CONFORMAL GEOMETRY IN MATHEMATICAL PHYSICS

Dedicated to 70th anniversary of V.V. Kozlov

Preliminaries

Pleriminaries. Immersions

An immersion of smooth two-dimensional manifold Λ is a mapping $\Phi : \Lambda \to \mathbb{R}^3$ such that for every $X \in \Lambda$, the differential $D\Phi(X)$ has the rank 2.

We say that Φ is *a weak immersion* if it is Lipschitz and for a.e. $X \in \Lambda$, the mapping $D\Phi(X)$ has the rank 2 and defines two-dimensional tangent space to $\Sigma = \Phi(\Lambda)$ at the point $x = \Phi(X)$.

If the mapping Φ is a bijection, we will say that Φ is an embedding. Notice that $\Sigma = \Phi(\Lambda)$ is a rectifiable surface in $\mathbb{R}^3$.

Preliminaries. Fundamental forms.

Hereinafter the notation $X = (X_1, X_2)$ stands for points in $\mathbb{R}^2$. We will write

$$\partial_i := \partial_{X_i} := \frac{\partial}{\partial X_i}, \quad i = 1, 2.$$

Two fundamental forms play important role in the further considerations. Their definitions are local and it suffices to define them for immersions of open set $\omega \subset \mathbb{R}^2$.

The first fundamental form

Let $\Phi : \omega \to \mathbb{R}^3$ be a Lipschitz immersion. The first fundamental form $\mathbf{g}$ at point $X \in \omega$ is defined by the equality

$$g_{ij}(X) = \partial_i \Phi(X) \cdot \partial_j \Phi(X), \quad X \in \omega, \quad i, j = 1, 2. \tag{1}$$

We will identify $\mathbf{g}$ with the symmetric matrix with the entries g_{ij}. Set

$$g := \det \mathbf{g}.$$

Recall that $\sqrt{g}\, dX_1 dX_2$ is the area element of $\Sigma := \Phi(\omega)$. We also denote by $\mathbf{g}^{-1}$ the inverse matrix.

Isothermal immersions

The immersion Φ is isothermal if and only if

$$g_{11} = g_{22}, \quad g_{12} = 0. \tag{2}$$

In this case we have

$$g_{ii}(X) = e^{2f(X)}, \quad \partial_i \Phi(X) = e^{f(X)}\ \mathbf{e}_i(X)$$

Here the unit orthogonal vectors $\mathbf{e}_i$,

$$|\mathbf{e}_i(X)| = 1, \quad \mathbf{e}_1(X) \cdot \mathbf{e}_2(X) = 0, \tag{3}$$

form an orthogonal basic in the tangential space $T_x\Sigma$ at the point $x = \Phi(X) \in \mathbb{R}^3$. The quantity e^f is named as the conformal factor.

The second fundamental form.

For the immersion $\Phi : \omega \to \mathbb{R}^3$, $\omega \subset \mathbb{R}^2$ the unit normal vector is defined by

$$\mathbf{e}_3 := \mathbf{n} = \frac{\partial_1 \Phi \times \partial_2 \Phi}{|\partial_1 \Phi \times \partial_2 \Phi|}. \tag{4}$$

In the differential geometry, the classic second fundamental form $\mathbf{b}$ is defined as a 2×2 matrix with the entries

$$b_{\alpha\beta} = -(\partial_\alpha \Phi \cdot \partial_\beta \mathbf{n}) \equiv (\partial_{\alpha,\beta} \Phi \cdot \mathbf{n}). \tag{5}$$

Notice that the differential form $b_{\alpha\beta} dX_\alpha dX_\beta$ defines the second order tensor.

Curvatures. The length of the second fundamental form

Next, introduce an orthonormal basic $\mathbf{e}_1, \mathbf{e}_2$ in $\mathrm{Tan}\Sigma_x$ such that the triplet $(\mathbf{e}_i)$ has the positive orientation. The coordinates dY of the vectors in the tangent space in the basis $\mathbf{e}_i$ are connected with the coordinates dX in the basis $\mathbf{t}_i = \partial_i \mathbf{\Phi}$ by the relation $dX = \mathbf{G}dY$. Here $\mathbf{G}$ is some matrix, depending on the choice of $\mathbf{e}_i$. *The second fundamental form* $\mathbf{A}(X)$ *is connected with the form* $\mathbf{b}$ *by the relation* $\mathbf{A} = \mathbf{G}^*\mathbf{b}\mathbf{G}$.

The principal curvatures k_1, k_2 of Σ are defined as the eigenvalues of the matrix $\mathbf{A}$.

The curvatures k_i *coincide with the eigenvalues of the matrix* $\mathbf{g}^{-1/2}\mathbf{b}\mathbf{g}^{-1/2}$, *or equivalently with the eigenvalues of the matrices* $\mathbf{g}^{-1}\mathbf{b}$ *and* $\mathbf{b}\mathbf{g}^{-1}$.

