# Quantum Mechanics of Large N Fermionic Tensors

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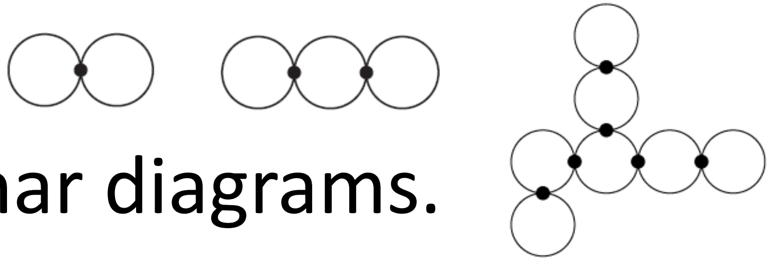
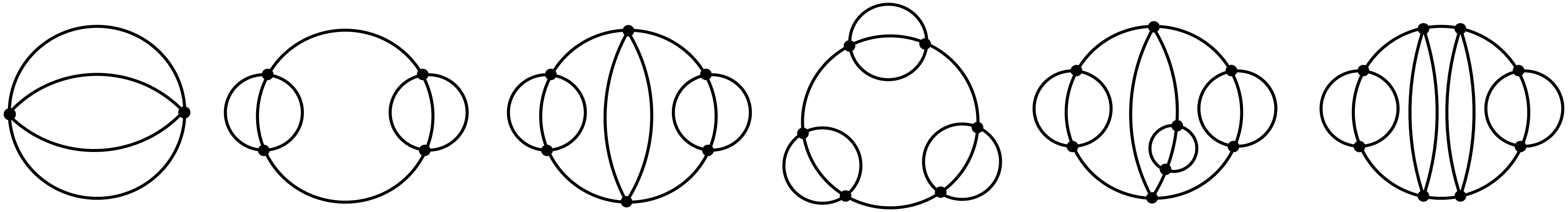
- IRK, G. Tarnopolsky, arXiv: 1611.08915
- IRK, F. Popov, G. Tarnopolsky, "TASI Lectures on Large N Tensor Models," arXiv: 1808.09434
- IRK, A. Milekhin, F. Popov, G. Tarnopolsky, arXiv: 1802.10263
- K. Pakrouski, IRK, F. Popov, G. Tarnopolsky, PRL 122 (2019) 1, 011601
- G. Gaitan, IRK, P. Pallegar, K. Pakrouski, F. Popov, arXiv: 2002.02066

# Three Large N Limits

- O(N) Vector: solvable because the "cactus" diagrams can be summed.

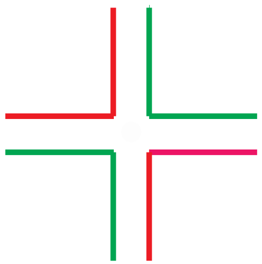


- Matrix ('t Hooft) Limit: planar diagrams. Solvable only in special cases.
- Tensor of rank three and higher. When interactions are specially chosen, dominated by the "melonic" diagrams. Bonzom, Gurau, Riello, Rivasseau; Carrozza, Tanasa; Witten; IK, Tarnopolsky

