On the Boundary Layers of a Rheologically Complex Fluid

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.

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Newtonian and Non-Newtonian fluids

Isaak Newton "Philosophiæ Naturalis Principia Mathematica" (Sectio IX) Hypothesis: "Resistentiam quœ oritur ex defectu lubricitatis partium fluidi, cœteris paribus, proportionalem esse velocitati , qua partes fluidi separantur ab invicem."

Dependence of stress in medium on strain rates is called the RHEOLOGICAL law.

Equations of viscous incompressible fluid

Denote by Ω a bounded domain in $\mathbb{R}^m$,
$u(x) = \{u_1(x), u_2(x), \dots, u_m(x)\}$, $x \in \Omega$ is a vector–function,
$\partial_i \cdot \equiv \partial \cdot / \partial x_i$ is a partial derivative in x_i and $\partial_t \cdot \equiv \partial \cdot / \partial t$.
In $\mathbb{R}^3$ we have

$$\begin{cases} \partial_t u_i + \sum\limits_{j=1}^{3} \left(u_j \partial_j u_i - \dfrac{1}{\rho} \partial_j \sigma_{ij}(u) \right) = \dfrac{1}{\rho} f_i, \ i = 1, 2, 3, \\ \sum\limits_{j=1}^{3} \partial_j u_j = 0, \end{cases} \tag{1}$$

where $u(x,t) = \{u_1(x,t), u_2(x,t), u_3(x,t)\}$ is the velocity vector in (x_1, x_2, x_3), ρ is the density of fluid,
$f(x,t) = \{f_1(x,t), f_2(x,t), f_3(x,t)\}$ is the vector of forces, $\sigma_{ij}(u)$ is the stress tensor in medium.

Equations of viscous incompressible fluid

Rheological properties of the medium are characterizing by the dependence of $\sigma_{ij}(u)$ on the strain tensor $B_{i,j}(u) = \partial_j u_i + \partial_i u_j$. For incompressible medium we have the Reiner–Rivlin equation

$$\sigma_{ij}(u) = -p_* \delta_{ij} + \varphi_1(I_2, I_3) B_{ij} + \varphi_2(I_2, I_3) \sum_{k=1}^{3} B_{ik} B_{kj}, \qquad (2)$$

where p_* is a hydrostatic pressure, φ_j are the viscosity, $I_2 = \sum_{i,k=1}^{3} B_{ik} B_{ki}$ and $I_3 = \sum_{i,j,k=1}^{3} B_{ij} B_{jk} B_{ki}$ are the second and the third invariants of the strain tensor, δ_{ij} is the Kronecker delta.

Equations of viscous incompressible fluid

The Newtonian fluid is a partial case of the Reiner–Rivlin fluid, i.e. $\varphi_1 = \mu = \mathrm{const} > 0$, $\varphi_2(I_2, I_3) \equiv 0$. For generalized newtonian fluid

$$\sigma_{ij}(u) = -p_* \delta_{ij} + \varphi(I_2(u)) B_{ij}(u). \tag{3}$$

Consider the Ostwald–De Vale nonnewtonian medium

$$\sigma_{ij}(u) = -p_* \delta_{ij} + k_0 \left| \frac{1}{2} \sum_{k,l=1}^{3} B_{kl}(u) B_{lk}(u) \right|^{\dfrac{n-1}{2}} B_{ij}(u), \tag{4}$$

where $0 < n < \infty$, $k_0 = \mathrm{const} > 0$.

Equations of viscous incompressible fluid

For $n = 1$ the relation (4) corresponds to the usual Newtonian fluid, for $0 < n < 1$ the medium is called PSEUDOPLASTIC, and for $n > 1$ it is called DILATANT.