Curvatures. The length of the second fundamental form

The mean curvature

$$H = \frac{1}{2}(k_1 + k_2),$$

The Gauss curvature

$$K = k_1 k_2,$$

The length of the second fundamental form

$$|\mathbf{A}|^2 = k_1^2 + k_2^2.$$

Useful formulae

Let $\mathbf{\Phi} : \omega \to \mathbb{R}^3$, $\omega \subset \mathbb{R}^2$, be an isothermal immersion and $\Sigma = \mathbf{\Phi}(\omega)$. Then

$$\int_\Sigma K \, d\Sigma = \int_\omega \mathbf{n} \cdot (\partial_1 \mathbf{n} \times \partial_2 \mathbf{n}) \, dX.$$

$$\int_\Sigma |\mathbf{A}|^2 \, d\Sigma = \int_\omega |\nabla \mathbf{n}|^2 \, dX$$

$$-\Delta f \; = \; \mathbf{n} \cdot (\partial_1 \mathbf{n} \times \partial_2 \mathbf{n}) \equiv \partial_1 \mathbf{e}_1 \cdot \partial_2 \mathbf{e}_2 - \partial_2 \mathbf{e}_1 \cdot \partial_1 \mathbf{e}_2.$$

Application. Geometry.

The Willmore (Thomsen (1923), Blaschke (1929), Willmore (1965)) functional

$$W(\Sigma) = \int_{\Sigma} |H|^2 \, d\Sigma$$

Applications

Application. Elasticity.

Germain (1821), Poisson (1816), Kirchhoff(1850), Landau & Lifschitz (1986), Friesecke, James & Muller, (2002),

$$a\int_\Sigma |\mathbf{A}|^2\, d\Sigma + b\int_\Sigma |H|^2\, d\Sigma$$

Application. Helfrich functional

$$\mathcal{H}(\Sigma) = \frac{k_c}{2} \int_{\Sigma} (H - c_0)^2 \, d\Sigma + 2k\pi\chi(\Sigma).$$

$$\mathcal{H}(\Sigma) + \lambda \text{ vol } G, \quad \Sigma = \partial G.$$

Helfrich (1973)
Lipowsky and Sackman (1995)
Keller, Mondino, Tristan Riviero (2014)

Application. Hawking mass-energy

$$\frac{\text{vol } \Sigma^{1/2}}{(16\pi)^{3/2}}\Big(16\pi - \int_\Sigma H^2\, d\Sigma\Big)$$

Huisken and Ilmanen (2001)
Koerber (2018)

Application. Hydroelastic waves

Characterized by the Lagrangian A periodic elastic shell S is a boundary of flow domain $G \subset \mathbb{R}^3$.

Σ, Ω are periodic cells of S and G

$$\mathcal{L} = \mathcal{E}_{el} + \mathcal{E}_{gr} - \mathcal{E}_{kin},$$

$$\mathcal{E}_{el} = c_b \int_{\Sigma} |\mathbf{A}|^2 ds + c_s \text{ area } \Sigma,$$

$$\mathcal{E}_{gr} = \frac{\lambda}{2} \int_{\Sigma} (x^3)^2 n^3 \, d\Sigma \tag{6}$$

$$\mathcal{E}_{kin} = \frac{1}{2} \int_{\Omega} |\nabla u|^2 + \int_{\Sigma} u n^1 \, d\Sigma + \frac{1}{2} \int_{\Sigma} x^3 n^3 \, d\Sigma.$$

$$\mathcal{E}_{el} \equiv c_e \mathcal{E}_e, \quad \mathcal{E}_e = \int_{\Sigma} |\mathbf{A}|^2 \, d\Sigma + 8\pi^2 \text{area } \Sigma.$$

Application. Hydroelastic waves

Ambrose DM. & Siegel M. 2017; Deacon N., Părău EI.& Whittaker R. 2015; Gao T., Wang Z. & Vanden-Broeck J.-M. 2016; Groves MD., Hewer B. & Wahlen E. 2016; Guenne P. & Părău EI. 2012 ; Liu S. & Ambrose DM. 2017; Milewski P. A. Vanden-Broeck JM. Wang Z. 2011; Milewski P. A. & Wang Z. 2013; Milewski P. A., Vanden-Broeck J. M. & Wang Z. 2013; Părău EI. & Vanden-Broeck J-M. 2011; Toland JF. 2008; Plotnikov PI. & Toland JF. 2012; Plotnikov PI. & Toland JF. 2011; Vanden-Broeck J-M. & Părău EI. 2011.

Periodic surfaces. Periods

Let $\mathbf{l}_1, \mathbf{l}_2 \in \mathbb{R}^2$ be a linearly independent vectors, which form a positive oriented frame in the plane. Denote by Γ the corresponding lattice in $\mathbb{R}^2$,

$$\Gamma = \{m\mathbf{l}_1 + n\mathbf{l}_2, \quad (m,n) \in \mathbb{Z}^2\}, \tag{7}$$

and by Π_Γ the fundamental cell of the lattice Γ,

$$\Pi_\Gamma = \{\alpha\mathbf{l}_1 + \beta\mathbf{l}_2, \quad \alpha, \beta \in [0,1]\}. \tag{8}$$

Further we restrict our considerations by the lattices Γ satisfying the following condition.

$$\mathbf{l}_1 = (l_1, 0), \quad \mathbf{l}_2 = (l_2, l_1^{-1},), \quad l_1 > 0. \tag{9}$$

In particular, we have area $\Pi_\Gamma = 1$.
Next we consider the rectangular lattice

$$\{m\mathbf{i} + n\mathbf{j}, \quad (m,n) \in \mathbb{Z}^2\}, \quad \mathbf{i} = (1,0,0), \quad \mathbf{j} = (0,1,0). \tag{10}$$

Periodic surfaces.

