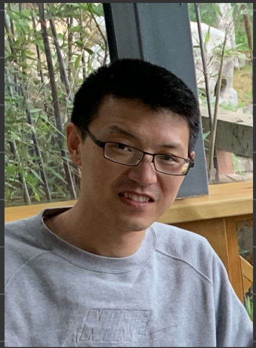
Tate-Hochschild cohomology, the singularity category and applications

Plan: 1. Hochschild cohom.

2. Tate-Hochsch. cohom & the main thm

3. Applications: 2 reconstruction thms (jt with Zheng Hua)

1. Hochschild cohomology by its history



k a field (for simplicity)

A a k-algebra (assoc., with 1, non com.)

$HH^*(A)$ = Hochschild cohom. (1945)

$\cong H^* C(A,A)$

$Hom_k(A^{\otimes p}, A)$

$C(A,A) = \left(A \to Hom_k(A,A) \to Hom_k(A \otimes A, A) \to \dots \to \quad \dots \right)$

$a \mapsto [a, ?]$, $D \longmapsto \left(a \otimes b \mapsto (Da)b - D(ab) + aDb \right)$

We see: $HH^0(A)$ = center of $A = Z(A)$: a com. alg.

$HH^1(A) \cong OutDer(A)$: a Lie alg.

$A^e = A \otimes A^{op}$, ${}_A A_A$ = identity bimodule

Cartan-Eilenberg (1956): $HH^*(A) \cong Ext^*_{A^e}(A,A)$

an algebra for the cup product: $\cup$

Gerstenhaber (1963): • $HH^*(A)$ is graded com.

Modern argument: A is the unit in the triang. mon. cat. $(\mathcal{D}(A^e), \overset{L}{\underset{A}{\otimes}})$.

- $HH^{*+1}(A)$ is a graded Lie alg.: Gerst. bracket.

Getzler–Jones (1994): $(C(A,A), \cup, \text{brace op.})$ is a B_∞-algebra.

B_∞: "B" stands for Baues (1981): $C^*_{sg}(X,\mathbb{Z})$ is a B_∞-algebra.

Brace op. (Kadeishvili 1988):

$$c\{u, v, \dots, w\} = \sum \pm$$

Rks: 1) The B_∞-str. contains all the info, e.g.

$$[c,u] = c\{u\} \mp u\{c\}.$$

2) These constructions generalize from k-algebras to k-categories (= k-alg. with several objects, Mitchell 1972) and to differential graded (= dg) categories.

Thm (Toën, Lowen-VdB 2005):

$$HH^*(A) \xleftarrow{\sim} HH^*_{ab}(\mathrm{Mod}\,A) \xleftarrow{\sim} HH^*(\mathcal{D}_{dg}A).$$

$$\begin{array}{c} HH^*_{ab}(\mathrm{Mod}\,A) \\ \parallel \\ HH^*(\mathrm{Inj}\,A) \end{array}$$

Moreover, these isom. lift to the B_∞-level (K2003)

Notation: $\mathrm{Mod}\,A = \{\text{all right } A\text{-modules}\}$

$\mathrm{Inj}\,A = \{\text{inj. } A\text{-modules}\}$

$\mathcal{D}A$ = unbounded der. cat. $\mathcal{D}\,\mathrm{Mod}\,A$

$\mathcal{D}_{dg}A$ = its can. dg enhancement

Rks: 1) The isom. $HH^*(A) \xleftarrow{\sim} HH^*(\mathcal{D}_{dg}A)$

is a derived version of $Z(A) \xleftarrow{\sim} Z(\mathrm{Mod}\,A)$.

2) In particular, we have $Z(A) \xleftarrow{\sim} Z(\mathcal{D}_{dg}A) := HH^0$...

