Magnetic billiards in a strong constant magnetic field

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Classical mechanics, dynamical systems and mathematical physics,
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M. Bialy, A. Mironov, L. Shalom, *Magnetic billiards: Non-integrability for strong magnetic field; Gutkin type examples*, arXiv:2001.02119

M. Bialy, A. Mironov, *Algebraic non-integrability of magnetic billiards* J. of Physics A: Math. and Theoretical. 2016. Vol. 49. No. 45

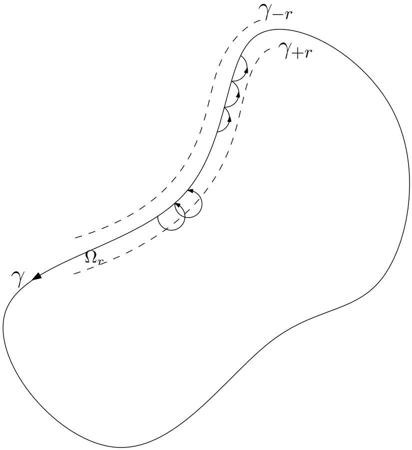


Рис.: Magnetic billiard.

Robnik, M.; Berry, M.V. *Classical billiards in magnetic fields.* J. Phys. A 18 (1985), no. 9, 1361–1378.

Kozlov, V.V.; Polikarpov, S.A. *On periodic trajectories of a billiard in a magnetic field.* (Russian) Prikl. Mat. Mekh. 69 (2005), no. 6, 942–949; translation in J. Appl. Math. Mech. 69 (2005), no. 6, 844–851 (2006).

Kozlov, V.V.; Treshchev, D.V. *Billiards. A Genetic Introduction to the Dynamics of Systems with Impacts.* Translated from the Russian by J. R. Schulenberger. Translations of Mathematical Monographs, 89. American Mathematical Society, Providence, RI, 1991.

Example

Let γ be a circle centered at the origin. Then the function which measures the distance of the center of Larmor circle to the origin is invariant under reflections and hence is an integral Φ of the billiard flow g^t. Specifically, Φ has the form:

$$\Phi(x, v) = x_1^2 + x_2^2 + \frac{2}{\beta}(v_1 x_2 - v_2 x_1).$$

where $x \in \gamma$ and v the unit inward vector at x.

Bolotin,S.V. *Integrable Birkhoff billiards.* Vestnik Moskov. Univ. Ser. I Mat. Mekh. 1990, no. 2, 33–36

Theorem

Let Ω be a bounded domain with a smooth boundary $\gamma = \partial\Omega$ such that the curvature assumption

$$\max|k| < \beta/2$$

is satisfied. Suppose that the magnetic billiard flow g^t in Ω admits a non-constant polynomial in momenta integral Φ. Then the curves $\gamma_{\pm r}$ are real ovals of affine algebraic curves which are non-singular in $\mathbf{C}^2$.

Corollary

For any non-circular domain Ω in the plane, the magnetic billiard inside Ω has no non-constant polynomial in momenta integral for all but finitely many values of β.

Corollary

Let Ω be the interior of the standard ellipse

$$\partial\Omega = \left\{ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \right\}, \quad 0 < b < a.$$

Then for any magnitude of the magnetic field $\beta > 2k_{\max} = \frac{2a}{b^2}$, the magnetic billiard in the ellipse does not admit a non-constant polynomial in momenta integral.

$$a^8(b^4+(r^2-y^2)^2-2b^2(r^2+y^2))+b^4(r^2-x^2)^2(b^4-2b^2(r^2-x^2+y^2)$$
$$+(x^2+y^2-r^2)^2)-2a^6(b^6+(r^2-y^2)^2(r^2+x^2-y^2)-b^4(r^2-2x^2+3y^2)$$
$$-b^2(r^4+3y^2(x^2-y^2)+r^2(3x^2+2y^2)))+2a^2b^2(-b^6(r^2+x^2)-$$
$$(-r^2+x^2+y^2)^2(r^4-x^2y^2-r^2(x^2+y^2))+b^4(r^4-3x^4+3x^2y^2$$
$$+r^2(2x^2+3y^2))+b^2(r^6-2x^6+x^4y^2-3x^2y^4+r^4(-4x^2+2y^2)$$
$$+r^2(5x^4-3x^2y^2-3y^4)))+a^4(b^8+2b^6(r^2+3x^2-2y^2)$$
$$+(r^2-y^2)^2(-r^2+x^2+y^2)^2-2b^4(3r^4-3x^4+5x^2y^2-3y^4+4r^2(x^2+y^2))$$
$$+2b^2(r^6-3x^4y^2+x^2y^4-2y^6+2r^4(x^2-2y^2)$$
$$+r^2(-3x^4-3x^2y^2+5y^4)))=0.$$