We say that a weak immersion $\Phi : \mathbb{R}^2 \to \mathbb{R}^3$ belongs to the class $\mathcal{S}$ if there is a lattice Γ with the properties. The mapping Φ admits the representation

$$\Phi(X) = (X \cdot \mathbf{l}'_1)\,\mathbf{i} + (X \cdot \mathbf{l}'_2)\,\mathbf{j} + \Phi_0(X), \tag{11}$$

where $\Phi_0 : \mathbb{R}^2 \to \mathbb{R}^3$ is a Lipschitz Γ- periodic mapping, the vectors $\mathbf{l}'_i$ form a lattice dual to Γ, i.e.,

$$\mathbf{l}'_k \cdot \mathbf{l}_m = \delta_{km} \quad \text{or simply} \quad \mathbf{l}'_1 = (l_1^{-1}, -l_2), \quad \mathbf{l}'_2 = (0, l_1).$$

Furthermore, the first fundamental form of the immersion Φ is bounded and has the bounded inverse, and the second fundamental form of Φ is square integrable over period. This means that

$$\|\mathbf{g}\|_{L^\infty(\Pi_\Gamma)} + \|\mathbf{g}^{-1}\|_{L^\infty(\Pi_\Gamma)} < \infty, \quad \int_{\Pi_\Gamma} |\nabla \mathbf{n}(X)|^2\, dX < \infty \tag{12}$$

Periodic surfaces.

The invariant geometric characteristics of the surface $\Sigma = \Phi(\Pi_\Gamma)$ are connected with the parametrization $x = \Phi(X)$ by the equalities

$$\int_\Sigma |\mathbf{A}|^2 \, d\Sigma = \int_{\Pi_\Gamma} \mathbf{G} \, \partial_i \mathbf{n} \cdot \partial_i \mathbf{n} \, dX, \quad \mathbf{G} = g^{-1/2}\mathbf{g}, \qquad \text{area } \Sigma = \int_{\Pi_\Gamma} g^{1/2} \, dX. \tag{13}$$

In isothermal (conformal) coordinates

$$\int_\Sigma |\mathbf{A}|^2 \, d\Sigma = \int_{\Pi_\Gamma} |\nabla \mathbf{n}|^2 \, dX, \qquad \text{area } \Sigma = \int_{\Pi_\Gamma} e^{2f} \, dX. \tag{14}$$

Modified Lagrangian

Introduce the smooth convex function $\Xi : [8\pi^2, 32\pi) \to \mathbb{R}$ such that

$$\Xi'(s) > 0, \quad \Xi(8\pi^2) = 0, \quad \Xi(s) \to \infty \text{ as } s \nearrow 32\pi.$$

Modified energy functional

Define the modified energy functional $\mathcal{L}_{mod}$ as follows

$$\mathcal{L}_{mod}(S) = \Xi(\mathcal{E}e) + \mathcal{E}_g - \mathcal{E}_{kin},$$

where

$$\mathcal{E}_e = \int_\Sigma |\mathbf{A}|^2 \, d\Sigma + 8\pi^2 \cdot \text{ area } \Sigma.$$

$$\mathcal{E}_g(\Sigma) = -\frac{1}{2} \int\limits_\Sigma (x_3)^2 \, d\Sigma$$

$$\mathcal{E}_{kin} = \frac{1}{2} \int\limits_\Omega |\nabla\phi|^2 \, dx + \int\limits_{\Omega \cap \{x_3 > -N\}} \nabla\phi \cdot \mathbf{i} \, dx + \frac{1}{2} \int\limits_\Sigma x_3 \; n_3 \, d\Sigma.$$

Variational problem.

Denote by $\mathcal{S}_0$ the class of weak immersions $x = \mathbf{\Phi}(X)$ of the class $\mathcal{S}$ such that

$$\int_\Sigma x_3\, n_3\, d\Sigma = 0.$$

and consider the following variational problem

$$\min_{\mathcal{S}_0} \mathcal{L}_{mod}(S) \tag{15}$$

Notice that every solution of problem (15) *is a critical point of the lagrangian* $\mathcal{L}$ *with* $c_b = \Xi'(\mathcal{E}_{el})$

