Dynamics of slow-fast Hamiltonian systems near a homoclinic set

Sergey Bolotin
Moscow Steklov Mathematical Institute

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Slowly time dependent Hamiltonian systems

- The Hamiltonian slowly depends on time:

$$H = H(q, p, \tau), \quad \omega = dp \wedge dq, \qquad \dot{\tau} = \varepsilon \ll 1.$$

- The energy $E(t) = H(q(t), p(t), \tau(t))$ changes slowly:

$$\dot{E} = \varepsilon \partial_\tau H \sim \varepsilon.$$

- Similar results for slow-fast Hamiltonian systems:

$$H = H(q, p, x, y), \qquad \omega_\varepsilon = dp \wedge dq + \varepsilon^{-1} dy \wedge dx.$$

- For slowly time dependent system:

$$\mathcal{H}(q, p, \tau, E) = H(q, p, \tau) - E, \quad \omega_\varepsilon = dp \wedge dq - \varepsilon^{-1} dE \wedge d\tau.$$

Adiabatic invariant

- $\varepsilon = 0$ – frozen autonomous system with Hamiltonian $H_\tau(q,p) = H(q,p,\tau)$ depending on a parameter τ.
- Suppose the system has 1 dof and the frozen energy levels $\{H_\tau = E\}$ are closed curves. Averaging implies that for small ε the adiabatic invariant

$$I(\tau, E) = \oint_{\{H_\tau = E\}} p\, dq$$

changes much slower than energy. Average rate of change of $I(t) = I(\tau(t), E(t))$ is $\sim \varepsilon^2$:

$$|I(t) - I(0)| \le C\varepsilon, \qquad 0 \le t \le T/\varepsilon.$$

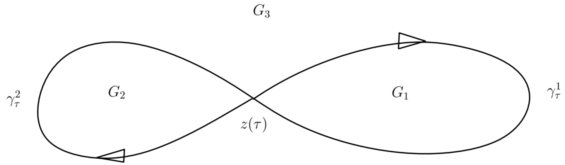
- Energy changes gradually: (τ, E) approximately follows a level curve $I(\tau, E) = \mathrm{const}$.

Adiabatic invariant near a separatrix

- Averaging doesn't work for trajectories passing near a hyperbolic equilibrium $z_0(\tau)$ of the frozen system.
- Suppose $z_0(\tau)$ has a figure eight separatrix $\Gamma_\tau = \gamma_\tau^1 \cup \gamma_\tau^2$. It divides the plane in 3 regions G_k. In the interior of G_k, there is an adiabatic invariant I_k.
- Near a separatrix,

$$I_k(\tau, E) = P_k(\tau) - \frac{E \ln |E|}{\lambda(\tau)} + O(|E|), \qquad P_k(\tau) = \oint_{\gamma_\tau^k} p\, dq.$$

  $\pm\lambda(\tau)$ are eigenvalues. P_k is the Poincaré function.
- Pendulum: $H_\tau = p^2/2 - \lambda^2(\tau)(1 - \cos q)$, $P_k(\tau) = 8\lambda(\tau)$.



Jumps of the adiabatic invariant

- A. Neishtadt (1986) showed that for trajectories crossing the separatrix, the adiabatic invariant has quasi-random jumps of order
$$\Delta I \sim \varepsilon, \qquad \Delta t \sim |\ln \varepsilon|.$$
Then the energy will have similar quasirandom jumps.
- Goal: obtain a partial analog of Neishtadt' result for n dof systems with a hyperbolic equilibrium possessing several homoclinics.
- Under certain conditions we construct trajectories with $|E| \gtrsim \varepsilon$ having quasirandom jumps of energy
$$\Delta E \sim \varepsilon, \qquad \Delta t \sim |\ln \varepsilon|.$$
while staying close to the homoclinic set.

Remarks

- The problem is related to the problem of Arnold's diffusion near a multiple resonance. The main difference is that in this problem the system is nearly integrable. For our results to work, the frozen system needs to be nonintegrable.
- V. Gelfreich and D. Turaev (2008) showed that if the frozen system has a compact uniformly hyperbolic chaotic invariant sets on regular energy levels, then there exist trajectories with energy having quasirandom jumps of order $\Delta E \sim \varepsilon$ over time intervals $\Delta t \sim 1$. The energy may grows with maximal rate $\sim \varepsilon$.
- This result does not work on critical energy levels, where dynamics near a homoclinic set is slow.

