On families of constrictions in model of overdamped Josephson junction and Painlevé 3 equation

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Circle homeomorphisms and rotation number

$S^1 := \mathbb{R}/2\pi\mathbb{Z}$ oriented circle; $f : S^1 \to S^1$ **a positive homeomorphism.**

Definition of the **Poincaré rotation number** $\rho(f) \in \mathbb{R}/\mathbb{Z}$.
H.Poincaré. *Sur les courbes définies par les équations différentialles.* J. Math. Pures App. **I 167** (1885).

$\pi : \mathbb{R} \to S^1 :=$ the **universal covering projection:** $x \mapsto x(\operatorname{mod} 2\pi)$.
$F :=$ a **continuous lifting** of f to $\mathbb{R}$.
Defined up to postcomposition
with translation by $2\pi m$, $m \in \mathbb{Z}$.

$\rho(F)(x) := \lim_{k\to+\infty} \frac{F^k(x)}{2\pi k}$.

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{F} & \mathbb{R} \\ \downarrow \pi & & \downarrow \pi \\ S^1 & \xrightarrow{f} & S^1 \end{array}$$

Henri Poincaré (1854–1912)

The limit exists and is independent ! of x.
$\rho(f) := \rho(F)(\operatorname{mod} \mathbb{Z})$.

Rotation numbers: first examples

$\rho(F)(x) := \lim_{k\to+\infty} \frac{F^k(x)}{2\pi k}$.

$\rho(f) := \rho(F)(\mathrm{mod}\,\mathbb{Z})$.

1) $f(x) = x + 2\pi a \quad => \quad \rho(f) = a(\mathrm{mod}\,\mathbb{Z})$ for every $a \in \mathbb{R}$.

2) $f(x)$ has a **fixed point** $<=> \quad \rho(f) = 0(\mathrm{mod}\,\mathbb{Z})$.

3) $f(x)$ has a ***q*-periodic point** $<=> \quad \rho(f) = \frac{p}{q}(\mathrm{mod}\,\mathbb{Z})$ for some $p \in \mathbb{Z}$.

A **family** $f = f(x,u)$ of positive circle homeomorphisms $S^1 \to S^1$, $x \mapsto f(x,u)$;
$u :=$ the **parameter**; u lies in a domain $U \subset \mathbb{R}^n$.

The **rotation number function:** $\rho(u) := \rho(f(.,u))$: $U \to S^1 = \mathbb{R}/\mathbb{Z}$.

Main definition (a version of **Arnold tongues**)
Phase-lock areas: subsets $\{u \in U \mid \rho(u) = r\} \subset U$ with **non-empty interiors.**

Example. Consider a family $f(x, u)$ of **circle diffeomorphisms.** Let for some $u_0 \in U$ the diffeomorphism $f_{u_0}(x) = f(x, u_0)$ have a q-periodic point x_0 with $(f_{u_0}^q)'(x_0) \neq 1$. Then

$$\rho(u_0) = \frac{p}{q} \text{ for some } p = p(u_0) \in \mathbb{Z};$$

$$L_{\frac{p}{q}} := \{u \in U \mid \rho(u) = \frac{p}{q}\} \text{ is a } \textbf{phase-lock area.}$$

This example illustrates a **classical fact:** **stability** of the above q-periodic point.

Rotation number and Poincaré map of flow on 2-torus

Differential equation on 2-torus $\mathbb{T}^2 := \mathbb{R}^2_{(\phi,\tau)}/2\pi\mathbb{Z}^2$

$$\begin{cases} \dot{\phi} = f(\phi, \tau) \\ \dot{\tau} = 1 \end{cases} \qquad f \in C^1, \quad f(\phi + 2\pi, \tau) = f(\phi, \tau + 2\pi) = f(\phi, \tau).$$

Solution $\phi = \phi(\tau) \in \mathbb{R}$. Uniquely defined by $\phi_0 = \phi(0)$.
If $\phi(\tau)$ is a solution, then $\phi(\tau + 2\pi)$ is also a solution.

