# On integrability of geodesic flows on 3-dimensional manifolds

Alexey Bolsinov
Loughborough University and
Moscow State University

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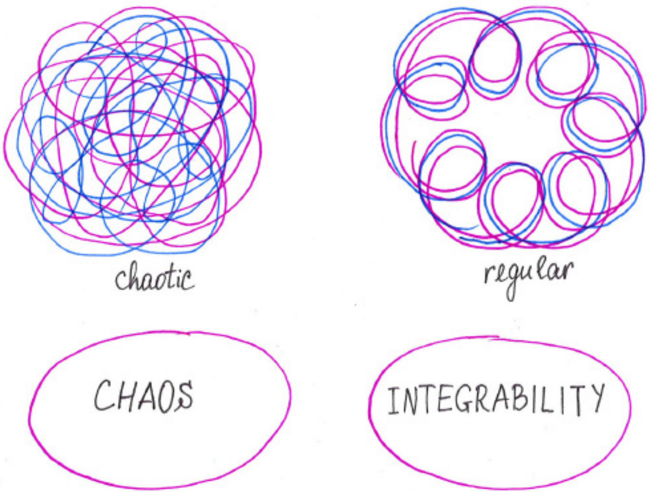
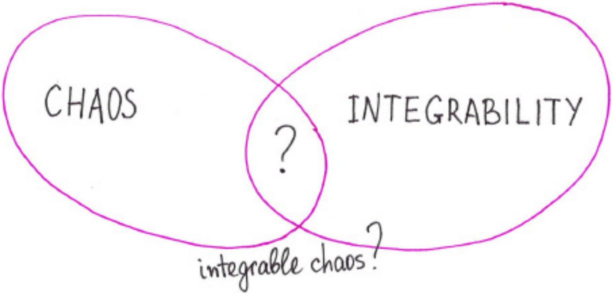
# Chaos and Integrability




CHAOS
INTEGRABILITY
?
integrable chaos?

Let $M$ be a manifold with a Riemannian metric $g = \sum g_{ij}(x)dx^i dx^j$.

### Definition

Geodesic line on $(M, g)$ is a trajectory of a point moving on $M$ freely, i..e, by inertia:

$$\nabla_{\frac{d\gamma}{dt}} \frac{d\gamma}{dt} = 0. \qquad (1)$$

### Hamiltonian form of the equation of geodesics.

In canonical coordinates $(x, p)$ on $T^*M$, the equations of geodesics (1) can be rewritten in Hamiltonian form:

$$\frac{dp_i}{dt} = \frac{\partial H}{\partial x^i}, \quad \frac{dx^i}{dt} = -\frac{\partial H}{\partial p_i}, \qquad (2)$$

where $H = \frac{1}{2} \sum g^{ij}(x) p_i p_j$.

### Properties of geodesics:

- if the metric is Euclidean, then the geodesics are straight lines.
- existence and uniqueness theorem;
- geodesic completeness and Hopf-Rinow theorem;
- existence of closed geodesics in each homotopy class;
- kinetic energy $H = \sum g_{ij}(x)\dot{x}^i \dot{x}^j$ as a first integral.

# Integrability and Liouville-Arnold Theorem

Consider the geodesic flow on a Riemannian manifold $(M, g)$.

Complete integrability: There exist $F_1 = H, F_2, \dots, F_n$ which

- are first integrals (i.e., preserved by the flow),
- pairwise commute $\{F_i, F_j\} = 0$,
- are functionally independent almost everywhere.