Transverse homoclinics

- Since the frozen system is autonomous consider first an autonomous system with a hyperbolic equilibrium z_0.
- $W^\pm = \{x : \phi^t(x) \to z_0 \text{ as } t \to \pm\infty\}$ – stable and unstable manifolds. ϕ^t – phase flow.
- A homoclinic orbit $\gamma : \mathbb{R} \to W^- \cap W^+$ is transverse if

$$T_{\gamma(t)} W^+ \cap T_{\gamma(t)} W^- = \mathbb{R}\,\dot\gamma(t).$$

- Generically all homoclinics are transverse.
- For Tonelli Hamiltonians transverse homoclinics may be replaced by minimal homoclinics.

Minimal homoclinics

- For simplicity consider a natural system:

$$H(q,p) = \frac{1}{2}\|p\|^2 + V(q).$$

  $\|\ \|$ – Riemannian metric on a compact non simply connected manifold Q. If q_0 is a point of nondegenerate maximum of V, then $z_0 = (q_0, 0)$ is a hyperbolic equilibrium.

- Let $r(Q)$ be the minimal number of multiplicative generators of $\pi_1(Q)$. There exist at least $2r(Q)$ minimal homotopy classes $\Omega \in \pi_1(Q)$ containing minimal homoclinics (B-Kozlov 1978):

$$J(\gamma) = \inf_{\Omega} J, \qquad J(\gamma) = \oint_{\gamma} \sqrt{2(V(q_0) - V(q))}\,\|dq\|.$$

Leading homoclinics

- Let $\pm\lambda_i$ be the eigenvalues of the hyperbolic equilibrium z_0. Suppose the leading eigenvalues are real and simple:

$$0 < \lambda = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n.$$

- The strong stable and unstable invariant manifolds $S^\pm$ are hypersurfaces in $W^\pm$ corresponding to nonleading eigenvalues.
- We call a homoclinic orbit strong if $\gamma(\mathbb{R}) \subset S^+ \cup S^-$. Generically there are no strong homoclinics.
- We call a homoclinic orbit leading if it does not lie in $S^+ \cup S^-$.

Positive homoclinics

- Let $\zeta_\pm$ be eigenvectors associated to the leading eigenvalues $\mp\lambda$ such that

$$\omega(\zeta_+, \zeta_-) = 1, \qquad \omega = dp \wedge dq.$$

- If γ is a leading homoclinic, then

$$v_\pm(\gamma) = \lim_{t\to\pm\infty} e^{-\lambda|t|}\dot\gamma(t) = \alpha_\pm(\gamma)\zeta_\pm \neq 0.$$

- We call γ positive (negative) if

$$\alpha_+(\gamma)\alpha_-(\gamma) > 0 \qquad (\alpha_+(\gamma)\alpha_-(\gamma) < 0).$$

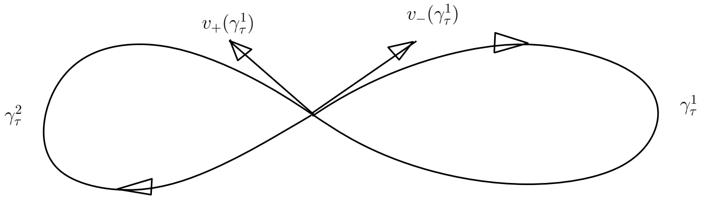


Figure: Positive homoclinics for 1 dof

Positive homoclinics for a natural system

For a natural system a homoclinic $(q(t), p(t))$ is positive (negative) if

$$\lim_{t\to+\infty} \frac{\dot q(t)}{\|\dot q(t)\|} = \mp \lim_{t\to-\infty} \frac{\dot q(t)}{\|\dot q(t)\|}.$$

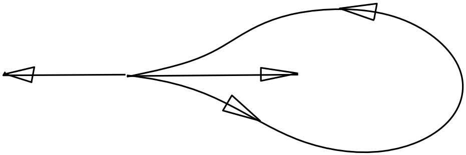


Figure: Positive homoclinics in the configuration space

Trajectories with given energy

Consider a natural Hamiltonian system. Let $\Omega_k \in \pi_1(Q)$ be minimal homotopy classes. Suppose there are no strong homoclinics, then $\alpha_\pm(\gamma) = \alpha_\pm^k$ is independent of a minimal homoclinic $\gamma \in \Omega_k$.

