Negation as a modality in a quantified setting

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On intuitionistic negation

In predicate intuitionistic logic, proving Φ provides a verification of Φ, while proving $\neg\Phi$ gives us a demonstration that

each verification of Φ yields a verification of $\perp$,

and since $\perp$ must not be verified, it implies that Φ cannot be verified. This leads to a number of suggestions on how intuitionistic logic may be improved. Here are two examples.

- Kreisel and others criticized 'trivialization' and 'irrelevance of witnesses' — this motivates modifying BHK semantics.
- Nelson suggested that we should add 'strong negation' which corresponds directly to falsification.

Došen's perspective

On the other hand, Došen proposed to take seriously the idea that

negation can be thought of as a negative modality

— or more precisely, as a modal operator of impossibility. In the standard possible world semantics for intuitionistic logic we have

$$w \Vdash \neg\Phi \quad \Longleftrightarrow \quad u \nVdash \Phi \text{ for all } u \geqslant w.$$

However, $\leqslant$ is too strong for our purposes, so it should be replaced by an accessibility relation R which argees with $\leqslant$ in a suitable way.

On ‘minimal’ negation

Axiomatically, we shall use the contraposition rule, which is rendered as

$$\frac{\Phi \to \Psi}{\neg\Psi \to \neg\Phi} \quad (\mathtt{CR})$$

Clearly, it can be viewed as an antimonotonicity rule. As in modal logic, CR — although trivially admissible — should not be derivable, viz.

$$(\Phi \to \Psi) \to (\neg\Psi \to \neg\Phi) \qquad (\mathtt{CR}')$$

should not belong to our logic. Thus even Johansson’s minimal logic is too strong, because it treats $\neg\Phi$ as $\Phi \to \bot$, and hence contains CR′.

A quantified version QN of Došen's N

We begin by expanding Došen's propositional logic N, which yields the weakest quantified logic we shall be concerned with.

Our predicate calculus naturally expands the propositional Hilbert-type system for N. It employs the following axiom schemata:

I1. $\Phi \to (\Psi \to \Phi)$;

I2. $(\Phi \to (\Psi \to \Theta)) \to ((\Phi \to \Psi) \to (\Phi \to \Theta))$;

C1. $\Phi \wedge \Psi \to \Phi$;

C2. $\Phi \wedge \Psi \to \Psi$;

C3. $\Phi \to (\Psi \to \Phi \wedge \Psi)$;

D1. $\Phi \rightarrow \Phi \vee \Psi$;

D2. $\Psi \rightarrow \Phi \vee \Psi$;

D3. $(\Phi \rightarrow \Theta) \rightarrow ((\Psi \rightarrow \Theta) \rightarrow (\Phi \vee \Psi \rightarrow \Theta))$;

N. $\neg\Phi \wedge \neg\Psi \rightarrow \neg(\Phi \vee \Psi)$;

Q1. $\forall x\, \Phi \rightarrow \Phi(x/t)$ where t is free for x in Φ;

Q2. $\Phi(x/t) \rightarrow \exists x\, \Phi$ where t is free for x in Φ.

Thus we have the 'positive' axioms of predicate intuitionistic logic plus all instances of N. Also, we employ four inference rules:

MP. the modus ponens rule.

CR. the contraposition rule.

BR1. the Bernays rule for $\forall$, which is rendered as

$$\frac{\Phi \to \Psi}{\Phi \to \forall x\, \Psi} \quad \text{provided that } x \text{ is not free in } \Phi;$$

BR2. the Bernays rule for $\exists$, which is rendered as

$$\frac{\Phi \to \Psi}{\exists x\, \Phi \to \Psi} \quad \text{provided that } x \text{ is not free in } \Phi.$$

These are the standard rules of predicate intuitionistic logic plus CR.

Denote by QN the least set of formulas containing the axioms of our calculus and closed under its rules of inference.

Given a set Γ of sentences and a formula Φ, we write $\Gamma \vdash \Phi$ iff Φ can be obtained from elements of $\Gamma \cup \mathrm{QN}$ by means of $\mathtt{MP}$, $\mathtt{BR1}$ and $\mathtt{BR2}$.