Flow on torus

$$\begin{cases} \dot{\phi} = f(\phi, \tau) \\ \dot{\tau} = 1 \end{cases} \qquad f \in C^1, \quad f(\phi + 2\pi, \tau) = f(\phi, \tau + 2\pi) = f(\phi, \tau).$$

Rotation number of flow:

$$\rho := \lim_{k \to +\infty} \frac{\phi(2\pi k)}{2\pi k} \in \mathbb{R}.$$

The **Poincaré map** of the circle $S^1 = \mathbb{R}_\phi / 2\pi\mathbb{Z} = S^1_\phi \times \{0\} \subset \mathbb{T}^2_{(\phi,\tau)}$:

$$h : S^1_\phi \to S^1_\phi, \ (\phi(0), 0) \mapsto (\phi(2\pi), 0).$$

Properties of the rotation number ρ:
1) Independent on choice of the initial condition $\phi_0 = \phi(0)$.
2) the rotation number **of flow mod $\mathbb{Z}$,** equals
the rotation number of the **Poincaré map.**

Phase-lock areas in family of dynamical systems on 2-torus

$$\begin{cases} \dot{\phi} = f(\phi, \tau; u) \\ \dot{\tau} = 1 \end{cases} ; \quad u \in U \subset \mathbb{R}^n \text{ is the parameter.} \qquad \text{(B)}$$

The rotation number function $\rho : U \to \mathbb{R}$**:**

$$\rho(u) := \lim_{k \to +\infty} \frac{\phi(2\pi k; u)}{2\pi k} \in \mathbb{R}.$$

Phase-lock areas: subsets $\{u \in U \mid \rho(u) = r\} \subset U$ with **non-empty interiors.**

Example. Let for $u = u_0$ (B) have an attracting (or repelling) **periodic orbit** (<=> h have attracting (repelling) q-periodic point). Then

Period of the orbit $= 2\pi q, \; q \in \mathbb{Z}; \qquad \rho(u_0) = \frac{p}{q} \in \mathbb{Q}$ for some $p \in \mathbb{Z}$;

$L_{\frac{p}{q}} := \{u \in U \mid \rho(u) = \frac{p}{q}\}$ is a **phase-lock area.**

Goal: study phase-lock areas in a model of **Josephson junction (superconductivity).**

Superconductivity

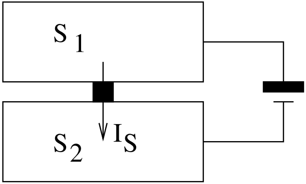
Phenomenon, when the **electric resistance** becomes **exactly zero.**

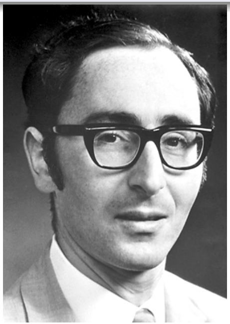
Occurs in **some** metals at temperature $\mathbb{T} < \mathbb{T}_{crit}$.

The resistance **jumps to zero,** once $\mathbb{T}$ becomes **less than** $\mathbb{T}_{crit}$.

The Josephson effect

Let two superconductors S_1, S_2 be separated
by a very narrow dielectric
or a very narrow metal layer,
thickness $\leq 10^{-5}$cm ($<<$ distance in Cooper pair).
There exists a **supercurrent I_S through the dielectric.**





Born in UK in 1940.
Nobel Prize in 1973
for "the discovery of tunnelling supercurrents".

Supercurrent is carried by coherent **Cooper pairs** of electrons.

Josephson effect

Quantum mechanics. State of S_j: wave function $\Psi_j = |\Psi_j| e^{i\chi_j}$;

χ_j are the **phases**, $\phi := \chi_1 - \chi_2$ is the **phase difference.**

The first Josephson relation: $I_S = I_c \sin\phi$, $I_c \equiv const$.

Josephson voltage relation: $V(t) = \frac{\hbar}{2e}\dot{\phi}$.

General mathematical model

$$\varepsilon_1 \frac{d^2\phi}{dt^2} + \varepsilon_2 \frac{d\phi}{dt} + \sin\phi = f(t); \quad \varepsilon_1, \varepsilon_2 = const.$$

In physical works this equation is called the **Langevin equation.**

Our main, special "overdamped" case: $\varepsilon_1 = 0$, $\varepsilon_2 = 1$, $f(t) = B + A\cos\omega t$.

$$\frac{d\phi}{dt} = -\sin\phi + B + A\cos\omega t.$$

$$\frac{d\phi}{dt} = -\sin\phi + B + A\cos\omega t.$$

V.M.Buchstaber, O.V.Karpov, S.I.Tertychnyi (2004): the model is equivalent to a family of dynamical systems on two-torus $\mathbb{T}^2_{(\phi,\tau)} = \mathbb{R}^2/2\pi\mathbb{Z}^2$, $\tau = \omega t$:

$$\begin{cases} \dot{\phi} = \frac{1}{\omega}(-\sin\phi + B + A\cos\tau) \\ \dot{\tau} = 1 \end{cases}. \tag{1}$$