Theorem (B-Rabinowitz 1998)

For small $\varepsilon > 0$ and any sequence $(k_i)_{i\in\mathbb{Z}}$ such that

$$\alpha_+^{k_i}\alpha_-^{k_{i+1}} > 0 \quad \textit{for all } i,$$

there exists a trajectory $x(t)$, $t \in \mathbb{R}$, with energy $H = -\varepsilon$ shadowing a chain $(\gamma_{k_i} \in \Omega_{k_i})_{i\in\mathbb{Z}}$.

The case of transversal homoclinics was studied by Turayev and Shilnikov (1997).

Corollary

If $r(Q) \geq 3$, there exist chaotic trajectories with energy $H = -\varepsilon$.

Poincaré function

- Consider a slowly time dependent system with Hamiltonian $H_\tau(q,p)$. Let $z_0(\tau)$ be a hyperbolic equilibrium of the frozen system and

$$W^\pm(\tau) = \{x : \phi_\tau^t(x) \to z_0(\tau) \text{ as } t \to \pm\infty\}$$

the stable and unstable manifolds.
- The action of a transverse homoclinic orbit $\gamma_\tau : \mathbb{R} \to W^-(\tau) \cap W^+(\tau)$ is called the Poincaré function:

$$P(\tau) = A(\gamma_\tau) = \oint_{\gamma_\tau} p\,dq.$$

- For a natural system the Poincaré function corresponding to a minimal homotopy class Ω is

$$P(\tau) = \inf_\Omega J_\tau(\gamma).$$

Multibump homoclinics

Assume that H is periodic in τ and $P(\tau) \neq \mathrm{const}$. Let S_k be ordered components of the set of local minimum points of P. Let us parametrize trajectories by the slow time $\tau = \varepsilon t$.

Theorem (B-MacKay 2003)

For small $\varepsilon > 0$ and any sequence $(k_i)_{i \in \mathbb{Z}}$ there exists a trajectory $x(\tau)$, $\tau \in \mathbb{R}$, and a sequence $\tau_i \in S_{k_i}$ such that

- *$x(\tau)$ shadows a homoclinic orbit $\gamma_{\tau_i} \in \Omega$ for $|\tau - \tau_i| \leq \delta$.*
- *$d(x(\tau), z_0(\tau)) \leq e^{-C/\varepsilon}$ for $\min |\tau - \tau_i| \geq \delta$.*

Same for non-natural systems and transverse homoclinics.

Comments

- If there are several minimal or transverse homoclinics, there are trajectories shadowing chains of homoclinics from these classes.
- $|E(\tau)| \le c\varepsilon$ and most of the time $E(\tau)$ is exponentially small:

$$|E(\tau)| \le e^{-C/\varepsilon} \quad \text{for} \quad \min_i |\tau - \tau_i| \ge \delta.$$

- At $\tau = \tau_i$ the energy has spikes of order ε. The spikes are rare:

$$\Delta\tau_i = \tau_{i+1} - \tau_i \ge \delta, \qquad \Delta t_i = \varepsilon^{-1}\Delta\tau_i \sim \varepsilon^{-1}.$$

- The energy doesn't grow substantially.

Trajectories with substantial change of energy

We fix arbitrary $0 < a < b$ and construct trajectories with $E \in -\varepsilon[a, b]$. Suppose the frozen system has leading transverse homoclinics γ_τ^k. We call a sequence (code) (k_i) positive if $\alpha_+(\gamma_\tau^{k_i})\alpha_-(\gamma_\tau^{k_{i+1}}) > 0$ for all i.