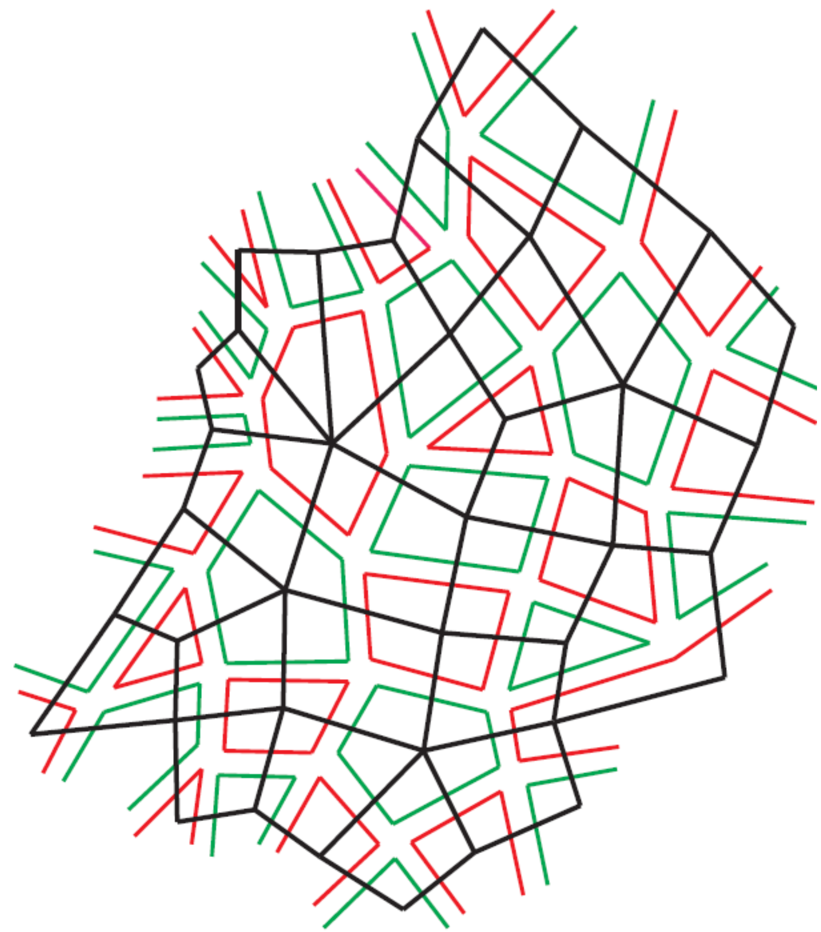


# O(N) x O(N) Matrix Model

- Theory of real matrices $\phi^{ab}$ with distinguishable indices, i.e. in the bi-fundamental representation of $O(N)_a \mathrm{x} O(N)_b$ symmetry.
- The interaction is at least quartic: $g \,\mathrm{tr}\, \phi\phi^T\phi\phi^T$
- Propagators are represented by colored double lines, and the interaction vertex is
- In d=0 or 1 special limits describe two-dimensional quantum gravity.



- In the large N limit where gN is held fixed we find planar Feynman graphs, and each index loop may be red or green.
- The dual graphs shown in black may be thought of as random surfaces tiled with squares whose vertices have alternating colors (red, green, red, green).



# From Bi- to Tri-Fundamentals

- For a 3-tensor with distinguishable indices the propagator has index structure

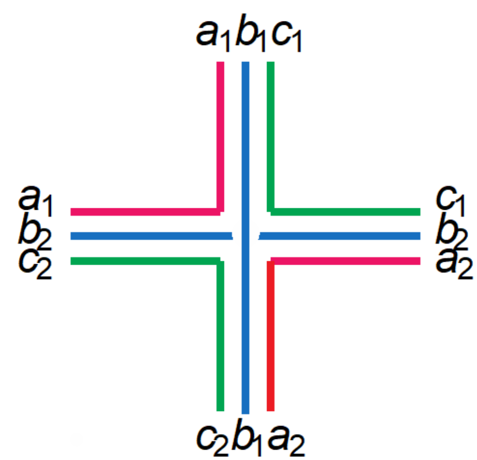
$$\langle \phi^{abc} \phi^{a'b'c'} \rangle = \delta^{aa'} \delta^{bb'} \delta^{cc'}$$

- It may be represented graphically by 3 colored wires 

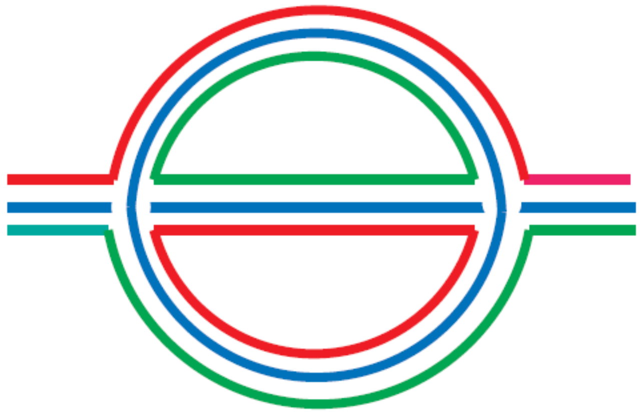
- Tetrahedral interaction with O(N)$_a$xO(N)$_b$xO(N)$_c$ symmetry

Carrozza, Tanasa; IK, Tarnopolsky

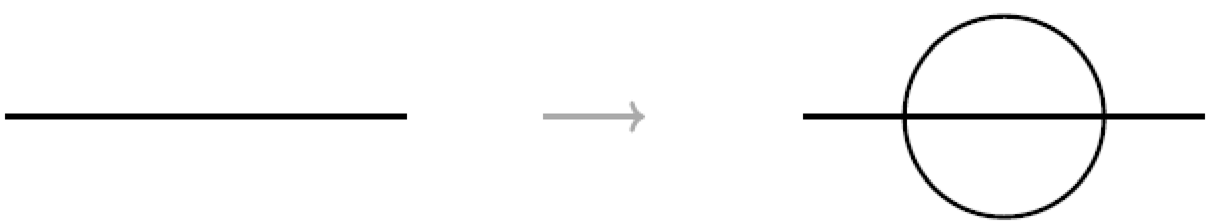
$$\frac{1}{4} g \phi^{a_1 b_1 c_1} \phi^{a_1 b_2 c_2} \phi^{a_2 b_1 c_2} \phi^{a_2 b_2 c_1}$$



- Leading correction to the propagator has 3 index loops

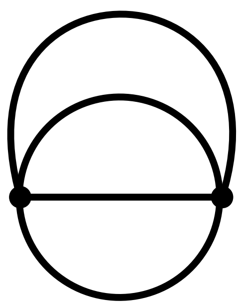


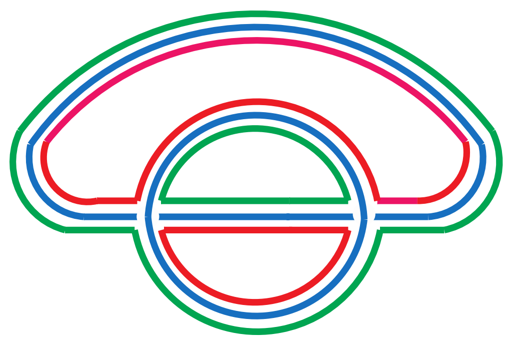
- Requiring that this "melon" insertion is of order 1 means that $\lambda = gN^{3/2}$ must be held fixed in the large N limit.
- Melonic graphs obtained by iterating