For media with a power rheological law, the system of equations (2) in $\mathbb{R}^m$ takes the form

$$\partial_t u_k - \nu \sum_{i=1}^{m} \partial_i \left\{ \left[\sum_{i,j=1}^{m} (\partial_j u_i + \partial_i u_j)^2 \right]^{\frac{p-2}{2}} (\partial_i u_k + \partial_k u_i) \right\} + \quad (5)$$

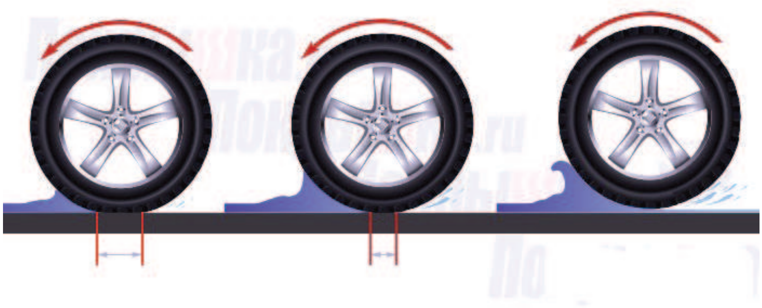
$$+ \sum_{i=1}^{m} u_i \partial_i u_k = f_k(x,t) - \frac{1}{\rho} \partial_k p_*(x,t), \; k = 1, 2, \cdots, m; \quad \sum_{j=1}^{m} \partial_j u_j = 0,$$

where $\nu = \frac{k_0}{\rho} \left(\frac{1}{2}\right)^{\frac{p-2}{2}}$, $p = n + 1$. For $p = 2$, i.e. $n = 1$, the system of equations (5) become the system of Navier — Stokes equations.

Generalized newtonian medium of O.A.Ladyzhenskaya

The need to build new models of a continuous medium, different from the classical ones, is due, in particular, to the fact that not all boundary and initial-boundary value problems for the Navier — Stokes system of equations are correctly solvable. One of the possible modifications of the Navier-Stokes equations was proposed by O. A. Ladyzhenskaya.

Generalized newtonian medium of O.A.Ladyzhenskaya



Aquaplaning

Generalized newtonian medium of O.A.Ladyzhenskaya

Consider the modified stationary system of equations of two-dimensional flow of viscous incompressible fluid

$$\begin{cases} -\nu \sum_{j=1}^{2} \frac{\partial}{\partial x_j}[(1 + k\, B^2(\mathbf{u}))B_{ij}(\mathbf{u})] + \sum_{j=1}^{2} u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i}, \\ \sum_{j=1}^{2} \frac{\partial u_j}{\partial x_j} = 0, \end{cases} \qquad (6)$$

where $i = 1, 2$, $B_{ij}(\mathbf{u}) = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}$, $B^2(\mathbf{u}) = \sum_{i,j=1}^{2} B_{ij}^2(\mathbf{u})$, ν is the kinetic viscosity of the medium, $0 < k \ll 1$, $k \sim \nu$, p is pressure.

Generalized newtonian medium of O.A.Ladyzhenskaya

Using the procedure proposed by L. Prandtl, from equations (6) we can derive a system of equations that describes the dynamics of a low-viscosity medium near a streamlined solid surface

$$\begin{cases} \nu \dfrac{\partial}{\partial y}\left(\left(1 + k\left(\dfrac{\partial u}{\partial y}\right)^2\right)\dfrac{\partial u}{\partial y}\right) - u\dfrac{\partial u}{\partial x} - v\dfrac{\partial u}{\partial y} = \dfrac{1}{\rho}\dfrac{\partial p}{\partial x}, \\ \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0, \end{cases} \tag{7}$$

where $u(x, y)$ and $v(x, y)$ are longitudinal and transverse to the streamlined surface components of the fluid velocity in the boundary layer.

Generalized newtonian medium of O.A.Ladyzhenskaya

System of equations (7) in the domain $D = \{0 < x < X,\ 0 < y < \infty\}$ is considered with boundary conditions

$$\begin{aligned} u(0,y) = u_0(y),\ u(x,0) = 0,\ v(x,0) = v_0(x), \\ u(x,y) \to U(x) \text{ при } y \to +\infty. \end{aligned} \tag{8}$$

Here the function $U(x)$ means the velocity of the fluid on the outer boundary of the boundary layer and is connected with the pressure $p(x)$ by the Bernoulli equation $\rho U^2(x) + 2p(x) = C = \text{const}$.