Our main problem: Describe the **rotation number** ρ of the flow (1) as a **function on the space** $(B, A; \omega)$ **with fixed** $\omega > 0$**.**

Phase-lock area:= a level set $\{\rho = r\}$ if it has a **non-empty interior.**

Quantization effect (Buchstaber, Karpov, Tertychnyi, 2010): **phase-lock areas** exist only for **integer rotation values.**

In a paper by I.A.Bizhaev, A.V.Borisov, I.S.Mamaev (2017) it was noticed that a big family of dynamical systems on torus in which the rotation number quantization effect realizes was introduced by **W.Hess** (1890).

It appears that in classical mechanics such systems were studied in problems on ridig body movement with fixed point in works by **W.Hess, P.A.Nekrassov, A.M.Lyapunov, B.K.Mlodzejewski, N.E.Zhukovsky** and others.

In relation to the problems presented in our talk let us mention two papers:

1) **B.C.Mlodzejewski and P.A.Nekrassov** *On conditions of existence of asymptotic periodic movement in Hess' problem.* Proc. Physical Science Department of the Imperial Society of Amateurs of Natural Sciences, **VI** (1893).

2) **P.A.Nekrassov.** *Étude analytique d'un cas de mouvement d'un corps pesant autour d'un point fixe.* Mat. Sb., **18** (1896), No. 2, 161–274,
where it was observed that the above-mentioned big family of dynamical systems considered by Hess can be equivalently described by a Riccati equation.

Let us return to our family of systems on $\mathbb{T}^2$:

$$\begin{cases} \dot\phi = \frac{1}{\omega}(-\sin\phi + B + A\cos\tau) \\ \dot\tau = 1 \end{cases} . \tag{1}$$

Its **Poincaré map** $h = h(B, A) : S^1 \to S^1$ is **Möbius.**

Quantization effect arises in families of **Möbius circle transformations** $h = h(B, A)$ that are **strictly monotonous** in a parameter B,
since for every $h \neq Id$:
- either h has a **fixed point on the circle** $=>$ (B, A) lies in a phase-lock area;
- or h is analytically **conjugated to a rotation** $=>$ $\rho(h)$ is locally strictly monotonous in B $=>$ (B, A) is not in a phase-lock area.

$$\begin{cases} \dot\phi = \frac{1}{\omega}(-\sin\phi + B + A\cos\tau) \\ \dot\tau = 1 \end{cases}. \qquad (1)$$

Many works concern systems (1)

A subfamily of these systems occurred in the work by **Yu.S.Ilyashenko, J.Guckenheimer** (2001) from the slow-fast system point of view.

In a paper by **R. Foote, M. Levi, S. Tabachnikov** (2013) it was noticed that family (1) arises

in the investigation of some systems with non-holonomic connections.

In Prytz planimeter model and in cinematics of bicycle moving

Analogous equation describes the observed direction to a given point at infinity while moving along a geodesic in the hyperbolic plane.

$$
\begin{cases} \dot{\phi} = \frac{1}{\omega}(-\sin\phi + B + A\cos\tau) \\ \dot{\tau} = 1 \end{cases} . \tag{1}
$$

Transversal Regularity Theorem (V.Buchstaber, A.Glutsyuk).

The **rotation number function** $\rho : \mathbb{R}^2_{(B,A)} \to \mathbb{R}$ with fixed ω gives a **regular analytic fibration over** $\mathbb{R} \setminus \mathbb{Z}$ by curves that are 1-to-1 analytically projected onto the vertical A-axis.

Boundaries of phase-lock areas

1). There exist functions $\psi_{r,\pm}(A)$ **analytic in** $A \in \mathbb{R}$ such that the **boundary** ∂L_r **is the union of their graphs:**

$$\partial L_r = \partial_+ L_r \cup \partial_- L_r, \ \partial_\pm L_r := \{B = \psi_{r,\pm}(A)\}.$$

$\psi_{r,\pm}(A)$ have asymptotics of **Bessel type** $r\omega \pm J_r(-\frac{A}{\omega}) + O(\frac{\ln|A|}{A})$**,** as $A \to \infty$.

