# On the stochasticity parameter of quadratic residues

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# The plan

- I. The stochasticity parameter
- II. Quadratic residues
- III. The stochasticity parameter of quadratic residues

Let  $\mathbb{T}_n=\mathbb{R}/n\mathbb{Z}$  be the circle of length n. Let  $k\in\mathbb{N}$  and  $U=\{0\leqslant u_1< u_2< ...< u_k< n\}$  be a k-element subset of  $\mathbb{T}_n$ .

Denote by  $s_i = u_{i+1} - u_i \in \mathbb{R}^+$ , i = 1, ..., k, consecutive distances between elements of U (we set  $s_k = u_1 + n - u_k$ ).

Following V.I.Arnold, define the stochasticity parameter S(U) of the set U to be

$$S(U) = \sum_{i=1}^{k} s_i^2.$$

Since  $\frac{n}{k} = \frac{1}{k} \sum_{i=1}^k s_i \leqslant \left(\frac{1}{k} \sum_{i=1}^k s_i^2\right)^{1/2}$  and  $\sum_{i=1}^k s_i^2 < \left(\sum_{i=1}^k s_i\right)^2 = n^2$ , it follows that

$$\inf_{|U|=k} S(U) = \frac{n^2}{k}$$

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$$\mathbb{P}(s_i > t) = \mathbb{P}(s_1 > t) = \left(\frac{n-t}{n}\right)^{k-1}$$

and, hence

$$\mathbb{E}s_i = \int_0^n \mathbb{P}(s_i > t) dt = \int_0^n (1 - t/n)^{k-1} dt = n \int_0^1 (1 - v)^{k-1} dv = n \int_0^1 v^{k-1} dv = n/k$$

(as should be, because  $\mathbb{E}(\sum_{i=1}^k s_i) = n$ ) and

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We see that

$$\mathbb{P}(s_i > t\mathbb{E}s_i) = \mathbb{P}(s_i > tn/k) = (1 - t/k)^{k-1} = e^{-t}(1 + o(1)), \quad k \to \infty,$$

uniformly for  $0\leqslant t\leqslant t_0$  for any fixed  $t_0$ , and so for large k the normalized gaps  $\tilde{s_i}=s_i/\mathbb{E}s_i$  have the exponential distribution with parameter 1.

Hence, the function  $N(t)=\#\{i:u_i\leqslant t\}$  behaves like the Poisson point process with constant rate 1 and the number N(a,b) of points  $u_i$  in an interval (a,b] has Poisson distribution with mean b-a.

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It is easy to see that S(U) is minimal when  $s_i$  are equal (or close) to n/k, and is maximal when U is an interval of length k (so  $\max_{|U|=k} S(U) = (n-k+1)^2 + k - 1$ ).

So too small or too large values of S(U) indicate that U is far from a random set

One can find the mean value s(k)=s(M,k) of S(U) over all k-element subsets of  $\mathbb{Z}_M$ .

Proposition 1. We have

$$s(k) = M \frac{2M - k + 1}{k + 1}$$
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Note that  $s(k) \sim \frac{2M^2}{k+1}$  whenever k = o(M).

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Recall that an element  $x\in\mathbb{Z}_n$  is called a quadratic residue if there is  $y\in\mathbb{Z}_n$  with  $x=y^2$ . Let  $R_n$  be the set of quadratic residues modulo n.

If  $p\geqslant 2$  is a prime, then there are (p+1)/2 quadratic residues modulo p: they are exactly

$$0^2, 1^2, 2^2, ..., \left(\frac{p-1}{2}\right)^2$$

(since  $k^2=(p-k)^2$  and if  $0\leqslant a< b\leqslant (p-1)/2$ , then  $(a-b)(a+b)\neq 0$  in  $\mathbb{Z}_p$ )

Also it is easy to show that

$$\frac{p^{r-1}(p-1)}{2} \leqslant |R_{p^k}| \leqslant \frac{p^{r-1}(p+1)}{2}$$

The function  $|R_n|$  is multiplicative, that is,  $|R_{nm}|=|R_n||R_m|$  whenever (n,m)=1 (because of the Chinese Remainder Theorem).



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#### Denote by $\omega(n) = \sum_{p|n} 1$ the number of prime divisors of n.

We see that

$$\frac{n}{|R_n|} = \prod_{p|n} \left(2 + O(1/p)\right)$$

and

$$rac{n}{R_n|} o\infty$$
 if and only if  $\omega(n) o\infty$ 

Note that if n is taken from  $[1,x]\in\mathbb{Z}$  uniformly at random, then

$$\mathbb{E}\omega(n) = \log\log x + O(1)$$

and

$$\operatorname{Var}\omega(n) \leq \log\log x + O(1).$$

Then by Chebyshev's inequality we obtain a theorem of Hardy and Ramanujan: if  $f(x) \to \infty$  as  $x \to \infty$ , then

$$\omega(n) = \log \log x + O(f(x)\sqrt{\log \log x})$$

for all but o(x) numbers  $n \leqslant x$ 



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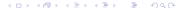
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## The limit distribution of spaces between quadratic residues

Let  $R_M=\{0=r_1< r_2<...< r_{|R_M|}\}$  be the set of quadratic residues modulo M. As earlier, set  $r_{|R_M|+1}:=r_1+M=M$ .