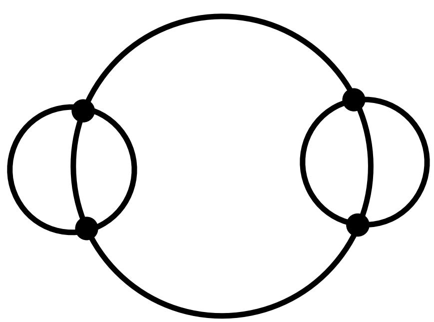
# Cables and Wires

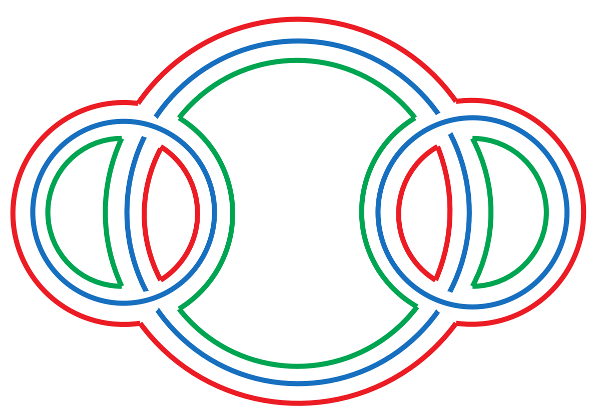
- The Feynman graphs of the quartic field theory may be resolved in terms of the colored wires (triple lines)