Observed by **S.Shapiro, A.Janus, S.Holly** (1964);
Confirmed **numerically** by **V.M.Buchstaber. O.V.Karpov, S.I.Tertychnyi** (2005).

Proved by **A.V.Klimenko** and **O.L.Romaskevich** (2014).

Explanation of graph decomposition. **Symmetry:** $(\phi, \tau) \mapsto (\pi - \phi, -\tau)$.

$=>$ If the Poincaré map h is **parabolic** (has **one fixed point** in the cirle), then the $\phi = \pm\frac{\pi}{2}(mod 2\pi\mathbb{Z})$ **is a fixed point** and **vice versa.**

$$=> \quad \partial L_r = \cup_\pm \partial_\pm L_r, \ \partial_\pm L_r = \{(B, A) \in L_r \mid h(\pm\frac{\pi}{2}) = \pm\frac{\pi}{2}\}.$$

2) Each L_r is an infinite chain (garland) of domains going to infinity, separated by points of intersection $\partial_+ L_r \cap \partial_- L_r$.

Observed numerically by **Buchstaber, Karpov, Tertychnyi.**
Proved by **Klimenko** and **Romaskevich** (2014).

The separation points with $A \neq 0$ are called **constriction points (constrictions).**

The separation points of L_r with $A = 0$ exist for $r \neq 0$, are called **growth points** and their abscissas B_r satisfy the equation $B_r^2 - r^2\omega^2 = 1$.

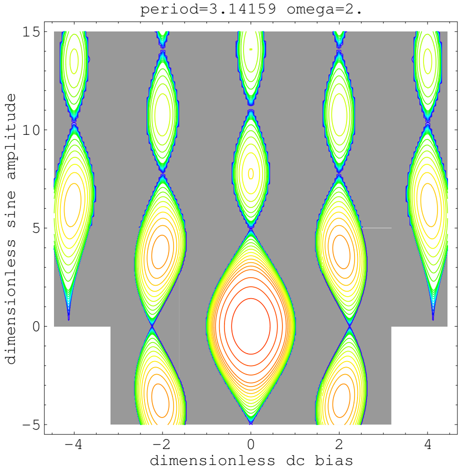
The phase-lock area L_0 has no growth points; it intersects the B-axis by $[-1, 1]$.

We present pictures of phase-lock areas for different values of ω. They are **symmetric** with respect to coord. axes: $(B, A) \mapsto (-B, A)$; $(B, A) \mapsto (B, -A)$.

Taking into account these symmetries, we present only upper parts of the pictures.

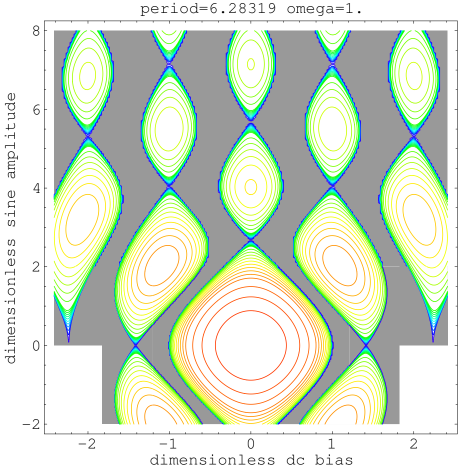
Phase-lock areas for $f(t) = B + A\cos\omega t$, $\omega = 2$

Each phase-lock area L_r is an infinite chain (garland) of domains going to infinity, separated by points of intersection $\partial_+ L_r \cap \partial_- L_r$.
The separation points with $A \neq 0$ are called **constriction points (constrictions).**



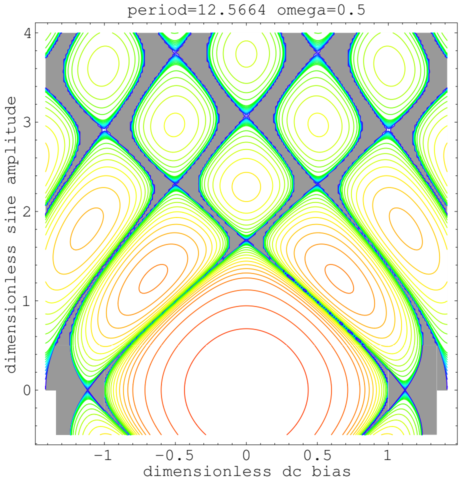
Phase-lock areas for $f(t) = B + A\cos\omega t$, $\omega = 1$

- infinitely many **constrictions** in every phase-lock area.