Take an index j randomly and uniformly in  $1,...,|R_M|$ . On average we of course have

$$\mathbb{E}(r_{j+1} - r_j) = \frac{M}{|R_M|}$$

In 1999/2000 P.Kurlberg and Z.Rudnick found the limit distribution of spaces between quadratic residues.

Proposition 2. We have

$$\mathbb{P}\left(r_{j+1} - r_j > u \frac{M}{|R_M|}\right) = e^{-u}(1 + o(1)), \qquad \omega(M) \to \infty$$

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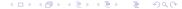
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It turns out to be false

**Theorem 1.** There exists absolute constant c>0 such that for any fixed A and M=Ap we have

$$S(R_M) = 2f_A(0.5)p + O(A^4p^{1-c})$$

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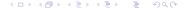
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#### Corollary. We have

$$\varliminf_{M \to \infty} \frac{S(R_M)}{s(|R_M|)} < 1 < \varlimsup_{M \to \infty} \frac{S(R_M)}{s(|R_M|)}$$

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We do not know, but

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# Examples of $f_A$

$$f_{11}(y) = \frac{27y^6 + 38y^5 + 34y^4 + 44y^3 + 34y^2 + 38y + 27}{1 + y + y^2 + y^3 + y^4 + y^5}$$
$$f_{13}(y) = \frac{37y^7 + 38y^6 + 54y^5 + 40y^4 + 40y^3 + 54y^2 + 38y + 37}{1 + y + y^2 + y^3 + y^4 + y^5 + y^6}$$

# The genuine definition of $f_A$

Let  $\{s_1,...,s_{|R_A|}\}$  be consecutive distances between quadratic residues modulo A . In fact,

$$f_A(y) = \frac{F(y)}{Q(y)}$$

where  $Q(y) = Q_A(y) = 1 + y + \ldots + y^{|R_A|-1}$  and

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is the reciprocal polynomial with the coefficients  $\beta_0 = \beta_{|R_A|} = \sum_i s_i^2 = S(R_A)$  and  $\beta_k = 2\sum_{i=1}^{|R_A|} s_i s_{i+k}$  for  $0 < k < |R_A|$  (we think of indices i as elements of  $\mathbb{Z}_{|R_A|}$ )

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### Refined conjecture: believed for almost all moduli

So, a more accurate question is the following. Do we have

$$S(R_M) \sim s(|R_M|), \quad M \to \infty,$$

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## $M = Ap_1 \dots p_t$

Let us turn to the second term in the asymptotics

**Theorem 1 (once again).** There exists absolute constant c>0 such that for any fixed A and M=Ap we have

$$S(R_M) = 2f_A(0.5)p + O(A^4p^{1-c})$$

where  $f_A$  is a function determined by the number A.

Why only one large prime factor? Why only a fixed A?

Let  $c_0$  and  $C_0$  be positive absolute constants,  $c_0$  is small,  $C_0$  is large. Let  $\Omega$  be the set of all positive integers M such that M=Am, where

- (i) A is square-free, (A, m) = 1 and  $A \leq 2^{c_0 t}$ ;
- (ii)  $m=p_1\dots p_t,\ t\geqslant 0.4\log\log M$  and  $p_1< p_2<\dots < p_t$  are primes greater than  $2^{C_0t},$

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#### Our main result is the following.

**Theorem 2.** There exists absolute constant c>0 such that for  $M\in\Omega$  we have

$$S(R_M) = m2^{t+1}A^2|R_A|^{-1} - A^2|R_A|^{-1}m + E,$$

where

$$E \ll m2^{3t}A^4p_1^{-c} + mA^2|R_A|2^{-t} = o(m), \quad M \to \infty, M \in \Omega.$$

Moreover, the set  $\Omega$  has positive lower density.