$$g^2 N^6 \sim N^3 \lambda^2$$

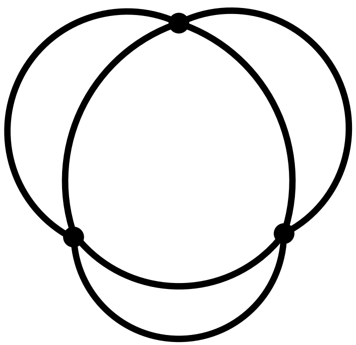


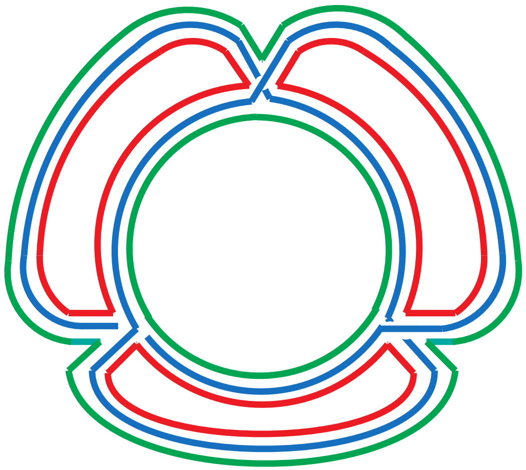


$$g^4 N^9 \sim N^3 \lambda^4$$

# Non-Melonic Graphs

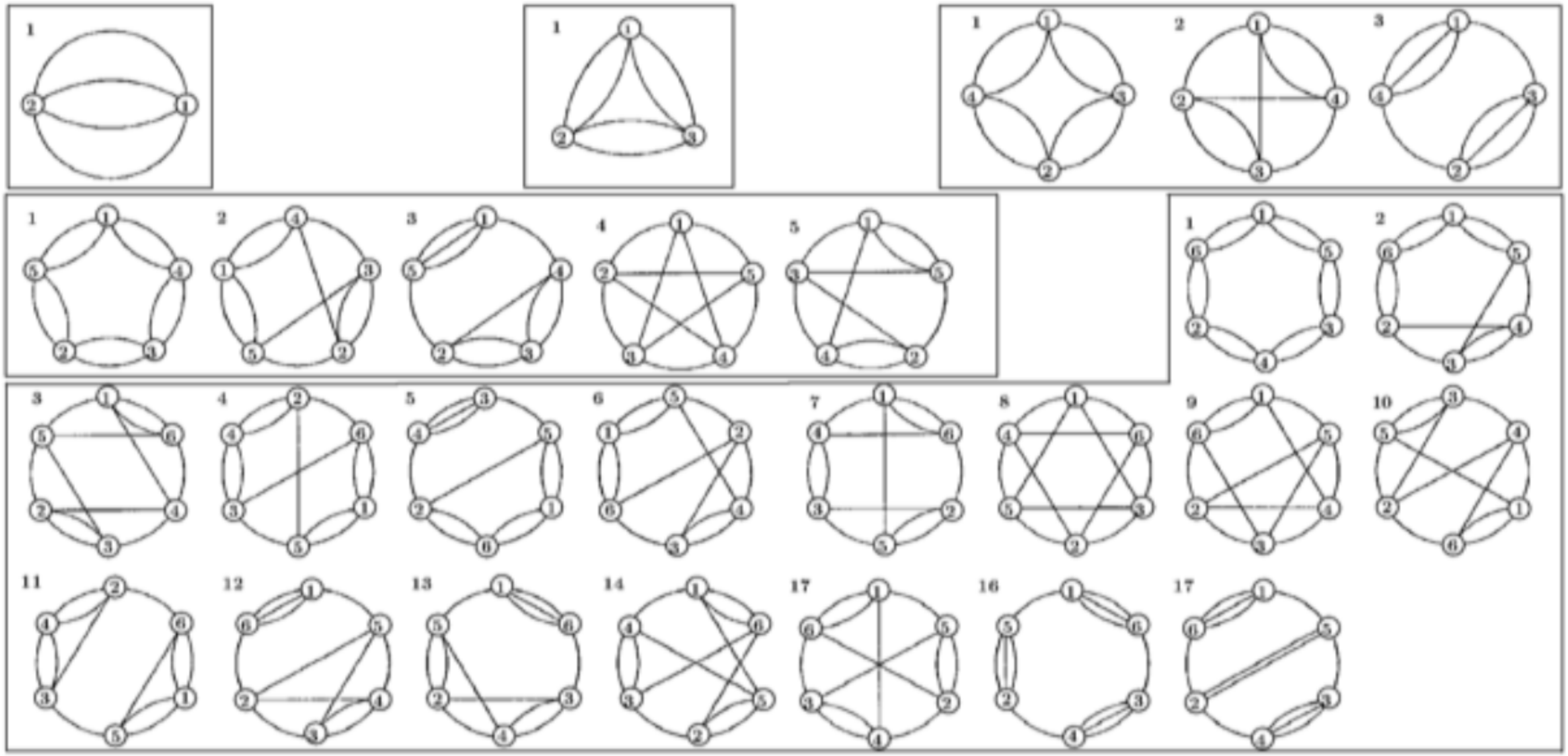
- Most Feynman graphs in the quartic field theory are not melonic are therefore subdominant in the new large N limit, e.g.





- Scales as $g^3 N^6 \sim N^3 \lambda^3 N^{-3/2}$
- None of the graphs with an odd number of vertices are melonic.

- Here is the list of snail-free vacuum graphs up to 6 vertices Kleinert, Schulte-Frohlinde



- Only 4 out of these 27 graphs are melonic.
- The number of melonic graphs with p vertices grows as $C^p$ Bonzom, Gurau, Riello, Rivasseau

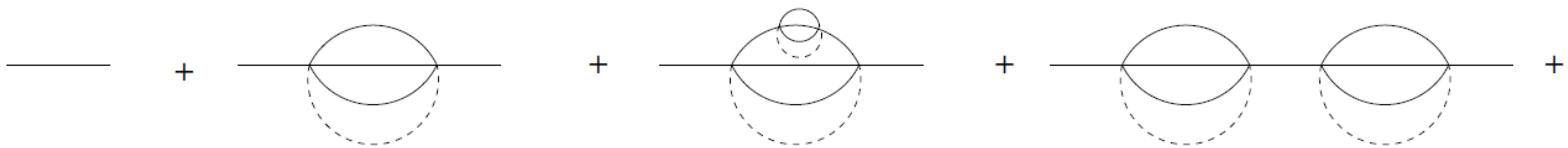
# The Sachdev-Ye-Kitaev Model

- Quantum mechanics of a large number $N_{SYK}$ of anti-commuting variables with action

$$I = \int \mathrm{d}t \left( \frac{\mathrm{i}}{2} \sum_i \psi_i \frac{\mathrm{d}}{\mathrm{d}t} \psi_i - \mathrm{i}^{q/2} j_{i_1 i_2 \ldots i_q} \psi_{i_1} \psi_{i_2} \ldots \psi_{i_q} \right)$$

- Random couplings j have a Gaussian distribution with zero mean.
- The model flows to strong coupling and becomes nearly conformal. Sachdev, Ye; Georges, Parcollet, Sachdev; Kitaev; Polchinski, Rosenhaus; Maldacena, Stanford; Jevicki, Suzuki, Yoon; Kitaev, Suh