It is known to be irreducible. Moreover, the parallel curves $\gamma_{\pm r}$ have singularities in the complex plane for every r such that $\frac{1}{r}>2k_{max}=\frac{2a}{b^2}$, namely

$$(0,\pm\frac{\sqrt{b^2-a^2}\sqrt{a^2-r^2}}{a}),\quad (\pm\frac{\sqrt{a^2-b^2}\sqrt{b^2-r^2}}{b},0).$$

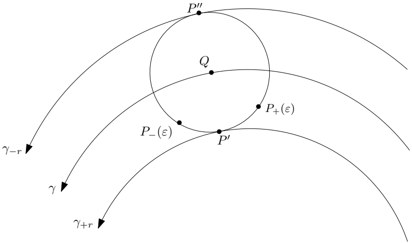


Рис.: Larmor centers $P_\pm$ lie symmetrically on the circle centered at $Q \in \gamma$.

The set of all Larmor centers fill the annulus Ω_r.

$$\mathcal{M} : \Omega_r \to \Omega_r, \ \mathcal{M}(P_-) = P_+.$$

The map $\mathcal{M} : \Omega_r \to \Omega_r$ preserves the standard symplectic form in the plane

Let us denote by $\mathcal{L}$ the mapping assigning the center of Larmor circle and any unit tangent vector to the circle:

$$\mathcal{L}(x, v) = x + rJv.$$

Given a polynomial integral $\Phi = \sum_{k+l=0}^{N} a_{kl}(x) v_1^k v_2^l$ of the magnetic billiard, we define the function $F : \Omega_r \to \Omega_r$ by the requirement

$$F \circ \mathcal{L} = \Phi.$$

This is a well-defined construction, since Φ is an integral of the magnetic flow, and therefore takes constant values on any Larmor circle. Moreover, since Φ is invariant under the billiard flow, F is invariant under the billiard map $\mathcal{M}$:

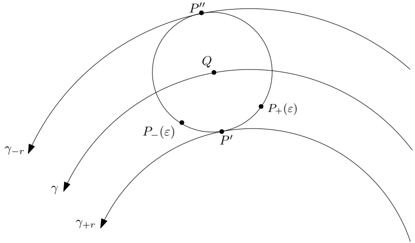
$$F \circ \mathcal{M} = F.$$

Theorem

F is a polynomial function in the variables (x_1, x_2) of degree at most $2N$ and

$$F|_{\gamma_{\pm r}} = \text{const.}$$

If $F|_{\gamma_{-r}} = c_1$ and $F|_{\gamma_{+r}} = c_2$, one can replace F by $F^2 - (c_1 + c_2)F + c_1 \cdot c_2$ to annihilate both constants c_1, c_2.



$$F(P_-(\epsilon)) = F(P_+(\epsilon))$$

$$F\left(x \pm r\frac{F_x(1-\cos\epsilon)+F_y\sin\epsilon}{|\nabla F|}, y \pm r\frac{F_y(1-\cos\epsilon)-F_x\sin\epsilon}{|\nabla F|}\right) -$$

$$F\left(x \pm r\frac{F_x(1-\cos\epsilon)-F_y\sin\epsilon}{|\nabla F|}, y \pm r\frac{F_y(1-\cos\epsilon)+F_x\sin\epsilon}{|\nabla F|}\right) =$$

The coefficient at ϵ^3 reads

$$(F_{xxx}F_y^3 - 3F_{xxy}F_y^2F_x + 3F_{xyy}F_yF_x^2 - F_{yyy}F_x^3)+$$

$$3\beta(F_x^2+F_y^2)^{\frac{1}{2}}(F_{xx}F_xF_y+F_{xy}(F_y^2-F_x^2)-F_{yy}F_xF_y) = 0, (x,y) \in \gamma_{+r}.$$