Geometry of surfaces with L^2 second fundamental form

Literature

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Isothermal coordinates

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Isothermal coordinates

Theorem. *Let Φ be an immersion of the class $\mathcal{S}$ with the lattice of periods Γ. Then there exist a lattice*

$$\Upsilon = \{m\mathbf{s}_1 + n\mathbf{s}_2, (m,n) \in \mathbb{Z}^2\}, \quad \mathbf{s}_1 = (\mu, 0), \mathbf{s}_2 = (\nu, \mu^{-1}), \ \mu > 0, \tag{16}$$

and bi-Lipschitz homeomorphism $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$c_1^{-1}|\varphi(X') - \varphi(X)| \le |X' - X| \le c_1|\varphi(X') - \varphi(X)| \tag{17}$$

for all $X, X' \in \mathbb{R}^2$. The mapping $\mathbf{\Psi}(X) := \mathbf{\Phi}(\varphi^{-1}(X))$ admits the representation

$$\mathbf{\Psi}(X) = (\mathbf{s}_1' \cdot X)\,\mathbf{i} + (\mathbf{s}_2' \cdot X)\,\mathbf{j} + \mathbf{\Psi}_0(X), \tag{18}$$

where the vectors $(\mathbf{s}_1', \mathbf{s}_2')$ are dual to $(\mathbf{s}_1, \mathbf{s}_2)$, and $\mathbf{\Psi}_0$ is the Lipschitz Υ-periodic mapping.

Isothermal coordinates

The immersion $\mathbf{\Psi} : \mathbb{R}^2 \to \mathbb{R}^3$ *is conformal, i.e.,*

$$\partial_j \mathbf{\Psi}(X) = e^{f(X)} \mathbf{e}_j(X), \quad \mathbf{e}_j \cdot \mathbf{e}_i = \delta_{ji}, \tag{19}$$

The Υ*- periodic vectors* $\mathbf{e}_i$*, the logarithm* f *of the conformal factor, and the parameters of the lattice* Υ *admit the estimates*

$$\|\mathbf{e}_i\|_{W^{1,2}(\Pi_\Upsilon)} + \|f\|_{L^\infty(\Pi_\Upsilon)} + \|f\|_{W^{1,2}(\Pi_\Upsilon)} \le c_2. \tag{20}$$

$$|\mu| + |\mu^{-1}| + |\nu| \le c_3. \tag{21}$$

Here the constants c_i *depend on* Φ *and* Γ*. The vector fields* $\mathbf{e}_i(X)$ *are orthogonal to* $\mathbf{n} := \mathbf{n}(\varphi^{-1}(X))$ *and the triplet* $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{n})$ *has the positive orientation. The immersion* $\mathbf{\Psi}$ *is unique with the accuracy up to the shift of the independent variable, i.e., if there are two immersions* $\mathbf{\Psi}$ *and* $\mathbf{\Psi}^*$ *which meet all requirements of the theorem, then* $\mathbf{\Psi}(X) = \mathbf{\Psi}^*(X + \text{const.})$

Moving frame method.

An orthonormal moving frame is the mapping

$$X \mapsto (\mathbf{b}_1(X), \mathbf{b}_2(X)), \quad X \in \mathbb{R}^2, \mathbf{b}_i \in \mathbb{R}^3$$

such that

$$\mathbf{b}_1 \cdot \mathbf{b}_2 = 0, \quad |\mathbf{b}_1| = |\mathbf{b}_2|.$$

It is easy to see that for every measurable $\theta(X)$, relations

$$\mathbf{e}_1 = \cos\theta \ \mathbf{b}_1 - \sin\theta \ \mathbf{b}_2, \quad \mathbf{e}_2 = \sin\theta \ \mathbf{b}_1 + \cos\theta \ \mathbf{b}_2$$

which can be rewritten in the complex form

$$\mathbf{e}_1 + i\mathbf{e}_2 = e^{i\theta}(\mathbf{b}_1 + i\mathbf{b}_2), \tag{22}$$

also define an orthonormal moving frame in $\mathbb{R}^2$.

Moving frame method.

Introduce the special vector field

$$\mathbf{b}_1 \cdot d\,\mathbf{b}_2 \;:= (\mathbf{b}_1 \cdot \partial_1 \mathbf{b}_2,\; \mathbf{b}_1 \cdot \partial_2 \mathbf{b}_2), \quad \mathbf{b}_1 \cdot d\,\mathbf{b}_2 : \mathbb{R}^2 \to \mathbb{R}^2. \quad (23)$$

For

$$\mathbf{e}_1 + i\mathbf{e}_2 = e^{i\theta}(\mathbf{b}_1 + i\mathbf{b}_2),$$

we have

$$\mathbf{e}_1 \cdot d\,\mathbf{e}_2 = \nabla\theta + \mathbf{b}_1 \cdot d\,\mathbf{b}_2.$$

We say that the orthonormal moving frame $(\mathbf{e}_1, \mathbf{e}_2)$ is a Coulomb frame if

$$\mathrm{div}\;\; \mathbf{e}_1 \cdot d\,\mathbf{e}_2 = 0$$

For every Coulomb frame there is a conformal factor $f(X)$ defined by

$$\nabla f = (\mathbf{e}_1 \cdot d\,\mathbf{e}_2)^{\perp}.$$

Coulomb moving frame.