A desirable property! The center $Z(\mathcal{D}A)$

is pathological, e.g.

$$Z(\mathcal{D}^b(k[\varepsilon]/(\varepsilon^2))) \xrightarrow{\sim} k \times k^{\mathbb{N}}$$

Krause - Ye 2011

2. Tate-Hochschild cohomology

A is a right Noetherian algebra (for simplicity)

$\operatorname{mod} A = \{\text{fin. gen. right } A\text{-modules}\}$

$\mathcal{D}^b A \;=\; \mathcal{D}^b(\operatorname{mod} A)$

$\operatorname{per} A \;=\; \{X \in \mathcal{D}^b A \mid X \text{ qis to bdd complex of fin. gen. proj. modules}\}$

$\operatorname{sg}(A) = \mathcal{D}^b(\operatorname{mod} A)/\operatorname{per} A$

$\simeq$ stable der. cat. (Buchweitz '86)

$\simeq$ singularity cat. (Orlov 2003)

$\simeq \mathcal{S}g(A)^c \cong \mathcal{K}_{ac}(\operatorname{Inj} A)^c$ (Krause 2005)

Assume A^e is also Noetherian.

Def.: Tate-Hochschild cohom. $= HH^*_{sg} A$

$$= \mathrm{Ext}^*_{sg(A^e)}(A, A)$$

Rk: $HH^*_{sg} A$ is still graded com. (although $sg(A^e)$ is <u>not</u> monoidal).

Thm (Zhengfang Wang): a) $HH^*_{sg}(A)$ carries a can. (but intricate!) Gerst. bracket ('18)

b) There is a B_∞-alg. $C_{sg}(A, A)$ computing $HH^*_{sg}(A)$.

Main thm: If $\mathcal{D}^b_{dg}(\mathrm{mod}\,A)$ is smooth, then *)

$$HH^*_{sg}(A) \simeq HH^*(\mathrm{sg}_{dg} A)$$

as graded algebras.

E.g.  : $k[\varepsilon]/(\varepsilon^2)$

Conj.: This isom. lifts to the B_∞-level.

Thm (Chen-Li-Wang 2020): True for $kQ/(kQ_1)^2$ where Q is a finite quiver w/o sinks or sources.

Rk: Can describe $\mathrm{sg}(kQ/(kQ_1)^2) \simeq \mathrm{per}(L(Q))$

$L(Q)$ = graded Leavitt path algebra.

(arrows in cohom. degree 1)

*) added on 03/06/20, cf counter-example at the end.

Isom. in the main thm: $\mathcal{M} = \mathcal{D}^b_{dg}(\mathrm{mod}\,A)$, $\mathcal{S} = \mathrm{sg}_{dg} A$

We have dg functors $A \xhookrightarrow{i} \mathcal{M} \xrightarrow{p} \mathcal{S}$, $p \circ i \simeq 0$.

$$\begin{tikzcd}
\mathcal{D}^b(\mathrm{mod}\,A \otimes A^{op}) \arrow[r, "(\mathbb{1}\otimes i)^*"] \arrow[d] & \mathcal{D}(A \otimes \mathcal{M}^{op}) \arrow[r, "(i \otimes \mathbb{1})^!"] & \mathcal{D}(\mathcal{M} \otimes \mathcal{M}^{op}) \arrow[d, "(p\otimes p)^*"] \\
\mathrm{sg}(A \otimes A^{op}) \arrow[rr, dashed] & & \mathcal{D}(\mathcal{S} \otimes \mathcal{S}^{op})
\end{tikzcd}$$

$A \longmapsto I_{\mathcal{S}}$

induces an isom. in Ext^*

$I_{\mathcal{S}} = \mathcal{S}$

✓

8. Applications: 2 reconstruction thms

Thm 1: $S = \mathbb{C}[[x_1, \dots, x_n]] \longrightarrow R = S/(f)$ isol. sing.

Then R is determined (up to isom.) by $\dim R$ and $sg_{dg}(R)$.

Sketch of proof:

diff. $\mathbb{Z}$-graded

$$Z(sg_{dg} R) = HH^0_{sg}(sg_{dg} R) \xrightarrow[\text{main}]{\sim} HH^0_{sg}(R)$$

$$S/\left(f, \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right)$$

$\shortparallel$

matrix fact. Eisenbud '80

2

$$\text{Tyurina alg.} \xrightarrow[\text{BACH } 1992]{\sim} HH^{2r}(R) \xrightarrow[\text{Buchweitz '86}]{\sim} HH^{2r}_{sg}(R), \quad \forall r \gg 0$$

dim R and the Tyurina alg. determine R
(Mather–Yau 1982, Greuel–Pham 2017) ✓

Rk: Dyckerhoff (2011): $Z(sg_{dg} R) \cong$ Milnor alg.

