = (A-B)q_2^2 + (\sqrt{A}q_3 - \sqrt{B}|q|)^2, \quad A > B > 0$$

is Liouville integrable with the additional integral

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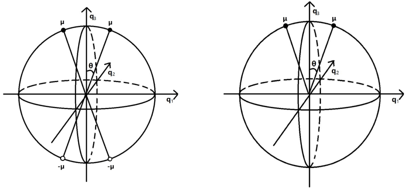

Рис.: Position of the fixed centres in the classical (left) and the new (right) systems

New integrable magnetic two-centre system on S^2

At the symplectic leaf with $(p,p)=1$, $(M,p)=\nu$ we have a new integrable system on S^2 with two locally Coulomb singularities with charge $\mu/\sqrt{A-B}$ fixed at the points $(\pm\sqrt{\frac{A-B}{A}}, 0, \sqrt{\frac{B}{A}})$ in the external field of Dirac magnetic monopole with charge ν.

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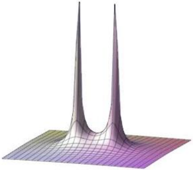

Рис.: Graph of the potential $U(q)$ with $\mu < 0$ after the stereographic projection

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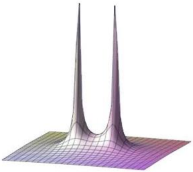

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One can show that this is the only integrable extension of Dirac magnetic monopole with additional integral quadratic in momenta (apart from the classical Clebsch case, **Veselov and Ye, to appear**).

Consider the unit sphere given by the equation $q_1^2 + q_2^2 + q_3^2 = 1$, and introduce the *spherical elliptic (Neumann) coordinates* as the roots u_1, u_2 of the quadratic equation

$$\Phi(u) = \frac{q_1^2}{A-u} + \frac{q_2^2}{B-u} + \frac{q_3^2}{C-u} = 0,$$

where $C = 0$ and $A > B > 0$ are the same as before.

New system in elliptic coordinates

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The Hamiltonian of the new system has the form

$$H = \frac{1}{2}\left(\frac{f(u_1)}{u_1 - u_2}\tilde{p}_1^2 + \frac{f(u_2)}{u_2 - u_1}\tilde{p}_2^2\right) - \frac{\mu}{\sqrt{u_2} - \sqrt{u_1}}.$$

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Note that the electric potential can be written in Stäckel form as

$$U = -\frac{\mu}{\sqrt{u_2}-\sqrt{u_1}} = -\frac{\mu(\sqrt{u_2}+\sqrt{u_1})}{u_2-u_1},$$

so that when magnetic charge is zero, then magnetic momenta $\tilde{p}_i = p_i$ and the system is separable and belongs to the class considered by **Kozlov and Harin**.

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Quantum version

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The modified operators $\hat{M}_{X_j} = i\nabla_j + \nu q_j$, satisfy the usual angular momentum relations (cf. **Fierz, 1944**): $[\hat{M}_k, \hat{M}_m] = i\epsilon_{kmn}\hat{M}_n$, $[\hat{M}_k, \hat{q}_m] = i\epsilon_{kmn}\hat{q}_n$.

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The quantum Hamiltonian and integral have the form

$$\hat{H} = \frac{1}{2}(\hat{M}_1^2 + \hat{M}_2^2 + \hat{M}_3^2) - \mu\frac{|q|}{\sqrt{R(q)}},\ R(q) = (A-B)q_2^2 + (\sqrt{A}q_3 - \sqrt{B}|q|)^2,$$

$$\hat{F} = A\hat{M}_1^2 + B\hat{M}_2^2 + \frac{2\sqrt{AB}}{|q|}(\hat{M}, q)\hat{M}_3 - 2\mu\sqrt{AB}\frac{q_3}{\sqrt{R(q)}}.$$

Hyperbolic version

Replace $E(3)$ by the group $E(2,1)$ of motion of pseudo-Euclidean space $\mathbb{R}^{2,1}$ with Lie algebra defined by

$$[M_1, M_2] = M_3,\ [M_2, M_3] = -M_1,\ [M_3, M_1] = -M_2,$$

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$$C_1 = (q, Jq) = -q_1^2 - q_2^2 + q_3^2 := ||q||^2,\ \ C_2 = \langle M, q\rangle := -M_1 q_1 - M_2 q_2 + M_3 q_3.$$

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The Hamiltonian of the natural hyperbolic analogue of the new system is

$$H = \frac{1}{2}(M_1^2 + M_2^2 - M_3^2) + \frac{\mu||q||}{\sqrt{R(q)}},\ \ R(q) = (B-A)q_2^2 + (\sqrt{A}q_3 - \sqrt{B}||q||)^2$$

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The corresponding additional integral has the form

$$F = A{M_1}^2 + B{M_2}^2 - 2\,\frac{\sqrt{AB}}{||q||}\langle M, q\rangle M_3 + 2\,\frac{\mu\sqrt{AB}q_3}{\sqrt{R(q)}}.$$

Some open problems

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Reference

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Многая лета, Валерий Васильевич!