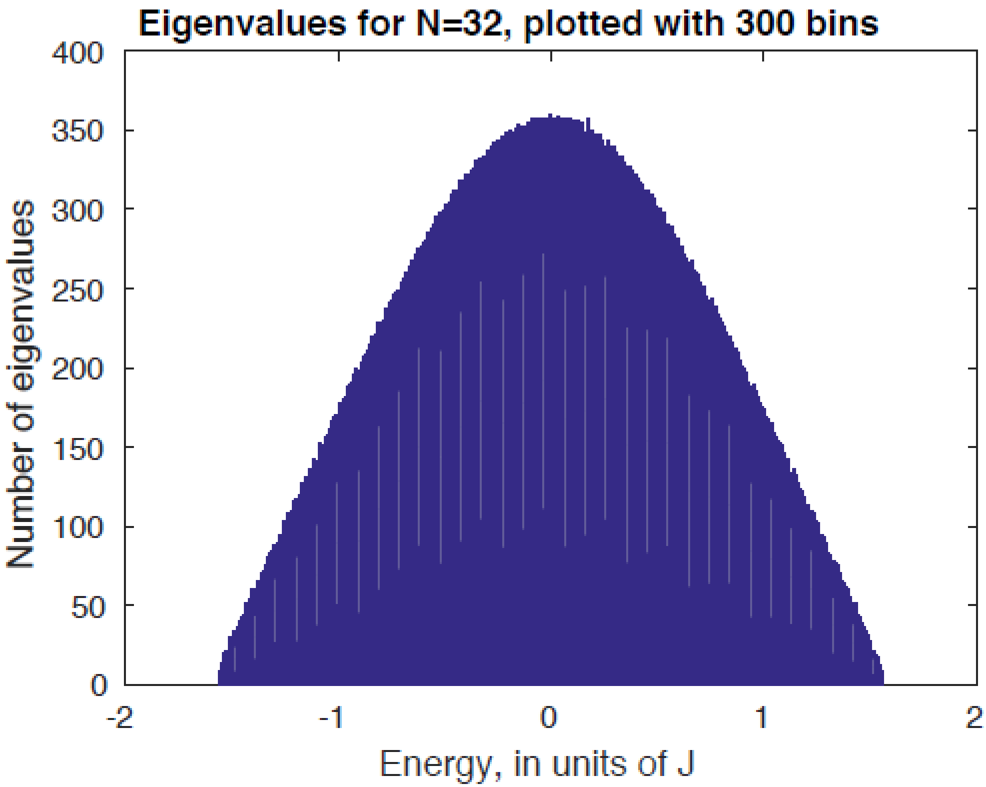
- The simplest dynamical case is q=4.
- Exactly solvable in the large $N_{SYK}$ limit because only the melonic Feynman diagrams contribute



- Solid lines are fermion propagators, while dashed lines mean disorder average.
- The exact solution shows resemblance with physics of certain two-dimensional black holes.

  Kitaev; Almheiri, Polchinski; Sachdev; Maldacena, Stanford, Yang; Engelsoy, Mertens, Verlinde; Jensen; Kitaev, Suh; …

- Spectrum for a single realization of $N_{SYK}$=32 model with q=4. Maldacena, Stanford
- No exact degeneracies, but the gaps are exponentially small. Large low T entropy.



# Random Matrix Behavior

- The square root behavior near the edge of eigenvalue distribution is ubiquitous for large N Hermitian matrix models, as seen first in the Wigner semicircle law $\rho_0(E) = \frac{1}{2\pi}\sqrt{4 - E^2}$
- For the SYK model one find the low-energy density of states
$$\rho_0(E) = \frac{\gamma}{2\pi^2} \sinh(2\pi\sqrt{2\gamma E})$$
- This corresponds to a "double-scaled" matrix model. Saad, Shenker, Stanford

# $O(N)^3$ Tensor QM

- Quantum Mechanics of $N^3$ Majorana fermions

  IK, Tarnopolsky

$$\{\psi^{abc}, \psi^{a'b'c'}\} = \delta^{aa'}\delta^{bb'}\delta^{cc'}$$

$$H = \frac{g}{4}\psi^{abc}\psi^{ab'c'}\psi^{a'bc'}\psi^{a'b'c} - \frac{g}{16}N^4$$

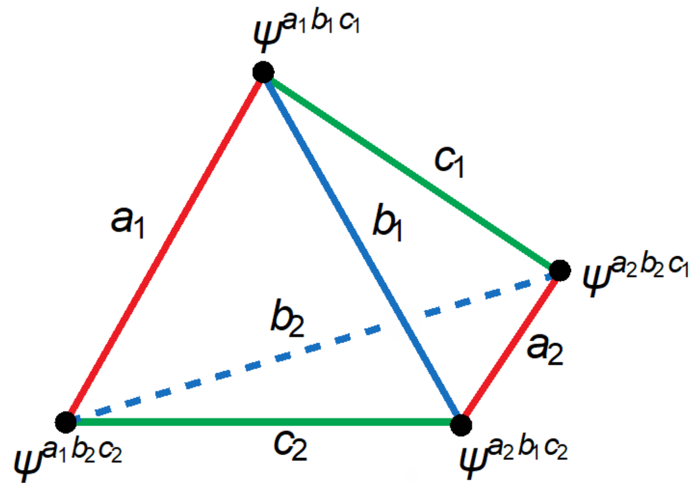
- Has $O(N)_a$x$O(N)_b$x$O(N)_c$ symmetry under

$$\psi^{abc} \to M_1^{aa'} M_2^{bb'} M_3^{cc'} \psi^{a'b'c'}, \quad M_1, M_2, M_3 \in O(N)$$

