On optimal stationary states of exploited populations

Alexey Davydov

Lomonosov Moscow State University,
NUST «MISiS» and IIASA

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Simple example

1. **Population dynamic:**

$$\dot{p} = p(1-p)$$

2. **Type of control:**

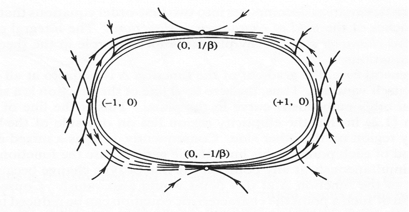
$$\dot{p} = p(1-p) - u,$$

$$\dot{p} = p(1-p) - up$$

3. **Objective of control: to maximize u or ux, respectively**
The result is the same $(= 1/4)$ but the first mode of control leads to unstable stationary state, while the second one leads to the stable one and provides long run optimal exploitation.

Averaged optimization

$$\dot{x}^2 + \dot{y}^2 \le 1 \qquad (\dot{x} - x)^2 + (\dot{y} - \beta y)^2 \le 1,\ \beta > 1$$



$$\overline{\lim} \frac{1}{T} \int_0^T f(x(t), u(t))dt \quad \text{by} \quad T \to \infty$$

Theorem

Let there exist a bounded trajectory of control system with limit point in the interior of nonlocal transitivity zone and with a value of averaged functional being close to its supremum. Then there exists periodic trajectory with a value of the functional which is also close to the supremum (F.Colonius, W.Kliemann, 1986).

Periodic harvesting with complete recovering I

V.I.Arnold, Averaged optimization and phase transition in control dynamical systems// Funct. Anal. and its Appl., 36 (2002), 1-11.
Motion along the cycle (on the circle) is defined by control system

$$\dot{x} = v(x, u), \quad \text{with} \quad u \in U$$

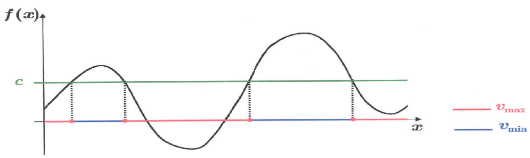
A profit density $\rho, \rho = \rho(x)$ does not depend on time (annual use of grassland, pastures,...)

The objective function $A = A(u)$ is $\overline{\lim}_{T\to\infty} \frac{1}{T} \int\limits_0^T \rho(x(t))dt$

$$\left(\text{or could be} \quad \overline{\lim}_{T\to\infty} \frac{1}{T} \int\limits_0^T \rho(x(t))\phi(x(t), u(t))dt \right)$$

Periodic harvesting with complete recovering II

Arnold's results and observations:



Remark

There could be many optimal solutions! It is easy to understand for constant profit density.

Theorem

For a generic sufficiently differentiable case with $v > 0$ there exists no more then one optimal cyclic (stationary) motion (A.D., 2005).

Condition: density ρ has critical points of finite multiplicity.

Harvesting with complete recovering III

The omitting of the assumption $v > 0$ could lead to appearance of arcs with complete controllability and the ones with rotation in one of two directions.

Theorem

For a continuous control system and profit density on the circle, the maximum averaged profit on infinite horizon always can be provided by level cycle or by staying at stationary point (H.Matos, A.D., 2004).

A. A. Davydov, H. Mena-Matos, *Singularity theory approach to time averaged optimization*, SINGULARITIES IN GEOMETRY AND TOPOLOGY, World Scientific Publishing Co. Pte. Ltd., 2007, 598 – 628.

Example: When the system is completely controllable or the maximum of a profit density is provided by some stationary point then the staying at such a point is the optimal motion.

Harvesting with complete recovering IV

Remark 1. How to find the maximum averaged profit A_{max}?

$A_s := \max_{x \in S} f(x), \qquad S := \{x \,:\, v_{min}(x) \leq 0 \leq v_{max}(x)\},$

A_c - supremum averaged profit among level cyclic motions, then

$$A_{max} := \max\{A_s,\, A_c\}$$

Remark 2. Generalization for modified objective functions (T.Shutkina; A.D.):

$$\frac{1}{T}\int_0^T f(x(t))dt \quad \longrightarrow \quad \frac{\int_0^T e^{-\sigma t} f(x(t))dt}{\int_0^T e^{-\beta t} dt}$$

Remark 3. Classification of generic singularities of A_{max} for parametric (low dimensional) cases (C.Moreira, H.Matos; T.Shutkina; A.D.)

Periodic harvesting with partial recovering I

A.O.Belyakov, A.A.Davydov, V.M.Veliov, *Optimal Cyclic Exploitation of Renewable Resources*, Journal of Dynamical and Control Systems, 21:3 (2015), 475 – 494

A.O.Belyakov, A.A.Davydov, *Efficiency optimization for the cyclic use of a renewable resource*, Proc. Steklov Inst. Math. (Suppl.), 299, suppl. 1 (2017), 14 – 21

Population area is S^1

$$p_t(t,x) = (a(x) - b(x)p(t,x))p(t,x), b \geq b_0 > 0.$$

The admissible effort (=harvesting density) $r, r(x) \geq 0, 0 < r_1 \leq r \leq r_2 \leq R$ with profit

$$P_k(r,T) = \int_{S^1} p(t_k,x)(1 - e^{-\gamma(x)r(x)})dx$$

Remark: $1/r(x)$ is velocity of motion for harvesting machine, and the period of harvesting is T, $T \geq \int_{S^1} r(x)dx$.