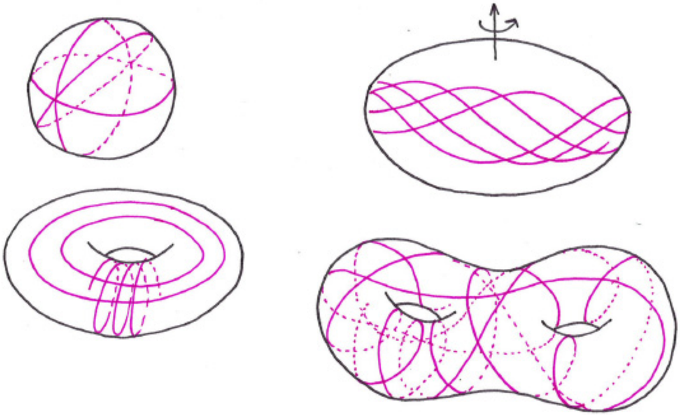
**Theorem (Liouville-Arnold)**

*Let $X = \{F_1 = c_1, \dots, F_n = c_n\}$ be a regular, compact and connected integral surface. Then $X$ is an n-dimensional torus and the dynamics on this torus is quasi-periodic.*

Superintegrability: in addition to $n$ commuting independent integrals, there are some more first integrals so that invariant tori have dimension less than $n$.

General problem: existence, construction and topological obstructions for integrable geodesic flows.

# Two-dimensional case



Theorem (Kozlov, 1979, real analytic case)

*On a 2-dimensional surface of genus $g \geq 2$, there exist no integrable geodesic flows.*

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|---|---|---|---|---|---|---|---|
| 5 | 4 | 3 | ? | ? | ? | 1 | *none* |

Observation. Let $G$ be a three-dimensional Lie group, then any left-invariant Hamiltonian system on $T^*G$ is Liouville integrable (with real analytic first integrals polynomials in momenta). Moreover, this system is superintegrable (i.e., invariant tori are two-dimensional).

The integrals are $H, f_1, f_2, f_3$ where $f_i$ are linear integrals that correspond to basis right-invariant vector fields.

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Question. What happens to the integrals after taking quotient $G \mapsto G/\Gamma$?

Answer. Instead of $f_1, f_2, f_3$ we should consider $h(f_1, f_2, f_3)$, where $h$ is an invariant of the (coadjoint) action of $\Gamma$ on $\mathfrak{g}^*$.

# Topological obstructions to integrability

**Theorem (Dinaburg, 1974)**

*If $\pi_1(M^n)$ is of exponential growth then $h_{top} > 0$ for any geodesic flow.*

**Theorem (Taimanov, 1987, geometrically simple case)**

*If $M^n$ admits an integrable geodesic flow, then $\pi_1(M^n)$ is almost abelian.*
*Moreover,* $\dim H_1(M^n, \mathbb{R}) < n$.
*In particular, this holds true in the real analytic case.*

# *Nil* (L. Butler) and *Sol* (I. Taimanov, AB)

## *Nil* case

$$M^3_{Nil} = G/\Gamma, \text{ where } G = \left\{ \begin{pmatrix} 1 & z & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \right\}, \Gamma = \left\{ \begin{pmatrix} 1 & k & m \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix}, k, n, m \in \mathbb{Z} \right\}$$

- integrable: $F_1 = p_x$, $F_2 = \exp\left(-\frac{1}{p_x^2}\right) \sin \frac{p_y}{p_x}$,
- not real analytic,
- $\pi_1(M^3_{Nil})$ is not almost commutative,
- $h_{top} = 0$.

## *Sol* case

$$M^3_{Sol} = G/\Gamma, \text{ where } G = \left\{ \begin{pmatrix} A^z & & x \\ & & y \\ 0 & 0 & 1 \end{pmatrix} \right\}, \Gamma = \left\{ \begin{pmatrix} A^k & & n \\ & & m \\ 0 & 0 & 1 \end{pmatrix}, k, n, m \in \mathbb{Z} \right\},$$

where $A^z = \exp(zB)$, $\exp B = A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$.

- integrable: $F_1 = p_x^2 - p_x p_y - p_y^2$, $F_2 = \exp\left(-\frac{1}{F_1^2}\right) \cdot \sin\left(2\pi \frac{\log\left|p_x - \frac{1+\sqrt{5}}{2} p_y\right|}{\log \lambda}\right)$.
- not real analytic,
- $\pi_1(M^3_{Sol})$ is not almost commutative and has exponential growth,
- $h_{top} > 0$.