By a logic is meant simply a collection of formulas closed under the four rules above and substitutions. Each logic that includes QN will be called a QN-extension. Given a QN-extension L, we define

$$\Gamma \vdash_L \Phi \quad :\Longleftrightarrow \quad L \cup \Gamma \vdash \Phi.$$

Following Došen, by a frame we mean a triple $\mathcal{W} = \langle W, \leqslant, R \rangle$ where:

- W is a non-empty set;
- $\leqslant$ is a preordering on W;
- R is a binary relation on W such that $\leqslant \circ R \subseteq R \circ \leqslant^{-1}$.

Naturally, one might wish to consider frame conditions which are stronger than $\leqslant \circ R \subseteq R \circ \leqslant^{-1}$. In particular, the condition

$$\leqslant \circ R \;\subseteq\; R.$$

corresponds to 'condensed frames' in Došen's works. (It also appears in a recent paper by Litak and Visser on 'constructive strict implication'.)

Similarly to predicate intuitionistic logic, we can define extended domain models for QN over such frames — or QN-models, for short. This leads, of course, to the logical consequence relation $\vDash$ for QN.

Theorem (Strong Completeness for QN)

For any set Γ of sentences and formula Φ,

$$\Gamma \vdash \Phi \iff \Gamma \vDash \Phi.$$

A quantified version QN* of N*

In the course of developing a framework for the study of logic programs with negation, Cabalar, Odintsov and Pearce introduced a propositional logic N* extending N. Let us briefly mention some of its nice features.

- It allows us to define $\perp$ as $\neg(\Psi \to \Psi)$.
- It has an elegant Routley-style semantics.
- It is intimately connected with Došen's view of intuitionistic modal logics as well as Vakarelov's classification of negations.

We are going to bring quantifiers into the picture.

Define the logic QN* to be QN plus the following axiom schemata:

N1*. $\neg(\Phi \to \Phi) \to \Psi$;

N2*. $\neg((\Phi \to \Phi) \to \neg(\Psi \to \Psi))$;

N3*. $\neg(\Phi \wedge \Psi) \to \neg\Phi \vee \neg\Psi$.

Clearly, if we fix a formula Φ_0, and take

$$\top \;:=\; \Phi_0 \to \Phi_0 \quad \text{and} \quad \bot \;:=\; \neg\top,$$

then N1*, N2* and N3* turn out to be equivalent respectively to

$$\bot \to \Psi, \quad \neg(\top \to \bot) \quad \text{and} \quad \neg(\Phi \wedge \Psi) \to \neg\Phi \vee \neg\Psi.$$

Denote by $\vdash^*$ the derivability relation of QN*.

We call a frame $\mathcal{W}$ special iff R is serial, and for every $w \in W$, $R(w)$ is directed with respect to $\leqslant$. The motivation for this comes from:

Proposition (Odintsov, 2010)

A frame $\mathcal{W}$ is special if and only if the propositional versions of $\mathtt{N1}^*$, $\mathtt{N2}^*$ and $\mathtt{N3}^*$ hold in each model for N based on $\mathcal{W}$.

By QN*-models are meant QN-models based on special frames. Take $\vDash^*$ to be the relativisation of $\vDash$ to the class of QN*-models.

Theorem (Strong Completeness for QN*)

For any set Γ of sentences and formula Φ,

$$\Gamma \vdash^* \Phi \quad \Longleftrightarrow \quad \Gamma \vDash^* \Phi.$$

Finally, it was shown by Odintsov that the $\{\wedge, \vee, \rightarrow, \bot\}$-fragment of N* is, in fact, precisely propositional intuitionistic logic. As might be expected, this generalises to QN*.

Proposition

The $\{\wedge, \vee, \rightarrow, \bot\}$-fragment of QN* is precisely predicate intuit. logic.

A useful extension QN$^\sharp$ of QN*

However, it seems that QN*, unlike N*, does not have a Routley-style semantics. So we are going to modify QN* appropriately.

Define the logic QN$^\sharp$ to be QN* plus the following axiom schemata:

N1$^\sharp$. $\forall x \neg \Phi \to \neg \exists x \, \Phi$;

N2$^\sharp$. $\neg \forall x \, \Phi \to \exists x \neg \Phi$;

CD. $\forall x \, (\Phi \vee \Psi) \to \Phi \vee \forall x \, \Psi$ for x not free in Φ.