Stationary boundary layer

Let us pass to the von Mises variables in problem (7), (8). To that end, we introduce the new independent variables

$$x = x, \quad \psi = \psi(x, y),$$

where

$$u = \frac{\partial \psi}{\partial y}, \quad v - v_0(x) = -\frac{\partial \psi}{\partial x}, \qquad \psi(x, 0) = 0$$

and the new unknown function

$$w(x, \psi) = u^2(x, y).$$

Stationary boundary layer

As a result, system (7) with the conditions (8) is reduced to a single quasilinear differential equation

$$\nu\sqrt{w}\left(1+\frac{3}{4}k\left(\frac{\partial w}{\partial\psi}\right)^2\right)\frac{\partial^2 w}{\partial\psi^2}-\frac{\partial w}{\partial x}-v_0\frac{\partial w}{\partial\psi}+2U\frac{\partial U}{\partial x}=0 \quad (9)$$

in the domain $G=\{0<x<X, 0<\psi<\infty\}$ with the conditions

$$w(0,\psi)=w_0(\psi),\ \ w(x,0)=0,\ \ w(x,\psi)\to U^2(x) \text{ as } \psi\to\infty. \quad (10)$$

The function $w_0(\psi)$ is determined from the equation

$$w_0\left(\int\limits_0^y u_0(\eta)\,d\eta\right)\equiv u_0^2(y).$$

Stationary boundary layer

Lemma

Suppose that problem (9), (10) in the domain $G = \{0 < x < X, 0 < \psi < +\infty\}$ admits a solution $w(x, \psi)$ with the following properties: the function $w(x, \psi)$ is bounded in $\overline{G}$; $w(x, \psi) > 0$ for $\psi > 0$; there exist constants M, m, ψ_1 depending only on X, u_0, v_0, $p(x)$ and such that

$$\left|\frac{\partial w}{\partial \psi}\right| < M, \quad \left|\sqrt{w}\frac{\partial^2 w}{\partial \psi^2}\right| < M, \quad (x, \psi) \in G, \tag{11}$$

moreover,

$$\left|\frac{\partial w}{\partial x}\right| < M\psi^{1-\beta}, \quad \frac{\partial w}{\partial \psi} \geqslant m > 0 \tag{12}$$

for $0 < \psi < \psi_1$, $0 < \beta < 1/2$.

Stationary boundary layer

Lemma

Then, problem (7), (8) *in the domain* $D = \{0 < x < X, 0 < y < +\infty\}$ *has a solution* $u(x, y)$, $v(x, y)$ *with the following properties: the function* $u(x, y)$ *is continuous and bounded in* $\overline{D}$, $u > 0$ *for* $y > 0$; $\frac{\partial u}{\partial y} > m_1 > 0$ *for* $0 < y < y_0$ (m_1 *and* y_0 *are constants);* $\frac{\partial u}{\partial y}$ *and* $\frac{\partial^2 u}{\partial y^2}$ *are continuous and bounded in* D; $\frac{\partial u}{\partial x}$, v, $\frac{\partial v}{\partial y}$ *are continuous and bounded in any finite part of* $\overline{D}$.

Stationary boundary layer

Theorem (Existence)

Under natural assumptions on the functions $U(x)$, $u_0(y)$, *and* $v_0(x)$, *problem* (9), (10) *has a solution* $w(x,\psi)$ *in the domain* G *for some* X, *and this solution has the following properties:* $w(x,\psi)$ *is bounded in* $\overline{G}$, $w(x,\psi) > 0$ *for* $\psi > 0$, *and* $w(x,\psi)$ *satisfies the inequalities* (11), (12).
If $U'(x) \geqslant 0$ *and* $v_0(x) \leqslant 0$ *or* $U'(x) > 0$, *then such a solution exists in* G *for any* $X > 0$.

Stationary boundary layer

Theorem (Uniqueness)

Problem (9), (10) *in* $\overline{G}$ *can have only one solution* $w(x,\psi)$ *with the following properties:* $w(x,\psi)$ *is continuous and bounded in* $\overline{G}$*;* $k_1\psi \leqslant w(x,\psi) \leqslant k_2\psi$ *for* $\psi \leqslant \psi_1$*;* $w(x,\psi) \geqslant a > 0$ *for* $\psi \geqslant \psi_1$*;* $\left|\sqrt{w}\frac{\partial^2 w}{\partial\psi^2}\right| \leqslant M$*. Here,* k_1*,* k_2*,* ψ_1*,* M *are positive constants.*

Stationary boundary layer

Note that these Theorems, together with Lemma, establishes the existence of the unique solution for the original problem (7), (8).

