Quasiclassical approximation for magnetic monopoles

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- Yu.A. Kordyukov, I.A. Taimanov. Quasiclassical approximation for magnetic monopoles. arXiv:1912.12444
- Yu.A. Kordyukov, I.A. Taimanov. Trace formula for the magnetic Laplacian. Russian Math. Surveys 74:2 (2019).

"Magnetic monopoles" in classical mechanics

Magnetic geodesics flows corresponding to monopoles appear in classical mechanics after the restriction of a system to a level of a certain first integral I:
(a) the Kirchhoff problem (Novikov–Schmelzer, 1981);
(b) the rigid body rotation with a fixed point in an axisymmetric field (Kozlov, 1976).

In particular,
in (a) it was showed that the magnetic geodesic flow is described by the classical Hamiltonian $p^2/2$ with the twisted symplectic form

$$\Omega = \sum dp_i \wedge dx^i + F$$

where F is a closed 2-form (the magnetic field) and the study of many-valued functionals (actions for the case $[F] \neq 0$) was started;
in (b) $\int_{S^2} F = 4\pi I$ and the system was restricted to the level $I = 0$ to obtain one-valued functional and to look for its critical points which correspond to periodic orbits.

Dirac's monopole (1931)

"Elementary classical theory allows us to formulate equations of motion for an electron in the field produced by an arbitrary distribution of electric charges and magnetic poles...
The object of the present paper is to show that quantum mechanics does not really preclude the existence of isolated magnetic poles.
On the contrary, the present formalism of quantum mechanics ... , when developed naturally without the imposition of arbitrary restrictions leads inevitably to wave equations whose only physical interpretation is the motion of an electron in the field of a single pole. This new development requires no change whatever in the formalism ... Under these circumstances one would be surprised if Nature had made no use of it.
The theory leads to a connection ... between the quantum of magnetic pole and the electronic charge."

The quantization condition by Dirac

Monopole's field:

$$\mathbf{H} = \frac{\mu \mathbf{r}}{r^3}, \quad \mu = \mathrm{const},$$

The quantization condition

$$4\pi\mu = 2\pi N \frac{\hbar c}{e},$$

with $N \in \mathbb{Z}$, leads to the relation

$$e\,\mu = \mathrm{const}\, N.$$

"... if there exists any monopole at all in the universe, all electric charges would have to be such that e times this monopole strength is equal to $\frac{1}{2}N\hbar c$*"*
(Dirac, 1978)

Monopole harmonics (Tamm, Wu–Yang)

- (Tamm, 1931) The eigenvalues (of the magnetic Laplacian):

$$\lambda_{N,j} = j(j+1) + \frac{N}{2}(2j+1), \quad j = 0, 1, 2 \ldots$$

  with the multiplicities

$$m_{N,j} = N + 2j + 1.$$

  For $N = 0$ we have spherical harmonics.
  For $N = 1$ and $\lambda_{1,0} = \frac{1}{2}$ the basis is given by

$$S_a = \cos\frac{\theta}{2}, \quad S_b = \sin\frac{\theta}{2} e^{i\phi}.$$

- (Wu–Yang, 1976) Monopole harmonics are non-singular sections of non-trivial line bundles.

Magnetic Laplacian

M is a Riemannian manifold with a closed 2-form F such that

$$\left[\frac{1}{2\pi}F\right] \in H^2(M;\mathbb{Z}).$$

L is an Hermitian line bundle with an Hermitian connection

$$\nabla^L = d - iA, \quad dA = F.$$

We have

$$c_1(L) = \left[\frac{1}{2\pi}F\right].$$

The magnetic Laplacians on L and L^N are

$$\Delta^L = (\nabla^L)^*\nabla^L,$$

$$\Delta^{L^N} = -\frac{1}{\sqrt{|g(x)|}}\left(\frac{\partial}{\partial x^j} - iNA_j(x)\right)\left[\sqrt{|g(x)|}g^{j\ell}(x)\left(\frac{\partial}{\partial x^\ell} - iNA_\ell(x)\right)\right]$$

The canonical operator for Δ^{L^N} (Kordyukov–T.)