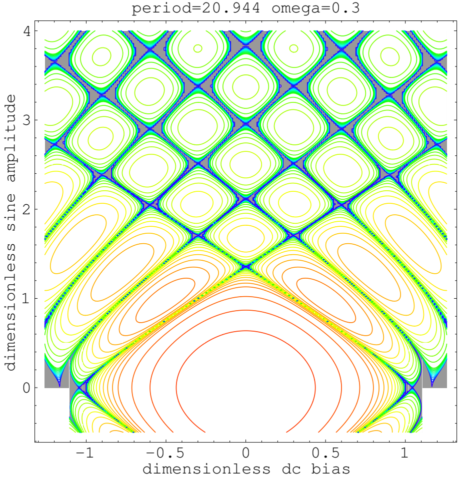
Phase-lock areas for $f(t) = B + A\cos\omega t, \omega = 0.5$

- infinitely many **constrictions** in every phase-lock areas.



Phase-lock areas for $f(t) = B + A\cos\omega t$, $\omega = 0.3$

- infinitely many **constrictions** in every phase-lock areas.



Constrictions:$=$ the separation points in L_r, $r \in \mathbb{Z}$, with $A \neq 0$.

The constrictions $\mathcal{A}_{r,1}, \mathcal{A}_{r,2}, \mathcal{A}_{r,3}, \dots$ with $A > 0$ are ordered by their A-coordinates.

.

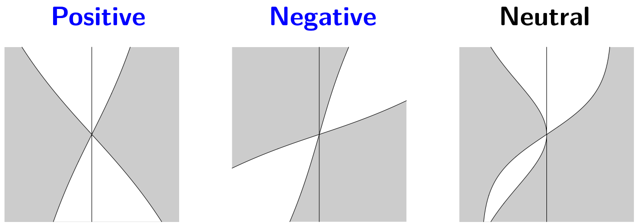
Theorem 1 (quantization of constrictions) (Yu.Bibilo, A.G.): *All the constrictions $\mathcal{A}_{r,k}$ lie in the line $\Lambda_r := \{B = r\omega\} :=$* **the axis** *of the area L_r.*

This is a confirmation of an experimental fact that was earlier discovered in numerical simulations (**Tertychnyi, Filimonov, Kleptsyn, Schurov, 2011**).

Previous results (Filimonov, Glutsyuk, Kleptsyn, Schurov, 2014) Each constriction $\mathcal{A}_{r,k}$ lies in a line $\{B = \ell\omega\}$, where $\ell \in [0, r]$ and $\ell \equiv r(mod 2\mathbb{Z})$.

Theorem 1 holds for $\omega \geq 1$.

Definition. **A priori possible types of constrictions:**



Theorem 2 (Yu.Bibilo, A.G.). Each constriction is **positive.**

Known: For every constriction $\ell := \frac{B}{\omega} \in \mathbb{Z}$
(Filimonov, Glutsyuk, Kleptsyn, Schurov, 2014).

Definition. **Ghost constriction:**$=$ a constriction satisfying one of two conditions:
- either the **rotation number** $\rho \neq \ell$,
- or the constriction is **non-positive.**

Theorems 1 and 2 state that there are **no ghost constrictions.**

Plan of proof of main results: no ghost constrictions

Known properties of constrictions: $\ell := \frac{B}{\omega} \in \mathbb{Z}$, $\ell \in [0, \rho]$, $\ell \equiv \rho (\mathrm{mod}\, 2)$.

Ghost constriction:= either non-positive, or $|\ell| < |\rho|$

Theorem A. For every given $\ell \in \mathbb{Z}$ each constriction $(\ell\omega, A; \omega)$ can be deformed to another constriction of the same type, ℓ, ρ, with **arbitrarily small** ω**.**

$=>$ **Ghost constrictions** are deformed to **ghost constrictions** with small ω.

Proof is based on equivalent description of the model by **complex linear equations on** $\overline{\mathbb{C}}$ and studying their **isomonodromic deformations.**

Theorem B. For every given $\ell \in \mathbb{Z}$ and every $\omega > 0$ small enough there are **no ghost constrictions** with $B = \ell\omega$.

Theorem B is proved by methods of theory of **slow-fast** families of dynam. systems.