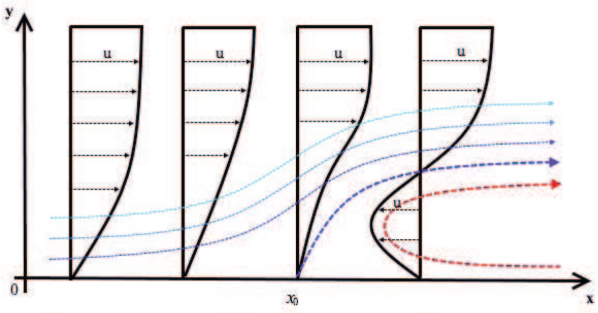
Separation of the boundary layer

Theorem

If the solution of problem (7), (8) *exists in the domain* D, *then* $X < x_0$, *where* x_0 *is determined by the conditions*

$$\max_y u_0^2(y) + 2\int\limits_0^{x_0} U(x)\frac{dU}{dx}\,dx = 0 \quad \textit{and} \quad \frac{dU(x_0)}{dx} < 0.$$

Separation of the boundary layer



$$\left\{\frac{\partial u}{\partial y}\right\}_{y=0} > 0 \qquad \left\{\frac{\partial u}{\partial y}\right\}_{y=0} = 0 \qquad \left\{\frac{\partial u}{\partial y}\right\}_{y=0} < 0$$

Stationary MHD–boundary layer

$$\begin{cases} \nu \dfrac{\partial}{\partial y}\left(\left(1 + k\left(\dfrac{\partial u}{\partial y}\right)^2\right)\dfrac{\partial u}{\partial y}\right) - u\dfrac{\partial u}{\partial x} - v\dfrac{\partial u}{\partial y} + B^2(U-u) = -U\dfrac{dU}{dx}, \\ \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0, \end{cases} \tag{13}$$

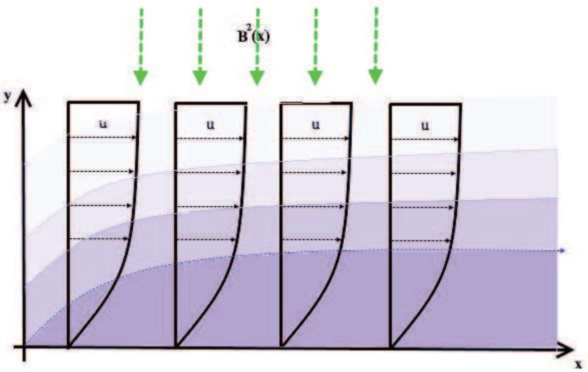
Separation of the MHD–boundary layer

Theorem

If the solution of problem (13), (8) *exists in the domain* D*, then* $X < x_0$*, where* x_0 *is determined by the conditions*

$$\max_y u_0^2(y) - \int_0^{x_0} \left(-2U(x)\frac{dU}{dx} - 2B^2(x)U(x) \right) dx = 0 \text{ and } \frac{dU(x_0)}{dx} < 0.$$

Separation of the MHD–boundary layer



Separation of the MHD–boundary layer

Due to Theorem there is no separation of the boundary layer provided that:

$$B^2(x) > \left|\frac{dU}{dx}\right|.$$

Nonstationary boundary layer

Consider the system of equations

$$\begin{cases} \nu \dfrac{\partial}{\partial y}\left(\left(1 + k\left(\dfrac{\partial u}{\partial y}\right)^2\right)\dfrac{\partial u}{\partial y}\right) - u\dfrac{\partial u}{\partial x} - v\dfrac{\partial u}{\partial y} - \dfrac{\partial u}{\partial t} = -\dfrac{dU}{dt} - U\dfrac{dU}{dx}, \\ \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0, \end{cases} \tag{14}$$

in domain $Q = \{0 < t < \infty,\ 0 < x < X,\ 0 < y < \infty\}$ with initial and boundary conditions

$$\begin{aligned} & u(0, x, y) = \mathcal{U}_0(x, y), \\ & u(t, 0, y) = 0, \quad u(t, x, 0) = 0, \quad v(t, x, 0) = v_0(t, x), \\ & u(t, x, y) \to U(t, x) \ \text{ as } \ y \to \infty, \end{aligned} \tag{15}$$

where $U(t, 0) = 0$, $U_x(t, 0) > 0$, $U(t, x) > 0$ as $x > 0$.

Nonstationary boundary layer

Assume that

$$U(t,x) = xV(t,x), \qquad V(t,x) = a + xa_1(t,x), \quad v_0(t,x) = b + xb_1(t,x), \tag{16}$$

where $a = const > 0$, $b = const$.