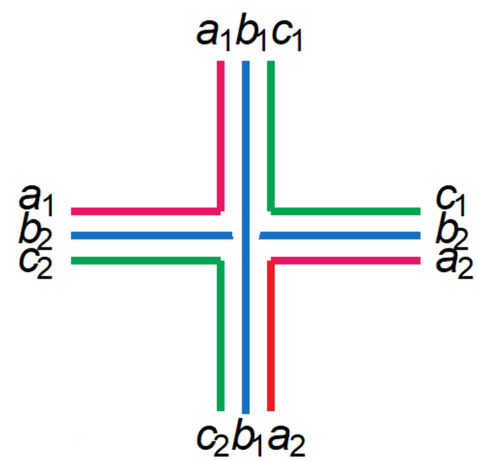
- The SO(N) symmetry charges are

$$Q_1^{aa'} = \frac{i}{2}[\psi^{abc}, \psi^{a'bc}], \qquad Q_2^{bb'} = \frac{i}{2}[\psi^{abc}, \psi^{ab'c}], \qquad Q_3^{cc'} = \frac{i}{2}[\psi^{abc}, \psi^{abc'}]$$

- The 3-tensors may be associated with indistinguishable vertices of a tetrahedron.

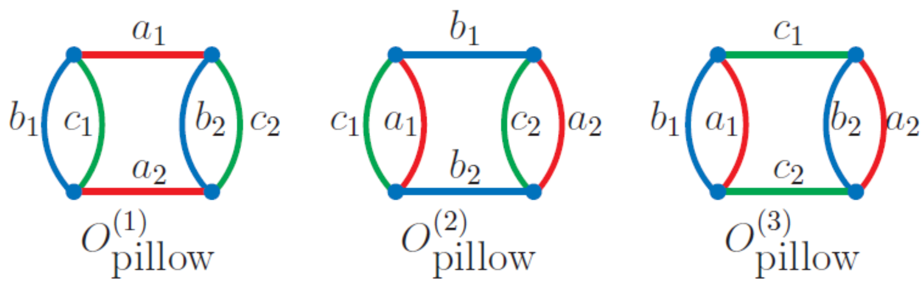



- This is equivalent to




- The triple-line Feynman graphs are produced using the propagator

- The tetrahedral term is the unique dynamical quartic interaction with $O(N)^3$ symmetry.
- The other possible terms are quadratic Casimirs of the three SO(N) groups.




$$O^{(1)}_{\text{pillow}} = \sum_{a_1<a_2} Q_1^{a_1a_2} Q_1^{a_1a_2}\,, \qquad O^{(2)}_{\text{pillow}} = \sum_{b_1<b_2} Q_2^{b_1b_2} Q_2^{b_1b_2}\,, \qquad O^{(3)}_{\text{pillow}} = \sum_{c_1<c_2} Q_3^{c_1c_2} Q_3^{c_1c_2}$$

- In the model where $SO(N)^3$ is gauged, they vanish.
- A baryonic operator: $\epsilon_{a_1\ldots a_N}\epsilon_{b_1\ldots b_N}\epsilon_{c_1\ldots c_N}\prod_{j=1}^{N}\psi^{a_jb_jc_j}$

# O(N)$^3$ vs. SYK Model

- Using composite indices $I_k = (a_k b_k c_k)$

$$H = \frac{1}{4!} J_{I_1 I_2 I_3 I_4} \psi^{I_1} \psi^{I_2} \psi^{I_3} \psi^{I_4}$$

The couplings take values $0, \pm 1$

$$J_{I_1 I_2 I_3 I_4} = \delta_{a_1 a_2} \delta_{a_3 a_4} \delta_{b_1 b_3} \delta_{b_2 b_4} \delta_{c_1 c_4} \delta_{c_2 c_3} - \delta_{a_1 a_2} \delta_{a_3 a_4} \delta_{b_2 b_3} \delta_{b_1 b_4} \delta_{c_2 c_4} \delta_{c_1 c_3} + 22 \text{ terms}$$

- The number of distinct terms is

$$\frac{1}{4!} \sum_{\{I_k\}} J^2_{I_1 I_2 I_3 I_4} = \frac{1}{4} N^3 (N-1)^2 (N+2)$$

- Much smaller than in SYK model with $N_{\text{SYK}} = N^3$

$$\frac{1}{24} N^3 (N^3 - 1)(N^3 - 2)(N^3 - 3)$$

# Gauged Model

- To eliminate large degeneracies, focus on the states invariant under SO(N)$^3$.
- Their number can be found by gauging the free theory IK, Milekhin, Popov, Tarnopolsky

$$L = \psi^I \partial_t \psi^I + \psi^I A_{IJ} \psi^J$$

$$A = A^1 \otimes \mathbb{1} \otimes \mathbb{1} + \mathbb{1} \otimes A^2 \otimes \mathbb{1} + \mathbb{1} \otimes \mathbb{1} \otimes A^3$$

$$\#\text{singlet states} = \int d\lambda_G^N \prod_{a=1}^{M/2} 2\cos(\lambda_a/2)$$

$$d\lambda_{SO(2n)} = \prod_{i<j}^{n} \sin\left(\frac{x_i - x_j}{2}\right)^2 \sin\left(\frac{x_i + x_j}{2}\right)^2 dx_1 \ldots dx_n$$