Equivalent description of model by linear systems on $\overline{\mathbb{C}}$.

$$\frac{d\phi}{d\tau} = \frac{1}{\omega}(\sin\phi + B + A\cos\tau) = \frac{\sin\phi}{\omega} + \ell + 2\mu\cos\tau. \tag{2}$$

V.M.Buchstaber, O.V.Karpov, S.I.Tertychnyi, 2004. The variable changes

$$z := e^{i\tau}, \quad \Phi := ie^{i\phi}$$

transforms (2) to Riccati equation

$$\frac{d\Phi}{dz} = z^{-2}((\ell z + \mu(z^2+1))\Phi + \frac{z}{2\omega}(\Phi^2+1)). \tag{3}$$

$\Phi(z)$ is a solution of (3) $<=>$ $\Phi(z) = \frac{v}{u}(z)$, where $Y = (u,v)(z)$ is a solution of

$$Y' = \left(\frac{\mathrm{diag}(-\mu,0)}{z^2} + \frac{R}{z} + \mathrm{diag}(-\mu,0)\right)Y, \quad R = \begin{pmatrix} -\ell & -\frac{1}{2\omega} \\ \frac{1}{2\omega} & 0 \end{pmatrix}, \tag{4}$$

In other words, (3) is the **projectivization** of (4).

$$\frac{d\phi}{d\tau} = \frac{1}{\omega}(\sin\phi + B + A\cos\tau) = \frac{\sin\phi}{\omega} + \ell + 2\mu\cos\tau. \qquad (2)$$

$$Y' = \left(\frac{\mathrm{diag}(-\mu, 0)}{z^2} + \frac{R}{z} + \mathrm{diag}(-\mu, 0)\right) Y, \quad R = \begin{pmatrix} -\ell & -\frac{1}{2\omega} \\ \frac{1}{2\omega} & 0 \end{pmatrix}, \qquad (4)$$

The monodromy operator M of system (4).
Acts on its local solution space $\mathbb{C}^2 = \mathbb{C}^2 \times \{z_0\}$ at $z_0 \in \mathbb{C}^*$
by analytic extension along counterclockwise circuit around 0.

Fact. $(B, A; \omega)$ is a **constriction** $<=>$ (4) has **trivial monodromy:** $M = Id$.
D.A.Filimonov, A.G., V.A.Kleptsyn, I.V.Schurov, 2014.

Linear systems with irregular nonresonant singularities.
Classical theory (Birkhoff, Jurkat, Lutz, Peyerimhoff, Balser, Sibuya).

$$Y' = \left(\frac{K}{z^2} + \frac{R}{z} + N + O(z)\right) Y, \quad Y = \begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{C}^2, \tag{5}$$

Germ at 0; $K, R, N \in \mathrm{Mat}_2(\mathbb{C})$, K has **distinct eigenvalues** $\lambda_1 \neq \lambda_2$.

Then we say that 0 is **irregular non-resonant singularity of Poincaré rank 1.**

Two germs (5), (5)' are **analytically equivalent,** if there exists a germ of holomorphic $\mathrm{GL}_2(\mathbb{C})$-valued function $H(z)$ such that $Y = H(z)\widetilde{Y}$ sends (5) to (5)'.

(5)$\simeq$(5)' **formally**, if this holds for a **formal** invertible matrix power **series** $\widehat{H}(z)$.

Theorem. System (5) is **formally equivalent** to a unique **formal normal form**

$$\widetilde{Y}' = \left(\frac{\widetilde{K}}{z^2} + \frac{\widetilde{R}}{z}\right) \widetilde{Y}, \ \widetilde{K} = \mathrm{diag}(\lambda_1, \lambda_2), \ \widetilde{R} = \mathrm{diag}(b_1, b_2). \tag{6}$$

Here $\widetilde{K} = \mathbf{H}^{-1} K \mathbf{H}$ for some $\mathbf{H} \in \mathrm{GL}_2(\mathbb{C})$; $\widetilde{R} =$ **the diagonal part** of $\mathbf{H}^{-1} R \mathbf{H}$.

Linear systems with irregular nonresonant singularities.
Classical theory (Birkhoff, Jurkat, Lutz, Peyerimhoff, Balser, Sibuya).

$$Y' = \left(\frac{K}{z^2} + \frac{R}{z} + N + O(z)\right) Y \tag{5}$$

is **generically not analytically equivalent** to its formal normal form

$$\widetilde{Y}' = \left(\frac{\widetilde{K}}{z^2} + \frac{\widetilde{R}}{z}\right)\widetilde{Y}, \ \widetilde{K} = \mathrm{diag}(\lambda_1, \lambda_2), \ \widetilde{R} = \mathrm{diag}(b_1, b_2): \tag{6}$$

for generic (5) the **normalizing series $\widehat{H}(z)$ diverges.**

There exists a covering $\mathbb{C}^* = S_0 \cup S_1$ by two sectors with vertex 0 and **analytic** functions $H_j : S_j \cap D_r \to \mathrm{GL}_2(\mathbb{C})$, $H_j \in C^\infty(\overline{S}_j \cap D_r)$, s.t. $Y = H_j(z)\widetilde{Y}$: (5)$\mapsto$(6).