Let us transform problem (14), (15) to initial boundary value problem for quasilinear equation. For this aim we define new independent variables τ, ξ, η (Crocco variables) and a new unknown function $w(\tau,\xi,\eta)$ in the following way:

$$\tau = t, \qquad \xi = x, \qquad \eta = \frac{u(t,x,y)}{U(t,x)}, \quad w(\tau,\xi,\eta) = \frac{u_y(t,x,y)}{U(t,x)}. \tag{17}$$

Nonstationary boundary layer

We obtain

$$\nu\big(1+3dU^2w^2\big)w^2w_{\eta\eta}-w_\tau-\eta Uw_\xi+(\eta^2-1)U_xw_\eta+ \\ +(\eta-1)\frac{U_t}{U}w_\eta-\eta U_xw+6\nu dU^2w_\eta^2w^3-\frac{U_t}{U}w=0 \tag{18}$$

in domain $\Omega=\{0<\tau<\infty,\,0<\xi<X,\,0<\eta<1\}$ with conditions

$$w(0,\xi,\eta)=\mathcal{W}_0(\xi,\eta),\quad w(\tau,\xi,1)=0, \\ \left(\nu\big(1+3dU^2w^2\big)ww_\eta-v_0w+\frac{U_t}{U}+U_x\right)\Big|_{\eta=0}=0, \tag{19}$$

where $\mathcal{W}_0(\xi,\eta)=\dfrac{\mathcal{U}_{0y}(x,y)}{U(0,x)}$.

Nonstationary boundary layer

Consider the auxiliary problem

$$\begin{aligned} \nu Y^2 Y_{\eta\eta} + (\eta^2 - 1) a Y_\eta - \eta a Y = 0, \quad 0 < \eta < 1, \\ \left(\nu Y Y_\eta - bY + a\right)\Big|_{\eta=0} = 0, \qquad Y(1) = 0. \end{aligned} \tag{20}$$

Nonstationary boundary layer

Lemma

Problem (20) has a solution $Y(\eta)$ with the following properties:

$$M_1(1-\eta)\sigma \leqslant Y(\eta) \leqslant M(1-\eta)\sigma \quad \textit{при} \quad 0 \leqslant \eta \leqslant 1, \tag{21}$$

$$M(1-\eta)(\sigma - M_2) \leqslant Y(\eta) \quad \textit{при} \quad 0 < \eta_0 \leqslant \eta < 1, \tag{22}$$

$$-M_3\sigma \leqslant Y_\eta(\eta) \leqslant -M_4\sigma, \tag{23}$$

$$|YY_{\eta\eta}| \leqslant M_5, \qquad YY_{\eta\eta} \leqslant -M_6, \tag{24}$$

$$\sigma = \sqrt{-\ln \mu(1-\eta)}, \quad \mu = const, \quad 0 < \mu < 1,$$

$$\nu M^2 = 2a, \quad \sigma(0) \geqslant \frac{|b|}{a} + 2, \quad \frac{M_7}{\sigma} < 1 \quad \textit{при} \quad \eta > \eta_0 = const \geqslant 0.$$

Nonstationary boundary layer

Let us denote

$$
\begin{aligned}
A &:= (\eta^2 - 1)(V + \xi V_x) + (\eta - 1)\frac{V_t}{V}, \\
B &:= -\eta(V + \xi V_x) - \frac{V_t}{V}, \\
C &:= V + \xi V_x + \frac{V_t}{V}.
\end{aligned}
\tag{25}
$$

Nonstationary boundary layer

Theorem

Suppose that $U_x > 0$ *as* $0 \leqslant x \leqslant X$*; the functions* V, V_x, v_0, v_{0x}, A_x, B_x, C_x *are bounded,* $|V_t| \leqslant Mx$, $|A_t| \leqslant Mx$, $|B_t| \leqslant Mx$, $|C_t| \leqslant Mx$, $|v_{0t}| \geqslant -Mx$,
$Ye^{-M_1\xi} \leqslant \mathcal{W}_0 \leqslant Ye^{M_2\xi}$, $\quad |\mathcal{W}_{0\xi}| \leqslant M(1-\eta)$; $\mathcal{W}_0$ *is continuously differentiable on* $\eta \in [0,1)$ *and*

