The Ergodic Theorem in the Kozlov–Treschev Form

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SOME CLASSICS:
$(X, \mathcal{A}, \mu)$ a probability space
$g_t \colon X \to X$ measure preserving, $t \geq 0$,
$g_t g_s = g_{t+s}$, $g_0 = Id$, $g_t(x)$ measurable in (t, x)
THEN for any μ-integrable f a.e. and in $L^1(\mu)$

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t f(g_s(x))\,ds = \overline{f}(x)$$

where $\overline{f}$ conditional expectation of f w.r.t.
σ-algebra $\mathcal{I}$ generated by all g_t-invariant functions

Kozlov V.V., Treschev D.V.: On new forms of the ergodic theorem. J. Dynam. Control Syst. 2003. V. 9, N 3.

$\nu = \varrho(s)\,ds$ absolutely continuous probability measure on $[0, +\infty)$

BOUNDED f:

$$\lim_{t\to\infty} \int_0^{+\infty} f(g_{ts}(x))\,\nu(ds) = \overline{f}(x), \qquad (KT)$$

$\overline{f}$ conditional expectation of f w.r.t. σ-algebra $\mathcal{I}$ generated by all g_t-invariant functions
Classics: $\nu =$ Lebesgue on $[0, 1]$, $\varrho(s) = I_{[0,1]}(s)$

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Generalizations of Kozlov–Treschev:

UNBOUNDED f

DYNAMICS OF MEASURES $\nu_{t,x}$=image of ν under $S_{t,x}\colon [0,+\infty) \to X$:

$$S_{t,x}(s) = g_{ts}(x),$$

$$\nu_{t,x}(A) = \nu(S_{t,x}^{-1}(A))$$

STOCHASTIC SYSTEMS:

$$d\xi_t^x = A(\xi_t^x)dW_t + b(\xi_t^x)dt, \quad \xi_0^x = x$$

$$F_t(x,w) = \int_0^\infty f(\xi_{ts}^x(w))\,\nu(ds)$$

EXAMPLE. $X = S^1$, μ Lebesgue, g_t rotations,

$$f(z) = |\sin\theta|^{-1}(\ln|\sin\theta| - 1)^{-2},$$

$z = \exp(i\theta),\ \theta \in [0, 2\pi)$
One can find integrable ϱ on $[0, +\infty)$ and f on X such that

$$\limsup_n \int_0^{+\infty} f(g_{ns}(x))\, \varrho(s)\, ds = +\infty$$

The effect due to $|\theta - n/k| \leq k^{-2}$ for infinitely many pairs n, k.

Theorem 1. $f \in L^p(\mu)$, $p \in [1, +\infty)$, $\varrho \in L^q[0, +\infty)$, $1/p + 1/q = 1$.
If ϱ has bounded support, then (KT) holds for a.e. x.
More generally, this is true if

$$\varrho(s) \leq \beta(s), \quad s \geq s_0,$$

$\beta \in L^q[0, +\infty)$ monotone decreasing
??? IS BOUNDED SUPPORT IMPORTANT**???**

Dynamics of $\nu_{t,x}$:

Theorem 2. X Souslin space, μ Borel probability measure on X, $\{g_t\}$ ergodic. Then

$$\nu_{t,x} \quad \text{converges weakly to } \mu \text{ as } t \to +\infty$$

for μ-a.e. x.

Theorem 3. Under the same assumptions, for each $\varepsilon > 0$ there is a compact set $K \subset X$ with

$$\mu(K) > 1 - \varepsilon$$

such that the family of measures

$$\{\nu_{t,x} \colon t \geq \varepsilon,\ x \in K\}$$

is uniformly tight, i.e.
for each $\delta > 0$ there is a compact set Q_δ with $\nu_{t,x}(Q_\delta) > 1 - \delta$ for all these measures.

$\varepsilon > 0$ is important:

$$\nu_{0,x} = \delta_x \quad \text{Dirac's mass at } x$$

Stochastic case: $Lf = \mathrm{trace}(AD^2 f)/2 + \langle b, \nabla f \rangle$

Diffusion matrix A locally Lipschitz, nondegenerate, drift b Borel, locally bounded,
there is a Lyapunov function V: as $|x| \to \infty$ we have $V(x) \to +\infty$ and $LV(x) \to -\infty$.
Then there is an invariant probability measure μ for this diffusion.
P be Wiener measure on the path space $W = C([0, +\infty), \mathbb{R}^d)$.

Theorem 4. f bounded Borel measurable. Then for each x

$$\lim_{t\to+\infty}\int_0^{+\infty} f(\xi^x_{ts}(w))\varrho(s)\,ds = \int_{\mathbb{R}^d} f(y)\,\mu(dy)$$

for P-a.e. $w \in W$.

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Measure Theory

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