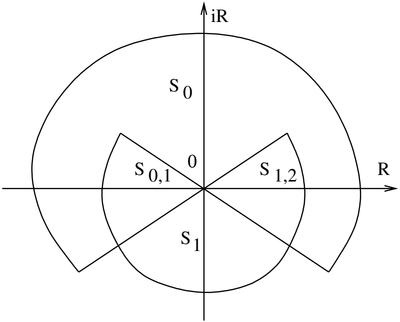
The matrix functions H_j are **unique** up to left multipl. by const diagonal matrices.

NF (6) has **canonical solution basis,** fund. matr. $W(z) = \mathrm{diag}(\widetilde{y}_1(z), \widetilde{y}_2(z))$.

$X^j(z) := H_j(z)W(z)$ are **canonical sectorial solution bases** for system (5) in S_j.

Classical theory (Birkhoff, Jurkat, Lutz, Peyerimhoff, Balser, Sibuya).

Example. Let $\lambda_1, \lambda_2 \in \mathbb{R}$, $\lambda_1 - \lambda_2 < 0$.



On intersect. component $S_{j,j+1}$ two canonical solution bases $X^0(z)$, $X^1(z)$ of (5).

$$X^1(z) = X^0(z) C_0 \text{ on } S_{0,1}; \quad X^0(z) \exp(2\pi i \widetilde{R}) = X^1(z) C_1 \text{ on } S_{1,2} \tag{7}$$

C_0, C_1 **Stokes matrices**. **Unipotent;** C_0 is **upper-triangular,** C_1 is **lower-triang.**

Theorem. (5) is **analytically** $\simeq$ to form. norm. form (6) $<=>$ $C_0 = C_1 = Id$; (5)$\simeq$(5)' anal. $<=>$ (6)=(6)', $(C_0', C_1') = D(C_0, C_1)D^{-1}$ for some diag. matr. D.

Linear systems on $\overline{\mathbb{C}}$. Monodromy–Stokes data.

$$Y' = \left(\frac{K}{z^2} + \frac{R}{z} + N\right) Y, \qquad K, R, N \in \mathrm{End}(\mathbb{C}^2), \tag{8}$$

where each one of the matrices K and N at 0, ∞ has **distinct real eigenvalues.**

Fix a $z_0 \in S_0$, e.g., $z_0 = 1$. Let $M : \mathbb{C}^2 \times \{z_0\} \to \mathbb{C}^2 \times \{z_0\}$ - **monodromy.**

In the sector S_0 two fund. solution matrices $X^{0,0}(z)$, $X^{0,\infty}(z)$: from 0 and ∞. Their 4 columns $f_{10}(z)$, $f_{20}(z)$, $f_{1\infty}(z)$, $f_{2\infty}(z)$ are solutions of (8).

$\pi : \mathbb{C}^2 \setminus \{0\} \to \mathbb{CP}^1 = \overline{\mathbb{C}}$ tautological projection.

$$q_{kp} := \pi(f_{kp}(z_0)) \in \mathbb{CP}^1 = \overline{\mathbb{C}}, \ q := (q_{10}, q_{20}, q_{1\infty}, q_{2\infty}) \in \overline{\mathbb{C}}^4.$$

$$(q, M) \simeq (q', M') \ := \ \text{ if there exists an } H \in \mathrm{GL}_2(\mathbb{C}),\ H(q) = q',\ H^{-1}M'H = M.$$

$$[q, M] = (q, M)/\simeq \ := \ \textbf{the monodromy–Stokes data.}$$

Definition. A family of systems (8) is **isomonodromic,** if $[q, M] = const$.