$$Y_\eta e^{M_3\xi} \leqslant \mathcal{W}_{0\eta} \leqslant Y_\eta e^{-M_4\xi},$$

$$-M(1-\eta) \leqslant \nu\big(1+3d\xi^2V^2\mathcal{W}_0^2\big)\mathcal{W}_0^2 w_{0\eta\eta} + Aw_{0\eta} + \\ + B\mathcal{W}_0 + 6\nu d\xi^2V^2\mathcal{W}_0^3 w_{0\eta}^2 \leqslant M\xi(1-\eta),$$

$$\Big(\nu(1+3d\xi V^2\mathcal{W}_0^2)\mathcal{W}_0\mathcal{W}_{0\eta} - v_0\mathcal{W}_0 + C\Big)\Big|_{\eta=0,\tau=0} = 0.$$

Nonstationary boundary layer

Theorem

Then problem (18), (19) *in* Ω *has a solution* $w(\tau, \xi, \eta)$, *such that:* w *is continuous in* $\overline{\Omega}$,

$$Ye^{-M_5\xi} \leqslant w \leqslant Ye^{M_6\xi};$$

w_η *is continuous in* $\eta < 1$, $Y_\eta e^{M_7\xi} \leqslant w_\eta \leqslant Y_\eta e^{-M_8\xi}$; w_τ, w_ξ, $w_{\eta\eta}$ *satisfy*

$$-M(1-\eta)\sigma \leqslant w_\tau \leqslant M\xi(1-\eta)\sigma, \ \left|w_\xi\right| \leqslant MY, \ -M_9 \leqslant ww_{\eta\eta} \leqslant -M_{10}.$$

A solution of problem (18), (19) *is unique.*

Nonstationary boundary layer

Theorem

Assume that $U(t,x) > 0$ *as* $x > 0$; $a_1(t,x)$, $a_{1x}(t,x)$, $a_{1xx}(t,x)$, $b_1(t,x)$, $b_{1x}(t,x)$ *are bounded as* $0 \leqslant t < \infty$, $0 \leqslant x \leqslant X$. *Suppose also that* $\mathcal{U}_0(x,y)$ *is such a function that* $U_x > 0$ *as* $0 \leqslant x \leqslant X$; V, V_x, v_0 *are bounded,* $\dfrac{V_t}{V} \leqslant Mx$ *and*
$M(1-\eta) \leqslant \mathcal{W}_0(\xi,\eta) \leqslant M(1-\eta)\sigma$, *where*
$\sigma = \sqrt{-\ln \mu(1-\eta)}$, $0 < \mu < 1$, *и* $\mathcal{W}_0(\xi,\eta)$ *has a continuous derivative on* $\eta \in [0,1)$.

Nonstationary boundary layer

Theorem

Then for some X *problem* (14), (15) *has a unique solution* u, v *in* Q *with the properties:* $u > 0$ *as* $y > 0$ *and* $x > 0$; $\frac{u}{U}$, $\frac{u_y}{U}$ *are bounded and continuous in* $\overline{Q}$; $u_y > 0$ *as* $y \geqslant 0$;
$u(t,x,y) \to U(t,x)$ *as* $y \to \infty$, $\frac{u_y(t,x,y)}{U(t,x)} \to 0$ *as* $y \to \infty$; u_x, u_y, u_{yy}, u_t, v_y *are bounded and continuous in* Q *on* y; v *is continuous in* $\overline{Q}$ *on* y *and bounded;*

Nonstationary boundary layer

Theorem

and

$$U(x)Y\Big(\frac{u}{U}\Big)e^{-M_{11}x} \leqslant u_y \leqslant U(x)Y\Big(\frac{u}{U}\Big)e^{M_{12}x},$$

$$Y_\eta\Big(\frac{u}{U}\Big)e^{M_{13}x} \leqslant \frac{u_{yy}}{u_y} \leqslant Y_\eta\Big(\frac{u}{U}\Big)e^{-M_{14}x},$$

$$\exp\Big[-My^2e^{M_{15}x}\Big] \leqslant 1-\frac{u}{U} \leqslant \exp\Big[-My^2e^{-M_{16}x}\Big],$$

where $Y(\eta)$ *is a solution of* (20).

Thank you for your attention!

С днём рождения, дорогой Валерий Васильевич!