Remarkably, the left-hand side is the complete derivative along the tangent vector field v to γ_{+r}, $v = (F_y, -F_x)$, of the following expression, which therefore must be constant:

$$H(F) \pm \beta|\nabla F|^3 = \mathrm{const}, \quad (x,y) \in \gamma_{+r};$$

here we used the notation

$$H(F) := F_{xx}F_y^2 - 2F_{xy}F_xF_y + F_{yy}F_x^2.$$

We say magnetic billiard satisfies δ-Gutkin property, $0 < \delta < \pi$, if any Larmor arc C entering Ω with the angle δ with γ exits Ω with the same angle δ with γ as well. Explicit examples of billiard tables with this property, which can be obtained by deformation of the circle were constructed for ordinary planar billiards by E. Gutkin. Gutkin found that for planar Gutkin billiards, the number δ satisfies the equation

$$\tan n\delta = n \tan \delta.$$

Theorem

For every $\delta \in (0, \pi)$ there exists a non-circular magnetic billiard in the plane with Gutkin property.

For the proof we reduce the question to a very beautiful Wegner examples which provide solutions to the so called floating problem (S.Ulam problem number 19 of the Scottish cafe book).

Let Ω be a magnetic billiard domain with the Gutkin property corresponding to the angle δ. Denote by $\Gamma \subset \Omega_r$ the curve consisting of Larmor centers of all the arcs having angle δ with the boundary. Magnetic billiard has δ-Gutkin property if and only if the curve Γ is invariant under $\mathcal{M}$.

Theorem

It then follows that Γ is a Zindler curve. Namely, moving the segment inscribed into Γ of constant length

$$L = 2r \sin \delta$$

around Γ is such that the velocity of the midpoint M of the segment is necessarily parallel to the segment. The midpoints of these segments form the curve $\tilde{\Gamma}$ which is parallel to γ at the distance $r \cos \delta$.

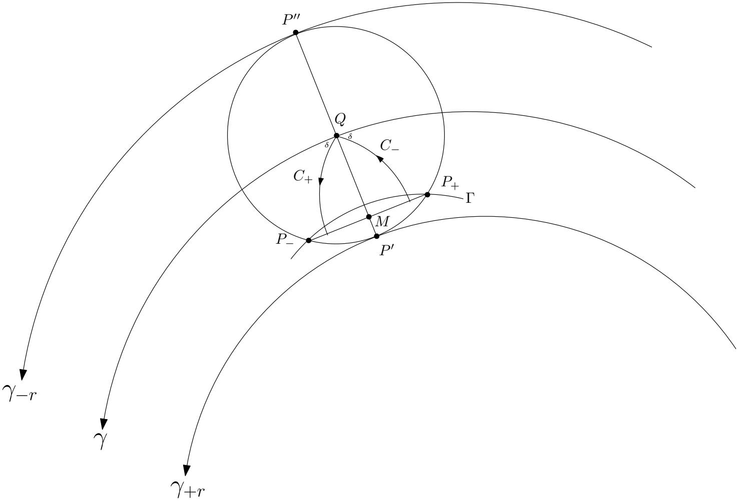


Рис.: Curve Γ of the Larmor centers.

The triangle $\Delta P_- P' P_+$ is rigid.

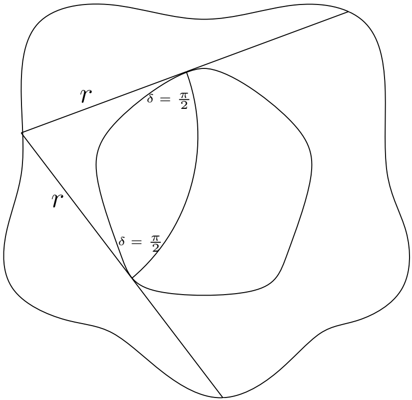


Рис.: Magnetic billiard with Gutkin property for $\delta = \pi/2$.

Corollary

Suppose $\delta = \pi/2$, $r\cos\delta = 0$. The curve $\tilde{\Gamma}$ coinsides with γ. Thus for Wegner curve Γ constructed for the length $2r$ of the inscribed segment, the curve of the midpoints of the segments is an example of Gutkin magnetic billiard.