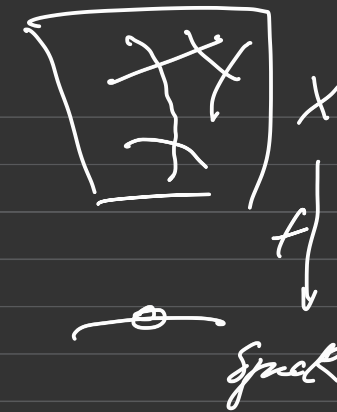
diff. $\mathbb{Z}/2$-gr.

$= S/(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n})$

Expl: k $\mathbb{Z}/2$-graded $HH^*_{\mathbb{Z}/2}(k) = k$

$k[u, u^{-1}]$, $|u| = 2$ $\mathbb{Z}$-gr. $HH_{\mathbb{Z}}(k[u, u^{-1}]) \neq k$.

R a complete local isol. compound Du Val sing.
(3-dim., normal, generic hyperplane section is a Kleinian sing.)



$f: X \to \operatorname{Spec} R$ small crepant resol.

f contracts a tree of rat. curves

Λ = contraction alg. (Donovan–Wemyss 2013): represents the non com. defo. of the exc. fibre.

Λ determines many invar. of the sing. (DW13, Toda '14, Hua–Toda '16, ..)

Conj. (DW13): The der. eq. class of Λ det. R.

$\Lambda = \mathrm{Jac}(Q, W)$ = Jacobian alg. of a quiver Q with potential W

(de Thanhoffer de Völcsey–VdB 2010)

$\overline{W}$ = image of W in $HH_0(\Lambda)$.

Thm: The der. equiv. class pair $(\Lambda, \overline{W})$
determines R (up to isom.)

Rks: 1) The proof is based on Thm and the
"non com. Mather–Yau thm" (Hua–Zhou '18).

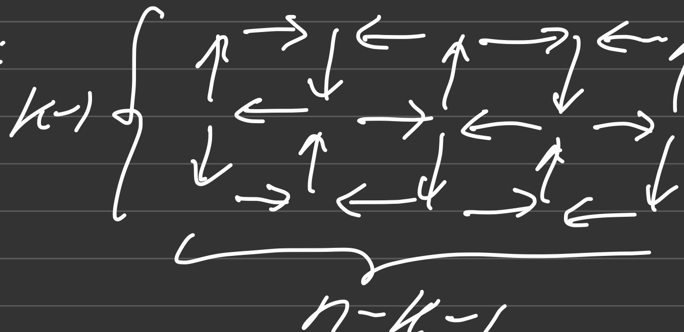
2) Link to cluster theory:

$sg(R) \simeq \mathcal{C}_{Q,W}$ = gen. cluster cat. (Amiot '09)

However, the quivers appearing in DW's theory
are quite different from those in cluster th.:

DW :

Cluster :



$\rightsquigarrow \widetilde{Gr}(k,n)$

bimodule
3-CY

$\Gamma(Q,W)$ = Ginzburg dg alg.

$\mathcal{C}_{Q,W} := \mathrm{per}(\Gamma) / \mathcal{D}_{fd}(\Gamma)$ ← 2-CY triang. cat.

Added on 03/06/20

Without the smoothness assumption, we still have a canonical algebra morphism

$$HH^*_{sg}(A) \longrightarrow HH^*(\mathrm{sg}_{dg}\, A)$$

but it is not an isomorphism in general.

Example (Xiao-Wu Chen): Suppose that k is a field of positive characteristic and $A \supset k$ a finite inseparable field extension. Then $\mathrm{sg}_{dg}(A) \simeq 0$ (because A is a field) but $HH^*_{sg}(A) \not\simeq 0$ (because A is not projective over A^e).

Thanks to the work of Elagin-Lunts-Schnürer (1810.07626), the smoothness holds if either

- k is perfect, A is a finitely generated module over its center and the center is a finitely generated k-algebra or
- A is finite-dimensional over k and $A/\mathrm{rad}A$ is a separable k-algebra.

Elagin-Lunts-Schnürer: https://arxiv.org/pdf/1810.07626.pdf