Periodic harvesting with partial recovering II

The problem is
to maximize outcome the averaged long run revenue of harvesting $P_k(r,T)/T$ with respect r and T.

Theorem

For admissible density r and period $T>0$ there exists limit

$$p_\infty(x,r,T) := \max\left\{0,\ \frac{a(x)}{b(x)}\,\frac{e^{a(x)T}-e^{\gamma(x)r(x)}}{e^{a(x)T}-1}\right\}$$

Thus the problem is to maximize

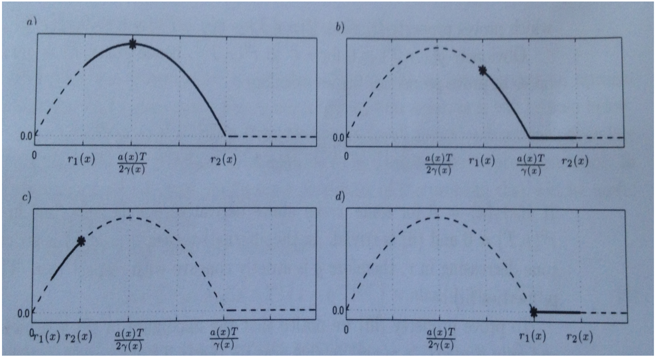
$$\frac{1}{T}\int_{S^1} p_\infty(x,r,T)(1-e^{-\gamma(x)r(x)})dx$$

Theorem

There exists optimal admissible harvesting policy T, r

Periodic harvesting with partial recovering III

Integrant for limit distribution looks like



The integrant graph (dashed line) and several configurations of leftmost maximizer with restriction to $[r_1(x), r_2(x)]$, $r_1(x) \leq r(x) \leq r_2(x)$ (solid line).

Fisher – Kolmogorov – Petrovskii – Piscounov equation

Колмогоров А.Н., Петровский И.Г., Пискунов Н.С., *Исследование уравнения диффузии, соединенной с возрастанием вещества, и его применение к одной биологической проблеме*, Бюллетень МГУ. Сер. А. Математика и Механика, 1:6 (1937),1 – 26

Fisher R. A., *The Wave of Advance of Advantageous Genes*, Annals of Eugenics,7(1937), 355 – 369

$$(A) \quad p_t = p_{xx} + p - p^2, \quad p = p(t,x),$$

$$(A) \quad p_t = p_{xx} + a(x)p - b(x)p^2 - u(x)p, \quad p = p(t,x),$$

with measurable control u, $0 \le U_1 \le u \le U_2$, $U_1, U_2 \in C^0$, or

$$(B) \quad p_t = p_{xx} + a(x)p - b(x)p^2, \quad p(kT+, x) = p(kT-, x)(1 - u(x)),$$

where measurable control u, $0 \le u \le 1$.

The questions are about the existence of nontrivial stationary states, stability of them and the search an optimal one.

Harvesting in the presence of diffusion (A)

Theorem

For any admissible control there exists stable stationary nonnegative solution.

Consider objective functional

$$\overline{\lim}_{T\to\infty} \frac{1}{T} \int_0^T \int_{S^1} p(t,x)dx$$

In the presence of global attractor of nonzero nonnegative solutions the possible maximum of this functional values is

$$\sup_u \int_{S^1} u(x)p_\infty(x)dx$$

Theorem

There exists admissible control u provided maximum averaged profit (on the respective steady state).

Harvesting in the presence of diffusion (A)

A sufficiently big positive constant is supersolution for the equation operator, which is monotone on initial data. If the principal eigenvalue λ for the problem

$$-\phi_{xx} - (a(x) - u(x))\phi = \lambda\phi$$

is negative then zero solution is unstable, and stationary nonnegative nonzero solution is attractor of nonzero nonnegative solutions. But if $\lambda \geq 0$ then zero solution is global attractor. (Berestycki at all 2005)

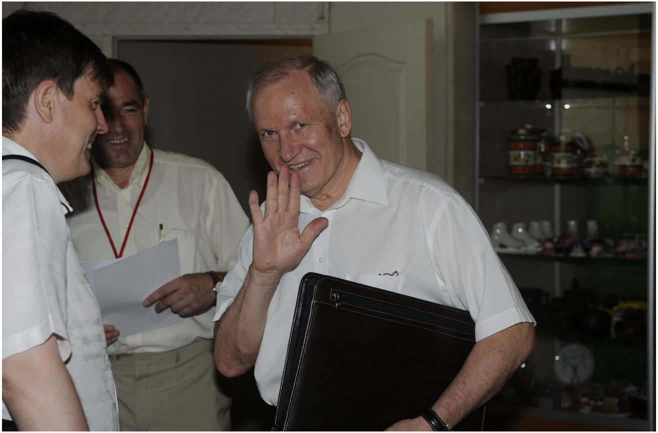
Harvesting in the presence of diffusion (A)

The objective functional is bounded. Hence there exists exact upper bound of its values on the attractors and sequences $\{u_k\}$ and $\{p_k\}$, which provides this bound by $k \to \infty$.

Selection of $\{p_{k_s}\}$ and $\{u_{k_s}\}$, which converge respectively uniformly with derivatives and weakly respectively. The derivative of limit function p'_∞ satisfies the Lipschitz condition.

That implies the existence of an admissible control provided the limit solutions p_∞.

Suzdal Conferences



Suzdal Conferences

Thanks for the attention