- There are no singlets for odd N due to a QM anomaly for odd numbers of flavors.
- The number grows very rapidly for even N

| $N$ | # singlet states |
|---|---|
| 2 | 2 |
| 4 | 36 |
| 6 | 595354780 |

Table 1: Number of singlet states in the $O(N)^3$ model

$$\#\text{singlet states} \sim \exp\left(\frac{N^3}{2}\log 2 - \frac{3N^2}{2}\log N + O(N^2)\right)$$

- The large low-temperature entropy suggests tiny gaps for singlet excitations ~ $c^{-N^3}$

# Discrete Symmetries

- Act within the $SO(N)^3$ invariant sector and can lead to small degeneracies.
- $Z_2$ parity transformation within each group like

$$\psi^{1bc} \to -\psi^{1bc}$$

- Interchanges of the groups flip the energy

$$P_{23}\psi^{abc}P_{23} = \psi^{acb} \ , \qquad P_{12}\psi^{abc}P_{12} = \psi^{bac}$$

$$P_{23}HP_{23} = -H \ , \qquad P_{12}HP_{12} = -H$$

- $Z_3$ symmetry generated by $P = P_{12}P_{23}$ , $\qquad P^3 = 1$

$$P\psi^{abc}P^\dagger = \psi^{cab} \ , \qquad PHP^\dagger = H$$

- At non-zero energy the gauge singlet states transform under the discrete group $A_4 \times Z_2$.
- Spectrum for N=4. Pakrouski, IK, Popov, Tarnopolsky