Method of proof of Theorem A (i.e., reduction to small ω).

$$Y' = \left(\frac{\mathrm{diag}(-\mu,0)}{z^2} + \frac{R}{z} + \mathrm{diag}(-\mu,0)\right) Y, \qquad R = \begin{pmatrix} -\ell & -\frac{1}{2\omega} \\ \frac{1}{2\omega} & 0 \end{pmatrix} \qquad \textbf{(Jos)}$$

lie in the **4-dimensional** family of **normalized** $\mathbb{R}_+$**-Jimbo type linear systems:**

$$Y' = \left(-\tau\frac{K}{z^2} + \frac{R}{z} + \tau N\right) Y, \ \tau \in \mathbb{R}_+, \ K, R, N \text{ are real 2x2-matrices}, \quad (\mathbf{J}(\mathbb{R}_+))$$

$$N = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix}, \ R = \begin{pmatrix} -\ell & -R_{21} \\ R_{21} & 0 \end{pmatrix}, \ K = -GNG^{-1}, \ N_{21} > 0, \ \ell \in \mathbb{R}, \qquad (9)$$

$$\text{where } G \in \mathrm{SL}_2(\mathbb{R}), \ G^{-1}RG = \begin{pmatrix} -\ell & * \\ * & 0 \end{pmatrix}; \text{ the elements } * \text{ are arbitrary.} \qquad (10)$$

Family $\mathbf{J}(\mathbb{R}_+)$ is foliated by **isomonodromic families** parametrized by τ, obtained from well-known **Jimbo isomonodromic families** via variable changes.
In isom. families $\ell \equiv const$, and $w(\tau) := -\frac{R_{12}(\tau)}{\tau K_{12}(\tau)}$ satisfies **Painlevé 3 equation:**

$$w'' = \frac{(w')^2}{w} - \frac{w'}{\tau} + w^3 - 2\ell\frac{w^2}{\tau} - \frac{1}{w} + (2\ell - 2)\frac{1}{\tau} \qquad \text{(P3)}$$

Method of proof of Theorem A (i.e., reduction to small ω).

Key argument 1. Systems Jos correspond to **1st order poles with res** $= 1$ of solutions $w(\tau) := -\frac{R_{12}(\tau)}{\tau K_{12}(\tau)}$ of P3: $K_{12} = 0$ and $R_{12} \neq 0$ on **Jos**.

$=>$ **Jos** is a **local cross-section** to the **isomonodromic foliation** of $\mathbf{J}(\mathbb{R}_+)$.

$$\Sigma_\ell := \{\text{systems in } \mathbf{J}(\mathbb{R}_+) \text{ with trivial monodromy and given } \ell \in \mathbb{Z}\}.$$

Key argument 2. Σ_ℓ is a 2-dim. submanifold foliated by isomonodromic leaves.

With **submersive** projection $\mathcal{R} : \Sigma_\ell \to \mathbb{RP}^1 = \mathbb{R} \cup \{\infty\}$ constant along the leaves.

$\mathcal{R}$ is the **cross-ratio** of $(q_{10}, q_{20}, q_{1\infty}, q_{2\infty})$; an **analytic invariant** of lin. system.

$Constr_\ell := \Sigma_\ell \cap$ **Jos** $=$ **constrictions** with $B = \ell\omega$;
Non-trivial fact: $\mathcal{R}(Constr_\ell)$ lies in $\mathbb{R}$: does not contain ∞.

Corollary of 1 and 2. $Constr_\ell$ is **local cross-section** to isomon. foliation of Σ_ℓ

$=>$ each component $\mathcal{C}$ in $Constr_\ell$ is **1-to-1 parametr.** by interval $(a, b) = \mathcal{R}(\mathcal{C})$.

$$\mathcal{R} := \frac{(q_{10} - q_{1\infty})(q_{20} - q_{2\infty})}{(q_{10} - q_{2\infty})(q_{20} - q_{1\infty})}.$$

For each component $\mathcal{C}$ in $Constr_\ell$ one has

$$\mathcal{R} : \mathcal{C} \to \mathcal{R}(\mathcal{C}) = (a, b) \subset \mathbb{R} \quad \textbf{is an analytic diffeomorphism.}$$

The **inverse:** $(\mathcal{R}|_{\mathcal{C}})^{-1} : x \in (a, b) \mapsto C(x) \in \mathcal{C}$.

Theorem. For every $c \in \{a, b\} \setminus \{0\}$ there exists a sequence $x_k \in (a, b)$, $x_k \to c$, such that $\omega_k := \omega(C(x_k)) \to 0$.

This allows to **deform** analytically a **ghost constriction** (if it exists) to another one, with **arbitrarily small** ω**.**