Theorem

For $\varepsilon > 0$, any positive code $(k_i)_{i=1}^N$, any τ_0 and $h_0 \in (a, b)$, there exist a sequence (τ_i) and a trajectory $x(\tau)$ such that:

- *The trajectory $x(\tau)$ shadows the homoclinic chain $(\gamma_{\tau_i}^{k_i})$.*
- *$d(x(\tau), z_0(\tau))$ has a local minimum $\sim \sqrt{\varepsilon}$ at $\tau = \tau_i$.*
- *$\Delta\tau_i = \tau_{i+1} - \tau_i = \frac{\varepsilon|\ln\varepsilon|}{\lambda(\tau_i)} + O(\varepsilon)$,*
- *$E(\tau_0) = -\varepsilon h_0$, and while $E(\tau) \in -\varepsilon(a, b)$,*

$$\Delta\eta_i = \eta_{i+1} - \eta_i = \varepsilon P'_{k_i}(\tau_i) + O(\varepsilon^2), \qquad \eta_i = E(\tau_i)\lambda(\tau_i).$$

Idea of the proof

- Trajectories of a slowly time dependent system may be represented as trajectories of an autonomous system with Hamiltonian

$$\hat H(q,p,\tau,h) = \frac{H(q,p,\tau)}{h}, \qquad \hat\omega = dp\wedge dq + dh\wedge d\tau,$$

on the energy level $\hat H = -\varepsilon$.

- Equilibrium $z_0(\tau)$ is replaced by a normally hyperbolic critical manifold

$$N = \{(z_0(\tau),\tau,h) : 0 < a \le h \le b\}.$$

- Then we can use the generalized Shilnikov theorem for normally hyperbolic critical manifolds (B-Negrini 2013).
- The problem is reduced to finding critical points of a discrete action functional ($h_i = -E(\tau_i)/\varepsilon$):

$$\mathcal{A} = \sum \left(\left(\tau_{i+1} - \tau_i + \frac{\varepsilon \ln \varepsilon}{\lambda(\tau_i)}\right) h_i + P_{k_i}(\tau_{i+1}) + O(\varepsilon) \right).$$

Relation to adiabatic invariants

- If $k_i \equiv k$, the sequence (τ_i, E_i), $E_i = E(\tau_i)$ shadows a trajectory of a Hamiltonian system with Hamiltonian

$$J_k(\tau, E) = P_k(\tau) - \frac{E \ln \varepsilon}{\lambda(\tau)}.$$

- For $|E| \sim \varepsilon$, this looks similar to the adiabatic invariant of a system with 1 dof near a homoclinic loop:

$$I_k(\tau, E) = P_k(\tau) - \frac{E \ln |E|}{\lambda(\tau)} + O(|E|).$$

- We can switch from moving along a curve $J_{k_1}(\tau, E) = \mathrm{const}$ to the curve $J_{k_2}(\tau, E) = \mathrm{const}$ as we please. For an appropriate choice of the code (k_i) the sequence (τ_i, E_i) moves in a prescribed way.

Remarks

- The time interval $0 \le t \le \mathcal{T} \sim N|\ln\varepsilon|$ in the theorem is relatively short: for longer time the trajectory will exit a neighborhood of the homoclinic set.
- To construct trajectories on time interval $0 \le t \le \varepsilon^{-1}T$, we have to choose the code (k_i) appropriately.
- If the set of positive codes is rich enough, we get "random" trajectories.

Suppose there are transverse positive homoclinics γ_τ^1 and γ_τ^2 and $\alpha_+(\gamma_\tau^1)\alpha_-(\gamma_\tau^2) > 0$, $\alpha_+(\gamma_\tau^2)\alpha_-(\gamma_\tau^1) > 0$. Suppose the Poincaré functions $P_k(\tau) = A(\gamma_\tau^k)$ satisfy $P_1'(\tau)P_2'(\tau) < 0$.

Corollary

Let $h : [0, T] \to (-\infty, 0)$ be a Lipschitz function. For small $\varepsilon > 0$ there exist sequences $(k_i)_{i=1}^N$, $(\tau_i)_{i=1}^N$ and a trajectory $x(\tau)$, $0 \leq \tau \leq T$, such that

- *$x(\tau)$ shadows the homoclinic chain $(\gamma_{\tau_i}^{k_i})_{i=1}^N$.*
- *$d(x(\tau), z(\tau))$ has a local min $\sim \sqrt{\varepsilon}$ at $\tau = \tau_i$.*
- $\Delta\tau_i = \tau_{i+1} - \tau_i = \frac{\varepsilon|\ln\varepsilon|}{\lambda(\tau_i)} + O(\varepsilon)$.
- $|E(\tau_i) - \varepsilon h(\tau_i)| \leq \frac{C\varepsilon}{|\ln\varepsilon|}$.

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