Notice that N1$^\sharp$ and N2$^\sharp$ may be thought of as playing the role of Barcan's formula. Denote by $\vdash^\sharp$ the derivability relation of QN$^\sharp$.

Following Cabalar, Odintsov and Pearce, by a Routley frame we mean a triple $\mathcal{W} = \langle W, \leqslant, * \rangle$ where:

- W is a non-empty set;
- $\leqslant$ is a preordering on W;
- $*$ is an anti-monotone function from W to W.

Evidently, Routley frames are special. These are practically equivalent to frames $\mathcal{W}$ satisfying the property that for every $w \in W$, $R(w)$ has a greatest element with respect to $\leqslant$.

By QN$^\sharp$-models we mean 'constant domain' models for QN which are based on Routley frames. Take $\vDash^\sharp$ to be the relativisation of $\vDash$ to the class of QN$^\sharp$-models.

Theorem (Strong Completeness for QN$^\sharp$)

For any set Γ of sentences and formula Φ,

$$\Gamma \vdash^\sharp \Phi \quad \Longleftrightarrow \quad \Gamma \vDash^\sharp \Phi.$$

Again, we have:

Proposition

The $\{\wedge, \vee, \rightarrow, \bot\}$-fragment of QN$^\sharp$ *is precisely predicate intuit. logic with constant domains.*

Concerning Leitgeb's quantified Hype

Recently Leitgeb advocated a certain system of hyperintensional logic; let Hype and QHype be its propositional and quantified versions.

Then it has been observed by Odintsov that, in effect, Hype coincides with the logic obtained from N* by adding the laws of double negation introduction and elimination; this gives us a Routley-style semantics for Hype which is simpler than the one suggested by Leitgeb.

On the other hand, the application to semantic paradoxes in Leitgeb's work requires QHype. Now I provide an alternative axiomatisation and semantics for QHype, and also prove the completeness result.

Define the logic QN° to be QN plus the following axiom schemata:

N1°. $\Phi \to \neg\neg\Phi$;

N2°. $\neg\neg\Phi \to \Phi$.

Denote by $\vdash^\circ$ the derivability relation of QN°.

Proposition

N *is redundant in* QN°*.*

Proposition

QN° *extends* $QN^\sharp$*.*

Corollary

QHype *(as presented by Leitgeb) coincides with* QN$^\circ$*.*

Leitgeb's axiomatisation is not 'minimal', and includes several redundant schemata (in particular, it assumes not only $\mathtt{N1}^\circ$ and $\mathtt{N2}^\circ$, but also all de Morgan's laws and $\mathtt{N2}^\sharp$); still, it leads to the same logic.

From now on we shall use QHype and QN$^\circ$ interchangeably.

We call a Routley frame $\mathcal{W} = \langle W, \leqslant, * \rangle$ involutive iff $w^{**} = w$ for every $w \in W$. By QN$^\circ$-models are meant QN$^\sharp$-models whose frames are involutive. Take $\vDash^\circ$ to be the relativisation of $\vDash$ to the class of QN$^\circ$-models.

Theorem (Strong Completeness for QHype)

For any set Γ of sentences and formula Φ,

$$\Gamma \vdash^\circ \Phi \quad \Longleftrightarrow \quad \Gamma \vDash^\circ \Phi.$$

In other words, QHype is strongly complete w. r. t. the class of involutive Routley frames. Thus it does not seem to be highly hyperintensional.

Proposition (Leitgeb; the above theorem gives a simpler proof)

The $\{\wedge, \vee, \rightarrow, \bot\}$-fragment of QN° *is precisely predicate intuit. logic with constant domains.*

Thus QN*, QN$^\sharp$, QN$^\circ$ are conservative (modal) enrichments of predicate intut. logic — with constant domains, if QN$^\sharp$ and QN$^\circ$ are concerned.

Henceforth we shall often write $-\Phi$ for $\Phi \rightarrow \bot$.

On disjunction

In his work Leitgeb mistakenly claimed that (Q)Hype has the disjunction property. It was observed by Drobyshevich that N* — and hence QN* — does not have the disjunction property, because

$$\neg p \vee \neg(-q \wedge (p \rightarrow q))$$

belongs to N*, but neither of its disjuncts does. We are going to provide a simpler counterexample. Consider the following scheme:

WEM. $\neg\Phi \vee \neg{-}\Phi$.