Let Γ be some lattice of periods and Π_Γ be a fundamental cell of the lattice Γ. We say that an orthonormal Γ-periodic moving frame has the finite energy if

$$\int_{\Pi_\Gamma} |\nabla \mathbf{b}_i|^2 \, dX < \infty.$$

Coulomb frame orthogonal to given vector field.

Theorem. *Let* Γ *be an arbitrary lattice generated by the periods*

$$\mathbf{l}_1 = (l_1, 0), \quad \mathbf{l}_2 = (l_2, l_1^{-1}), \quad l_1 > 0.$$

Let Π_Γ *be a fundamental cell of periods , and* $\mathbf{n} : \mathbb{R}^2 \to \mathbb{S}^2$ *be a* Γ*-periodic vector field satisfying the conditions*

$$\int_{\Pi_\Gamma} |\nabla \mathbf{n}|^2 \, dX < c < \infty, \int_{\Pi_\Gamma} \mathbf{n} \cdot (\partial_1 \mathbf{n} \times \partial_2 \mathbf{n}) \, dX = 0. \qquad (24)$$

Furthermore assume that there are $r > 0$ *and* $\delta > 0$ *such that*

$$\int_{D_r(a)} |\nabla \mathbf{n}|^2 \, dX \leq \frac{8\pi}{3} - \delta \quad \text{for all} \quad a \in \mathbb{R}^2. \qquad (25)$$

Coulomb frame orthogonal to given vector field.

Then there exists Coulomb frame $(\mathbf{e}_1, \mathbf{e}_2)$ *and* Γ*-periodic function* $f : \mathbb{R}^2 \to \mathbb{R}$ *with the following properties.*

$$\mathbf{e}_1(X) + i\mathbf{e}_2(X) = e^{i\boldsymbol{\mu}\cdot X}\ \big(\mathbf{b}_1(X) + i\mathbf{b}_2(X)\big), \quad X \in \mathbb{R}^2. \tag{26}$$

Here Γ*-periodic vector fields* $\mathbf{b}_i$ *satisfy the conditions*

$$\mathbf{b}_i \cdot \mathbf{b}_j = \delta_{ij}, \quad b_i \in W^{1,2}(\Pi_\Gamma), \tag{27}$$

and $\boldsymbol{\mu} \in \mathbb{R}^2$ *is a constant vector. The vector field* $\mathbf{e}_1 \cdot d\mathbf{e}_2$ *is* Γ*-periodic and*

$$\int_{\Pi_\Gamma} \mathbf{e}_1 \cdot d\mathbf{e}_2\, dx = 0. \tag{28}$$

The function f *and the vector fields* $\mathbf{e}_i$, $\boldsymbol{\mu}$ *satisfy the inequalities*

$$\int_{\Pi_\Gamma} |\nabla \mathbf{b}_i|^2\, dX + \int_{\Pi_\Gamma} |\nabla f|^2\, dx + |\mu| \le c_1, \quad |f| \le c_1, \tag{29}$$

where c_1 *depends only on* r, δ *and* $\max|l_i|$ *.*

Coulomb frame orthogonal to given vector field.

They satisfy the equations

$$\partial_1 f = -\mathbf{e}_1 \cdot \partial_2 \mathbf{e}_2, \quad \partial_2 f = \mathbf{e}_1 \cdot \partial_1 \mathbf{e}_2 \ \text{ in } \ \mathbb{R}^2, \tag{30}$$

$$-\Delta f = \partial_1 \mathbf{e}_1 \cdot \partial_2 \mathbf{e}_2 - \partial_2 \mathbf{e}_1 \cdot \partial_1 \mathbf{e}_2 \ \text{ in } \ \mathbb{R}^2, \quad \int_{\Pi_\Gamma} f \, dX = 0. \tag{31}$$

Moreover, the right hand side of equation (31) *satisfies the equality*

$$\partial_1 \mathbf{e}_1 \cdot \partial_2 \mathbf{e}_2 - \partial_2 \mathbf{e}_1 \cdot \partial_1 \mathbf{e}_2 = \mathbf{n} \cdot (\partial_1 \mathbf{n} \times \partial_2 \mathbf{n}). \tag{32}$$

the vectors $\mathbf{e}_i$ *are orthogonal to* $\mathbf{n}$ *and the triplet* $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{n})$ *has the positive orientation.*

Hadamard Theorem

Theorem *Let a Lipschitz mapping* $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ *satisfies the following conditions*

$$\|D\varphi(X)\| + \|(D\varphi(X))^{-1}\| \leq M < \infty, \quad \det D\varphi(X) > 0 \ \ a.e.\ in\ \ \mathbb{R}^2, \tag{33}$$

where the Jacobi matrix $D\varphi$ *of the map* φ *is the matrix with the entries* $\partial_j \varphi_i$*, the constant* M *is independent of* X*. Furthermore, we assume that the matrix* $D\varphi$ *has the representation*

$$D\varphi(X) = \begin{pmatrix} \cos(\boldsymbol{\mu} \cdot X), & \sin(\boldsymbol{\mu} \cdot X) \\ -\sin(\boldsymbol{\mu} \cdot X), & \cos(\boldsymbol{\mu} \cdot X) \end{pmatrix} \mathbf{G}(X), \tag{34}$$

where $\boldsymbol{\mu}$ *is a constant vector and* $\mathbf{G}$ *is* Γ*-periodic matrix.*