Consider the group $SL(2,\mathbb{R}) = \left\{ X = \begin{pmatrix} a & b \\ c & d \end{pmatrix},\ \det X = 0 \right\}$ and its Lie algebra $sl(2,\mathbb{R}) = \left\{ X = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix} = \alpha E_1 + \beta E_2 + \gamma E_3 \right\}$.

Denote by $\xi_1, \xi_2, \xi_3$ the left-invariant vector fields corresponding to the basis $E_1, E_2, E_3$.
Similarly, let $\eta_1, \eta_2, \eta_3$ be the corresponding right-invariant vector fields on $SL(2,\mathbb{R})$.

The left-invariant Hamiltonian: $H = (\xi_1^2 + \xi_2\xi_3) + 2(\xi_2 - \xi_3)^2$.

Before taking quotient w.r.t. $\Gamma$, we have 3 first integrals $\eta_1, \eta_2, \eta_3$.

Natural identification $PSL(2,\mathbb{R})$ with the unite tangent bundle $S\mathbb{H}^2$ gives the following formula in local coordinates $x, y, \phi$:

$$ds^2 = \frac{dx^2 + dy^2}{y^2} + \left( d\phi + \frac{dx}{y} \right)^2$$

Let $\Gamma \subset PSL(2,\mathbb{R})$ is a Fucshian group that acts on $\mathbb{H}^2$ freely and with compact quotient $M_g^2$ and consider $M_{sl}^3$ to be $PSL(2,\mathbb{R})/\Gamma$. Topologically, $M_{sl}^3 = SM_g^2$.

Apart from the Hamiltonian $H$ the geodesic flow on $M^3_{sl}$ possesses one more quadratic integral (Casimir function):

$$\Delta = \eta_1^2 + \eta_2\eta_3 = \xi^2 + \xi_2\xi_3.$$

But we need one more!

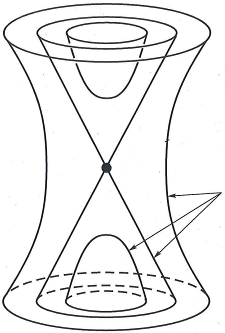
**Theorem**

*Geodesic flow on $M^3_{sl}$ is Liouville integrable in analytic sense in the open (invariant) region of the phase space $T^*M^3_{sl}$ defined by $\Delta < 0$.*

*In the region with $\Delta > 0$ there are no smooth integrals independent from $H$ and $\Delta$. At the fixed integral level of $H$ and $\Delta > 0$, the system has positive topological entropy $h_{top} \geq \sqrt{1-\mathcal{C}}$, where $\mathcal{C} = \frac{H-2\Delta}{H+2\Delta}$.*

# Explanation of this phenomenon

In $\mathbb{R}^3(\eta_1, \eta_2, \eta_3) = T^*SL(2,\mathbb{R}) \simeq sl(2,\mathbb{R})^*$, consider the level surfaces of the integral $\Delta = \eta_1^2 + \eta_2\eta_3$.



The Fucshian group $\Gamma$ naturally acts on this space (by means of the (co)adjoint representation).
What about the orbit space and invariant functions of this action?

## One more general result

Let us relax the geometric simplicity condition (real analyticity) condition, i.e., assume that the first integrals of a geodesic flow on $M^3$ are such that the complement to the regular set filled by 3D Liouville tori is a timely-embedded polyhedron in $T^*M^3$.

### Theorem (L. Butler, 2005)

*If an integrable geodesic flow on a closed manifold $M^3$ satisfies the above condition, then $\pi_1(M^3)$ is almost polycyclic.*
*"Conversely", if $\pi_1(M^3)$ is polycyclic, then $M^3$ admits a real-analytic metric with integrable geodesic flow.*

Thank you for your attention