For $\mathcal{U}$ on a Lagrangian manifold Λ such that it is invariant under the magnetic geodesic flow and uniquely projected onto $U \subset M$ the canonical operator takes the form

$$K^h_{\Lambda,\mathcal{U}} u(x) = e^{(i/h)S_{\mathcal{U}}(s)} \sqrt{\left|\frac{d\mu(s)}{dx_g}\right|} u(s), \quad \pi(s) = x \in \pi(\mathcal{U}),$$

where $d\mu$ is a smooth measure on Λ, invariant under the magnetic geodesic flow, dx_g is the Riemannian volume form,

$$S_{\mathcal{U}}(s) = \int_{s_0}^{s} d^{-1}\Omega = \int_{s_0}^{s} \left(\sum_{j=1}^{n} p_j dx^j + A_U \right), \quad s \in \mathcal{U}.$$

Since the magnetic field is quantized, the small parameter $h \in \{1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{N}, \dots\}$, and

$$K^{1/N}_{\Lambda} u \quad \text{is a section of } L^N.$$

Quantization condition

Assume the Lagrangian manifold Λ and the form $d\mu$ are invariant under the magnetic geodesic flow. Then

$$H|_\Lambda \equiv E = \mathrm{const.}$$

For a closed curve γ in M, its action S_γ is defined by

$$S_\gamma = \int_\gamma \sum_{j=1}^{n} p_j dx^j + h_A(\gamma) \mod 2\pi\mathbb{Z},$$

where $\exp(ih_A(\gamma)) \in U(1)$ is the holonomy of the projection of γ to M with respect to the connection ∇^L on L.
For any closed curve γ on Λ, its action satisfies the condition

$$S_\gamma = N^{-1}\frac{\pi}{2} I_\gamma \mod 2\pi\mathbb{Z},$$

where $I_\gamma \in \mathbb{Z}$ is the Maslov index of γ.

Quasimodes

If Λ satisfies the quantization condition, the section

$$U_N = K_\Lambda^{1/N}[1] \in C^\infty(M, L^N)$$

is an almost eigenfunction of Δ^{L^N} with the corresponding approximate eigenvalue

$$\hat{\lambda}_N = EN^2 + O(1),$$

that is, the following equality holds:

$$\Delta^{L^N} U_N = \hat{\lambda}_N U_N + O(1).$$

Approximate eigenvalues

The application of this method to finding the approximate eigenvalues of the magnetic laplacian for Dirac magnetic monopoles on the 2-sphere gives the following answer:
the approximate eigenvalues are

$$\widehat{\lambda}_{N,j} = j(j+1) + \frac{N}{2}(2j+1) + \frac{\mathbf{1}}{\mathbf{4}}$$

with the multiplicities

$$\widehat{m}_{N,j} = N + 2j + 1.$$

Analogous calculations for the Laplace–Beltrami operator were done by Kogan (1969) (they are reproduced in the book by Maslov and Fedoruk (1976)).

Remarks

The canonical operator for a general symplectic structure was introduced by Karasev and Maslov in the early 1980s and it takes values in a bundle of wave packets.

It was applied to a construction of spectral series for magnetic Laplacians on hyperbolic surfaces with constant magnetic fields (Bruening, Nekrasov, Shafarevich (2007)).

It would be interesting to compare these two approaches.

The trace formula for magnetic laplacian

$\lambda_{N,j}$ are the eigenvalues of Δ^{L^N}, $\eta_{N,j} = \sqrt{\lambda_{N,j} + N^2}$.
Fix $E > 1$ and put $X_E = \{\sqrt{|p|^2 + 1} = E\}$
The asymptotic distribution of $\eta_{N,j}$ in the intervals $(EN - \tau, EN + \tau)$ with arbitrary $\tau > 0$ is related with the geometry of the flow.
We assume that the set of the periods of the flow ϕ_t on X_E is discrete, for every period T the set of T-periodic points is a submanifold and the tangent spaces to it are exactly formed by fixed vectors of the tangent map ϕ^*_T (the cleanness condition)
Then for any $\varphi \in S(\mathbb{R})$ we have

$$Y_N(\varphi) = \sum_{j=0}^{\infty} \varphi(\eta_{N,j} - EN) \sim \sum_{j=0}^{\infty} c_j(N, \varphi) N^{d-j}, N \to \infty$$

(Guillemin–Uribe).