Hadamard Theorem

Then there are $m, n \in \mathbb{Z}$ *such that*

$$\boldsymbol{\mu} = 2\pi m\, \mathbf{l}'_1 + 2\pi n\, \mathbf{l}'_2,$$

where $\mathbf{l}'_i$ *are dual vectors to the periods* $\mathbf{l}_i$*, i.e.,* $\mathbf{l}'_i \cdot \mathbf{l}_j = \delta_{ij}$*. This means that the Jacobian* $D\varphi$ *is* Γ*- periodic. Moreover,* φ *takes homeomorphically* $\mathbb{R}^2$ *onto* $\mathbb{R}^2$*. There is a lattice*

$$\mathcal{C} = \{m\mathbf{c}_1 + n\mathbf{c}_2,\ \ m, n \in \mathbb{Z}\},$$

with linearly independent periods $\mathbf{c}_i$*,*

$$|\mathbf{c}_i| + |\mathbf{c}'_i| \le c(c_\gamma, M). \tag{35}$$

such that the inverse φ^{-1} *admits the representation*

$$\varphi^{-1}(Y) = \mathbf{l}_1(\mathbf{c}'_1 \cdot Y) + \mathbf{l}_2(\mathbf{c}'_2 \cdot Y) + \varphi_0(Y). \tag{36}$$

Here φ_0 *is the Lipschitz* $\mathcal{C}$*-periodic mapping.*

Local structure theorem.

Let us consider immersion $x = \mathbf{u}(X)$ of the unit disk D_1 to $\mathbb{R}^3$ and the surface $M = \mathbf{u}(D_1)$ with the properties

H.1

$$\partial_i \mathbf{u}(X) = e^f(X)\mathbf{e}_i(X), \quad \mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}, \text{ and } \mathbf{u}(0) = 0. \qquad (37)$$

$$|\mathbf{A}(X)| = e^{-f(X)}|\nabla \mathbf{n}(X)| \text{ for } X \in D_1. \qquad (38)$$

H.2

$$\|\nabla f\|_{L^2(D_1)} + \|\nabla \mathbf{e}_i\|_{L^2(D_1)} + \|\nabla \mathbf{n}\|_{L^2(D_1)} \leq \varepsilon. \qquad (39)$$

H.3

$$\lambda^{-1} \leq e^{f(X)} \leq \lambda < \infty \text{ in } D_1, \quad \int_{D_1} f(X)\, dX = 0. \qquad (40)$$

Local structure theorem.

Let us consider immersion $x = \mathbf{u}(X)$ of the unit disk D_1 to $\mathbb{R}^3$ and the corresponding surface $M = \mathbf{u}(D_1)$ with the following properties

$$\partial_i \mathbf{u}(X) = e^f(X)\mathbf{e}_i(X), \quad \mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}, \text{ and } \mathbf{u}(0) = 0. \qquad (41)$$

Notice that the normal vector $\mathbf{n} = \mathbf{e}_1 \times \mathbf{e}_2$. The vector fields $\mathbf{e}_i$ and $\mathbf{n}$ define a moving orthonormal frame on the surface M. In this framework, the length of the second fundamental form $\mathbf{A}$ of M is defined by the equality

$$|\mathbf{A}(X)| = e^{-f(X)}|\nabla \mathbf{n}(X)| \text{ for } X \in D_1. \qquad (42)$$

Local structure theorem.

$$\|\nabla f\|^2_{L^2(D_1)} + \|\nabla \mathbf{e}_i\|^2_{L^2(D_1)} + \|\nabla \mathbf{n}\|^2_{L^2(D_1)} \le \varepsilon_0^2, \tag{43}$$

where ε_0 is a small parameter.

$$\lambda^{-1} \le e^{f(X)} \le \lambda < \infty \quad \text{in} \quad D_1, \quad \int_{D_1} f(X)\, dX = 0. \tag{44}$$

Local structure theorem.

$$\overline{\mathbf{n}} = \Big| \int_{D_1} \mathbf{n}\, dX \Big|^{-1} \int_{D_1} \mathbf{n}\, dX. \qquad (45)$$

Lemma. *There exist an absolute constants ε_0 and c with the following property. For every $\varepsilon \in (0, \varepsilon_0)$ there is the orthonormal frame $(\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3)$ such that*

$$\mathbf{b}_3 = \overline{\mathbf{n}}, \quad \Big| \mathbf{b}_i - \frac{1}{\pi} \int_{D_1} \mathbf{e}_i\, dX \Big| \le c\varepsilon. \qquad (46)$$

Now choose a cartesian system of coordinates $x = (x_1, x_2, x_3)$ in the ambient space $\mathbb{R}^3$ such that the origin of this system is in the point $\mathbf{u}(0)$, and the directions of the coordinate axis Ox_i coincide with the directions of $\mathbf{b}_i$.