(Let us note that 
$$m2^{t+1}A^2|R_A|^{-1}=2M\cdot\frac{A}{|R_A|}2^t\sim\frac{2M^2}{|R_M|}).$$

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#### Corollaries

#### Corollary. We have

$$S(R_M) = s(|R_M|)(1 + o(1)), \quad M \to \infty, M \in \Omega.$$

It is a generalization of the mentioned result of Garaev, Konyagin, Malykhin

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# Another conjecture

Gravitation Conjecture. The set

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We can write

$$S(R_M) = \sum_{l \geqslant 1} N_l l^2,$$

where

$$N_l = \#\{x \in \mathbb{Z}_M : x, x + l \in R_M, x + 1, \dots, x + l - 1 \notin R_M\}.$$

Let M=p be a prime for simplicity. Consider the Legendre symbol  $\left(rac{\cdot}{p}
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It is a character  $\operatorname{mod} p$ , that is, a homomorphism between  $\mathbb{Z}_p^*$  and  $\mathbb{C}^*$ 

Denote the Legendre symbol  $\operatorname{mod} p$  by  $\chi_p$  for the brevity. Our benefit is that

$$R_p(x) = \frac{1}{2}(1 + \chi_p(x) + 1(x = 0))$$

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$$N_{l} = \sum_{x \in \mathbb{Z}_{p}} R_{p}(x) R_{p}(x+l) \prod_{i=1}^{l-1} (1 - R_{p}(x+i)) =$$

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Here we have the main term  $p2^{-l}$ , the error term O(l) and  $2^l-1$  character sums of the type  $\sum_{x\in\mathbb{Z}_p}\chi_p(x+a_1)...\chi_p(x+a_r)$  with distinct  $a_1,...,a_r$ . Such a sum is estimated by  $rp^{1/2}$  in magnitude (famous Weil's theorem).

Hence

$$N_l = p2^{-l} + O(l2^l p^{1/2})$$



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For composite moduli of the type  $p_1...p_t$  with distinct large  $p_j$  (larger than  $2^t$ ) and small l (roughly less than  $M/|R_M| \sim 2^t$ ) we can find the asymptotics for  $N_l$  using a simple version of sieve method and estimates of character sums. Large values of l give negligible contribution to the sum  $\sum_l N_l l^2$ . Small arbitrary factor A gives additional technical difficulties.

In fact, we prove that

$$S(R_M) = m2^t f_A(y_t) + O(m2^{3t} A^4 p_1^{-c}),$$

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#### We are able to prove that

$$f_A(1) = \frac{2A^2}{|R_A|} \text{ and } \quad f_A'(1) = \frac{A^2}{|R_A|}.$$

Hence by Taylor's expansion

$$m2^{t}f_{A}(y_{t}) = m2^{t+1}A^{2}|R_{A}|^{-1} - mA^{2}|R_{A}|^{-1} + \frac{1}{2}f_{A}''(\theta_{t})m2^{-t}.$$

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#### Recall the definition of $\Omega$ .

Let  $c_0$  and  $C_0$  be positive absolute constants,  $c_0$  is small,  $C_0$  is large. Let  $\Omega$  be the set of positive integers M such that M=Am where

- (i) A is square-free, (A, m) = 1 and  $A \leq 2^{c_0 t}$
- (ii)  $m=p_1\dots p_t,\ t\geqslant 0.4\log\log M$  and  $p_1< p_2<\dots < p_t$  are primes greater than  $2^{C_0t}$ .

It is well-known that  $\omega(M)$  is close to  $\log\log X$  most of the time  $(M\leqslant X)$ , so  $A\leqslant 2^{c_0t}\leqslant (\log X)^{\alpha}$ ; whereas it can be shown by sieve methods that

$$\Omega_0(X) := \#\{m \leqslant X : \mu(m) \neq 0, (m, P((\log X)^{\alpha})) = 1\} \sim \frac{X}{\alpha \log \log X}.$$

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$$\# \left(\Omega \cap [1, X]\right) \gg \sum_{A \leqslant (\log X)^{\alpha}} \Omega_0(X/A) \gg \sum_{A \leqslant (\log X)^{\alpha}} \mu^2(A) \frac{X}{A \log \log(X/A)} \gg X.$$

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# Lower bound for $S(R_M)$

Theorem 3. We have

$$S(R_M) \geqslant mA^2|R_A|^{-1}(2^{t+1}-1) + O\left(\frac{M^{2-c}}{|R_M|}\right)$$

for almost all M.

# Lower bound for $S(R_M)$

Let  $M\in[1,X]$  be "a standard" number and  $p_1\leqslant p_2\leqslant p_3\leqslant ...\leqslant p_k$  are all prime divisors of M written with multiplicity. Let u>0 be fixed. How often do we have

$$\prod_{i < j} p_i < p_j^u \quad ?$$

Erdös-Bovey result ( $\sim 1970$ ):

There exists a continious increasing function  $\tau(u)\colon [0,\infty)\to [0,1]$  with  $\tau(0)=0$ ,  $\lim_{u\to\infty}\tau(u)=1$  such that

$$\#\{j: \prod_{i < j} p_i < p_j^u\} \sim \tau(u) \log \log X$$

for almost all  $M \in [1, X]$ .



#### THANK YOU FOR YOUR ATTENTION!