Here 'WEM' stands for 'weak excluded middle'.

Proposition

$\mathtt{WEM}$ *is derivable in* QN^*.

Proof.

We argue as follows:

1.	$\Phi \wedge -\Phi \rightarrow (\top \rightarrow \bot)$	positive intuitionistic logic
2.	$\neg(\top \rightarrow \bot) \rightarrow \neg(\Phi \wedge -\Phi)$	from 1 using $\mathtt{CR}$
3.	$\neg(\Phi \wedge -\Phi)$	from $\mathtt{N2}^*$, 2
4.	$\neg\Phi \vee \neg-\Phi$	from 3, $\mathtt{N3}^*$.

(Notice that the argument involves no quantifier axioms or rules.) □

Call a QN*-extension subclassical iff it is a subset of predicate class. logic.

Corollary

No subclassical QN**-extension has the disjunction property.*

Proof.

Let L be a subclassical QN*-extension. Clearly, there is a formula Φ such that neither $\neg\Phi$ nor $\neg{-}\Phi$ is in L. But $\neg\Phi \vee \neg{-}\Phi$ belongs to L. □

Evidently, QN*, QN$^\sharp$, QN$^\circ$ are subclassical. Hence none of them has the disjunction property. The same applies to their prop. versions.

On the existential quantifier

The existential property is a bit more tricky. Assume, for simplicity, that σ includes exactly two constant symbols, say 0 and 1. Now consider the following scheme:

WCP. $\neg - (\Phi(x/0) \wedge \Phi(x/1)) \vee \exists x \neg \Phi$.

Here 'WCP' stands for 'weak choice principle'.

Proposition

WCP *is derivable in* QN*.

Proof.

We shall write $\Phi(0)$ and $\Phi(1)$ for $\Phi(x/0)$ and $\Phi(x/1)$ respectively. The informal argument is:

$$\begin{aligned} \text{not } \neg\!-(\Phi(0)\wedge\Phi(1)) \quad &\overset{\texttt{WEM}}{\Longrightarrow} \quad \neg(\Phi(0)\wedge\Phi(1)) \\ &\overset{\texttt{N3}^*}{\Longrightarrow} \quad \neg\Phi(0)\vee\neg\Phi(1) \\ &\Longrightarrow \quad \exists x\,\neg\Phi. \end{aligned}$$

This can be easily turned into a formal derivation. □

Corollary

No subclassical QN**-extension has the existential property.*

Proof.

Let L be a subclassical QN*-extension. Evidently, there is a formula $\Phi(x)$ (with exactly one free variable) such that

$$\neg\!-(\Phi(0) \wedge \Phi(1)) \vee \neg\Phi(0) \quad \text{and} \quad \neg\!-(\Phi(0) \wedge \Phi(1)) \vee \neg\Phi(1)$$

are not in L. On the other hand, predicate intuitionistic logic proves

$$\Psi \vee \exists x\, \Theta \rightarrow \exists x\,(\Psi \vee \Theta) \quad \text{for } x \text{ not free in } \Phi;$$

hence $\exists x\,(\neg\!-(\Phi(0) \wedge \Phi(1)) \vee \neg\Phi(x))$ belongs to L. □

So in particular, none of QN*, QN♯, QN° has the existential property.

Some directions

- Since N clearly plays an important role in studying propositional intuitionistic modal logics, it would be interesting to investigate predicate versions of such logics.
- It may be reasonable to develop predicate versions of the logics presented by Litak and Visser.
- It will be useful to introduce and study sequent systems for QN and its extensions; this should help us analyze interpolation and certain other metamathematical properties.
- We could try to employ QN^*-extensions to provide a framework for studying first-order logic programs with negation.

A curious technical remark

- If we are concerned with constant domain models (as in the cases of $\mathsf{QN}^\sharp$ and QN°), a rather specific kind of extension lemma is needed. It might be called Cresswell's lemma (for predicate modal logics; the name has been suggested by Shehtman) — as far as I remember, it was adapted to predicate intuitionistic logic by Skvortsov.
- The proof of this lemma — at least the version I found in [QNCL] — doesn't generalise readily to uncountable signatures.
- Of course, we can still obtain weak completeness results for arbitrary signatures. But then we need other ways of proving compactness.

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