Local structure theorem.

$$\overline{x} = (x_1, x_2). \tag{47}$$

We have

$$x = (\overline{x}, x_3). \tag{48}$$

In the planes of the variables $\overline{x}$ introduce the polar coordinates

$$\overline{x} = r(\cos\varphi, \sin\varphi). \tag{49}$$

Local structure theorem.

Theorem. *Let* $K \in (0,1)$. *Then there exist positive constants* ε_0, c_i, *depending only on* λ *and* K *such that the following assertions hold for every* $\varepsilon \in (0, \varepsilon_0)$.

(i) The surface M *is in the layer* $|x_3| \leq c_1\varepsilon$. *For every* $r \leq K$, *the intersection of the cylinder* $|\overline{x}| = r$ *with the boundary of the surface* M *is empty.*

(ii)There is a set $\mathcal{F} \subset [K/100, K]$ *with the positive measure* meas $\mathcal{F} \geq 98K/100$ *such that for every* $r \in \mathcal{F}$ *the intersection of the cylinder* $|\overline{x}| = r$ *with the surface* M *coincides with a* C^1 *Jordan curve* γ.

Local structure theorem.

(iv) In the cylinder coordinates (r, φ, x_3) *the curve* γ *is given by equation*

$$x_3 = \eta_r(\varphi), \quad \varphi \in [0, 2\pi], \quad \eta_r(0) = \eta_r(2\pi), \tag{50}$$

where

$$|\eta_r'(\varphi)| + |\eta_r(\varphi)| \le c_3\varepsilon, \quad \varphi \in [0, 2\pi]. \tag{51}$$

Local graph approximation theorem.

Theorem. *Let an immersion* $\mathbf{u} : D_1 \to \mathbb{R}^3$ *meets all requirements of the Local structure theorem. Then there exist a system of disjoint discs* $D_{\rho_n}(X_n)$, $X_n \in D_1$ *and the Lipschitz immersion* $\mathbf{u}^* : D_1 \to \mathbb{R}^3$ *with the following properties. For every* $\alpha \in (0,1)$,

$$\sum_m \rho_n^\alpha \le c(\alpha) e^{-\frac{1}{\sqrt{\varepsilon}}}$$

$$\mathbf{u} = \mathbf{u}^* \quad in \quad D_1 \setminus \Big(\bigcup_n D_{\rho_n}(X_n) \Big)$$

$\mathbf{u}^*$ *is a graph of the Lipschitz function*

Jordan-Brouwer theorem.

Jordan-Brouwer theorem.

Theorem. *Let an immersion* Φ *belongs to the class* $\mathcal{S}$ *and the surface* $S = \Phi(\mathbb{R}^2)$ *has no selfintersections. Then the open set* $\mathbb{R}^3 \setminus S$ *consists of two infinite connected components* $G^{\pm}$ *such that for all sufficiently large* N,

$$\{x_3 < -N\} \subset G^{-}, \quad \{x_3 > N\} \subset G^{+}.$$

Periodic surfaces. Preventing self- intersections

Let $\mathbf{\Phi} \in \mathcal{S}$ and $S = \mathbf{\Phi}(\mathbb{R}^2)$. If

$$\int_{\Sigma} |\mathbf{A}|^2 \, d\Sigma + 8\pi^2 \text{area } \Sigma < 32\pi - \delta.$$

then S has no self-intersections.
Li P. & Yau ST. 1982: A closed surface has no self intersections if its Willmore energy is less than 8π.

Sobolev spaces. Boundary value problem

Sobolev space.

Let $S \in \mathcal{S}$ satisfies the conditions of Jordan-Brouwer Theorem. Let G be a domain bounded by S and given by Jordan-Brouwer Theorem. Recall that

$$\{x_3 < -N\} \subset G, \quad G + m\mathbf{i} + n\mathbf{j} = G.$$

Let

$$\Omega = G \cap \{0 \leq x_1, x_2 \leq 1\} \subset G.$$

Let

$$\mathcal{D} = \{x : 0 \leq x_1, x_2 \leq 1, \quad -\infty < x_3 < \infty\}$$

i.e. $\Omega = G \cap \mathcal{D}$

Sobolev space.

Let $W^{1,2}$ be the set of all locally integrable functions $u : G \to \mathbb{R}$ such that u is 1-periodic in x_1, x_2,

$$\|u\|^2_{W^{1,2}} \equiv \int_\Omega |\nabla u|^2 \, dx < \infty, \quad \int_B u \, dx = 0,$$

where B be some fixed ball in G, $W^{1,2}$ is the Hilbert space.
Theorem. *Let $u \in W^{1,2}$ and $\delta > 0$. Then there is 1-periodic in x_1, x_2 function $u_{ext} \in C^\infty(\mathbb{R}^3)$ such that*

$$u_{ext}(x) = 0 \;\; \textit{for} \;\; x_3 > N, \quad \int_\Omega |\nabla u - \nabla u_{ext}|^2 < \delta.$$

Boundary value problem.