The trace formula for the magnetic laplacian on hyperbolic surfaces (Kordyukov–T.)

$$ds^2 = \frac{dx^2 + dy^2}{y^2}, F = \frac{dx \wedge dy}{y^2}$$

on $\widetilde{M}$, where $M = \mathcal{H}/\Gamma$, $g = \mathrm{genus}(M)$.

$$E < \sqrt{2},$$

i.e. $\frac{|p|^2}{2} < \frac{1}{2}$ (in this case all trajectories are closed). Then $d = 1$ and

$$c_0(N, \varphi) = (2g-2)E \sum_{k \in \mathbb{Z}} \hat{\varphi}\left(\frac{2\pi kE}{\sqrt{2 - E^2}}\right) \exp(ik\pi) \exp\left(2\pi ikN\sqrt{2 - E^2}\right).$$

Here $T_\gamma = \frac{2\pi kE}{\sqrt{2-E^2}}$ is the period and $S_\gamma = 2\pi k\sqrt{2 - E^2}$ is the action of a closed magnetic geodesic γ on X_E (with multiplicity k).

The Katok example

M is the two-sphere $\{x^2 + y^2 + z^2 = 1\}$ and in the spherical coordinates the Riemannian metric g on M is

$$g = \frac{d\theta^2}{1 - \varepsilon^2 \sin^2\theta} + \frac{\sin^2\theta}{(1 - \varepsilon^2 \sin^2\theta)^2} d\varphi^2$$

and the magnetic field form F is equal to

$$F = dA = \frac{\varepsilon \sin 2\theta}{(1 - \varepsilon^2 \sin^2\theta)^2} d\theta \wedge d\varphi,$$

where

$$A = -\frac{\varepsilon \sin^2\theta}{1 - \varepsilon^2 \sin^2\theta} d\varphi.$$

Modulo ε^2, this system describes the motion of a charged particle on the sphere of radius 1 in $\mathbb{R}^3$ in the external magnetic field

$$F = \frac{1}{2}\varepsilon\, dx \wedge dy.$$

For $\frac{|p|^2}{2} = \frac{1}{2}$ and $\varepsilon \in (0,1)$ irrational the only closed trajectories are the equators $\{z = 0\}$ passed in different directions.

The trace formula for the Katok example (Kordyukov–T.)

If 0 is a unique period in $\mathrm{supp}(\hat{\varphi})$, then

$$Y_N(\varphi) \sim \sum_{j=0}^{\infty} c_j(N,\varphi) N^{1-j}, \quad N \to \infty,$$

where $c_0(N,\varphi) = 2\sqrt{2}\hat{\varphi}(0)$ (Weyl's term).
If 0 is not in the support of $\hat{\varphi}$, then

$$Y_N(\varphi) \sim \sum_{j=0}^{\infty} c_j(N,\varphi) N^{-j}, \quad N \to \infty,$$

where

$$c_0(N,\varphi) = \sum_{k\neq 0} \frac{1}{\sqrt{2}(1-\varepsilon^2)}$$

$$\times \left(\frac{e^{ik\pi m_{k,+}/4} e^{-iNk\frac{2\pi}{1-\varepsilon}}}{\sin \frac{\pi k}{1-\varepsilon}} + \frac{e^{ik\pi m_{k,-}/4} e^{-iNk\frac{2\pi}{1+\varepsilon}}}{\sin \frac{\pi k}{1+\varepsilon}} \right) \hat{\varphi}\left(\frac{2\pi\sqrt{2}k}{1-\varepsilon^2}\right),$$

$$m_{k,\pm} = 2\left[\frac{2k}{1\mp\varepsilon}\right] + 2\,\mathrm{sign}\, k + 1.$$