$$\text{div } (\nabla\phi + \mathbf{i}) = 0 \ \text{ in } \ G, \quad (\nabla\phi + \mathbf{i}) \cdot \mathbf{n} = 0 \ \text{ on } \ S,$$
$$\phi(x + m\mathbf{i} + n\mathbf{j}) = \phi(x) \ \text{ in } \ G, \quad \nabla\phi(x) \to 0 \ \text{ as } \ x_3 \to -\infty. \tag{52}$$

Boundary value problem.Weak solutions

Definition 1. $\phi \in W^{1,2}$ is a weak solution to (52) if the integral identity

$$\int_{\Omega} \nabla\phi \cdot \nabla\xi \, dx + \int_{\Omega_N} \mathbf{i} \cdot \nabla\xi \, dx = 0, \quad \Omega_N = \Omega \cap \{x_3 \geq -N\}, \tag{53}$$

holds for all 1-periodic in x_1, x_2 function $\xi \in C^\infty(\mathbb{R}^3)$ such that $\xi(x) = 0$ for $x_3 > N > 0$. Here N is an arbitrary number such that $N > \sup_S |x_3|$.

Definition 2. $\phi \in W^{1,2}$ is a weak solution to (52) if the integral identity (53) holds for all $\xi \in W^{1,2}$

Definitions 1 and 2 are equivalent.

Boundary value problem.

Theorem. *Let $S \in \mathcal{S}$ satisfies the conditions of Jordan-Brouwer Theorem. Then problem* (52) *has a unique weak solution $\phi \in W^{1,2}$ such that for every $a \in S$ and $r > 0$,*

$$\int_{B(a,r)} |\nabla\phi|^2 dx \leq cr^{\alpha}, \quad 0 < \alpha < 1.$$

Regularity

$$\int_{D_r(X)} |\nabla \mathbf{n}|^2 \, dX \le c r^\beta \quad \beta > 0.$$

$$\mathbf{n} \in C^\gamma(\mathbb{R}^2)$$

Weak compactness

Compactness

Compactness Theorem. *Let a sequence of surfaces* $\{S_k\} \subset S$ *with the isothermal parametrizations* $x = \Psi_k(X)$, $X \in \mathbb{R}^2$, *has a common bound*

$$\int_{\Pi_{\Upsilon_k}} |\nabla \mathbf{n}_k|^2 \, dx \le M.$$

If, in addition, there is $\lambda > 2$, *such that*

$$\int_{\Pi_{\Upsilon_k}} e^{\lambda f_k} \, dX \le c < \infty,$$

then there is a subsequence, still denoted by S_k *and the isothermal immersion* $S \in \mathcal{S}$ *such that* $S_k \to S$ *in the uniform metric.*

Conditions of the Compactness Theorem are fulfilled if there are $r > 0$ and $\delta > 0$ such that for all $X_0 \in \mathbb{R}^2$,

$$\int_{|X - X_0| \le r} |\nabla \mathbf{n}_k|^2 \, dx \le 8\pi - \delta.$$

Helein's Theorem and Helein's conjecture

Theorem. *Let* $\mathbf{n} : D_1 \to \mathbb{S}^2$ *satisfies the condition*

$$\int_{D_1} |\nabla \mathbf{n}|^2 \, dX \le \frac{8\pi}{3} - \delta. \tag{54}$$

. Then there is an orthonormal frame $\mathbf{e}_i$, $\mathbf{e}_i \cdot \mathbf{n} = 0$, *and* f *such that*

$$\partial_1 f = -\mathbf{e}_1 \cdot \partial_2 \mathbf{e}_2, \quad \partial_2 f = \mathbf{e}_1 \cdot \partial_1 \mathbf{e}_2 \ \text{ in } \ D_1, \tag{55}$$

$$-\Delta f = \partial_1 \mathbf{e}_1 \cdot \partial_2 \mathbf{e}_2 - \partial_2 \mathbf{e}_1 \cdot \partial_1 \mathbf{e}_2 \ \text{ in } \ \mathbb{R}^2, \quad f = 0 \ \textit{ on } \ \partial D_1. \tag{56}$$

and

$$\int_{D_1} (|\nabla \mathbf{e}_i|^2 + |\nabla f|^2) \, dx + |f| \le c(\delta). \tag{57}$$

The Helein conjecture is that $8\pi/3$ can be replaced by 8π. It is known that it can be replaced by 4π. For mappings $\mathbf{n} : D_1 \to \mathbb{R}^k$, $k \ge 4$ the bound 4π is optimal. In the three-dimensions case the conjecture is true Kuwert, Li (2012). However , Li, Luo, Tang (2013)

Various

$$a\int_\Sigma H^2\,d\Sigma + b\int_\Sigma d\sigma.$$

$$a\int_{\mathbb{R}^3}\Big(-\varepsilon\Delta\varphi + \frac{1}{\varepsilon}W'(\varphi)\Big)^2\,dx + b\int_{\mathbb{R}^3}\frac{\varepsilon}{2}|\nabla\varphi|^2 + \frac{1}{\varepsilon}W(\varphi)\Big)^2\,dx$$