$$\pm\sqrt{32(447 \pm \sqrt{125601})}$$

$$\pm\sqrt{32(187 \pm \sqrt{11481})}$$

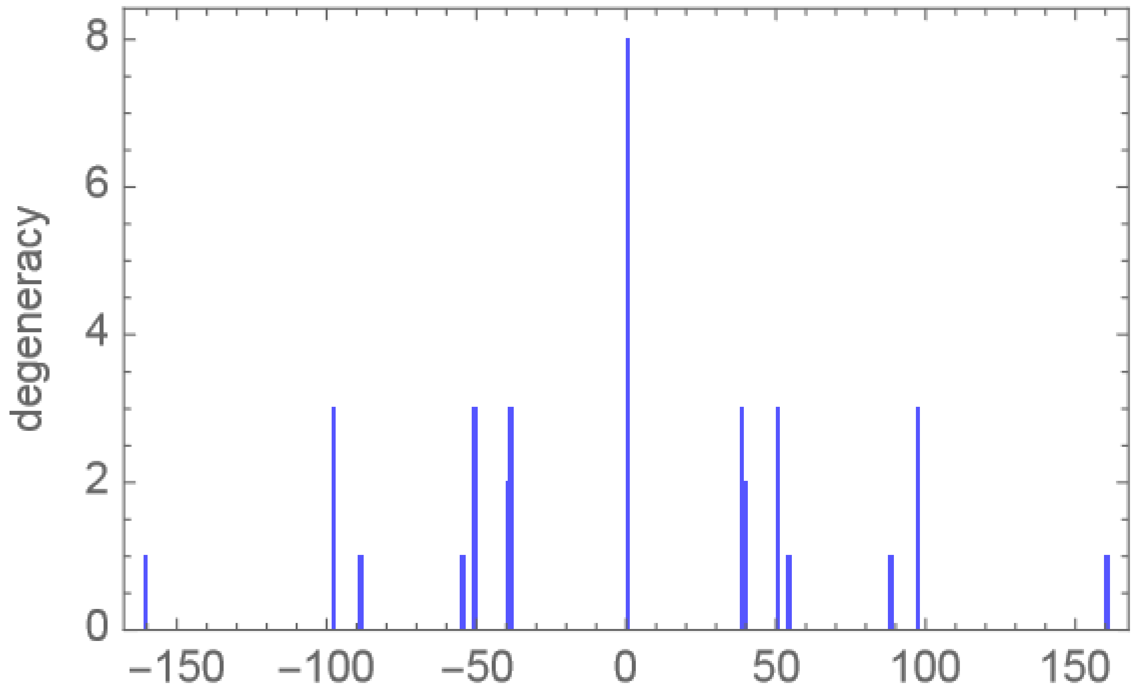
$$8\sqrt{24} =$$

$$8\sqrt{23} =$$

| $E$ | $P_1$ | $P_2$ | $P_3$ | $E$ | $P_1$ | $P_2$ | $P_3$ |
|---|---|---|---|---|---|---|---|
| $-160.140170$ | 1 | 1 | 1 | 160.140170 | 1 | 1 | 1 |
| $-97.019491$ | 1 | 1 | $-1$ | 97.019491 | 1 | 1 | $-1$ |
| $-97.019491$ | $-1$ | 1 | 1 | 97.019491 | $-1$ | 1 | 1 |
| $-97.019491$ | 1 | $-1$ | 1 | 97.019491 | 1 | $-1$ | 1 |
| $-88.724292$ | $-1$ | $-1$ | $-1$ | 88.724292 | $-1$ | $-1$ | $-1$ |
| $-54.434603$ | 1 | 1 | 1 | 54.434603 | 1 | 1 | 1 |
| $-50.549167$ | 1 | 1 | $-1$ | 50.549167 | 1 | 1 | $-1$ |
| $-50.549167$ | $-1$ | 1 | 1 | 50.549167 | $-1$ | 1 | 1 |
| $-50.549167$ | 1 | $-1$ | 1 | 50.549167 | 1 | $-1$ | 1 |
| $-39.191836$ | 1 | 1 | 1 | 39.191836 | 1 | 1 | 1 |
| $-39.191836$ | 1 | 1 | 1 | 39.191836 | 1 | 1 | 1 |
| $-38.366652$ | 1 | $-1$ | $-1$ | 38.366652 | 1 | $-1$ | $-1$ |
| $-38.366652$ | $-1$ | 1 | $-1$ | 38.366652 | $-1$ | 1 | $-1$ |
| $-38.366652$ | $-1$ | $-1$ | 1 | 38.366652 | $-1$ | $-1$ | 1 |
| 0.000000 | 1 | 1 | 1 | 0.000000 | $-1$ | $-1$ | $-1$ |
| 0.000000 | $-1$ | 1 | 1 | 0.000000 | 1 | $-1$ | $-1$ |
| 0.000000 | 1 | $-1$ | 1 | 0.000000 | $-1$ | 1 | $-1$ |
| 0.000000 | 1 | 1 | $-1$ | 0.000000 | $-1$ | $-1$ | 1 |

**arXiv:1808.07455**

# Energy Distribution for N=4




- For N=6 there will be over 595 million states packed into energy interval <1932. So, the gaps should be tiny. Near the edge should be similar to the SYK density of states.

# Tensors with Unequal Ranks

- Generalize the Majorana tensor model to have $O(N_1) \times O(N_2) \times O(N_3)$ symmetry
- The traceless Hamiltonian is

$$H = \frac{g}{4}\psi^{abc}\psi^{ab'c'}\psi^{a'bc'}\psi^{a'b'c} - \frac{g}{16}N_1N_2N_3(N_1 - N_2 + N_3)$$

$$\{\psi^{abc}, \psi^{a'b'c'}\} = \delta^{aa'}\delta^{bb'}\delta^{cc'}$$

$$a = 1, \ldots, N_1;\ b = 1, \ldots, N_2;\ c = 1, \ldots, N_3$$

- The Hilbert space has dimension $2^{[N_1N_2N_3/2]}$
- Eigenstates of H form irreducible representations of the symmetry.

# Energy Bounds

- There is a bound on the singlet ground state energy IK, Milekhin, Popov, Tarnopolsky

$$|E| \leq E_{bound} = \frac{g}{16} N^3 (N+2) \sqrt{N-1}$$

- In the melonic limit, this correctly scales as $N^3$.
- The gap to the lowest non-singlet state scales as 1/N.
- For unequal ranks the bound is

$$|E| \leq \frac{g}{16} N_1 N_2 N_3 (N_1 N_2 N_3 + N_1^2 + N_2^2 + N_3^2 - 4)^{1/2}$$

# A Fermionic Matrix Model

- For $N_3$=2 the bound simplifies to

$$|E|_{N_3=2} \leq \frac{g}{8} N_1 N_2 (N_1 + N_2)$$

- Saturated by the ground state.
- This is a fermionic matrix model with symmetry

$$O(N_1) \times O(N_2) \times U(1)$$

$$\bar{\psi}_{ab} = \frac{1}{\sqrt{2}} \left( \psi^{ab1} + i\psi^{ab2} \right), \qquad \psi_{ab} = \frac{1}{\sqrt{2}} \left( \psi^{ab1} - i\psi^{ab2} \right)$$

$$\{\bar{\psi}_{ab}, \bar{\psi}_{a'b'}\} = \{\psi_{ab}, \psi_{a'b'}\} = 0, \qquad \{\bar{\psi}_{ab}, \psi_{a'b'}\} = \delta_{aa'} \delta_{bb'}$$

- The traceless Hamiltonian is

$$H = \frac{g}{2}(\bar{\psi}_{ab}\bar{\psi}_{ab'}\psi_{a'b}\psi_{a'b'} - \bar{\psi}_{ab}\bar{\psi}_{a'b}\psi_{ab'}\psi_{a'b'}) + \frac{g}{8}N_1N_2(N_2 - N_1)$$

- May be expressed in terms of quadratic Casimirs

$$-\frac{g}{2}\left(4C_2^{SU(N_1)} - C_2^{SO(N_1)} + C_2^{SO(N_2)} + \frac{2}{N_1}Q^2 + (N_2 - N_1)Q - \frac{1}{4}N_1N_2(N_1 + N_2)\right)$$

- $SU(N_1) \times SU(N_2)$ is not a symmetry here but an enveloping algebra.
- For all $N_1$, $N_2$, the energy levels are integers in units of g/4.

| $(N_1, N_2)$ | (2,2) | (2,3) | (3,3) | (2,4) | (3,4) | (4,4) |
|---|---|---|---|---|---|---|
| $\frac{4}{g}E_{\text{degeneracy}}$ | $-8_{1}$ | $-13_{2}$ | $-20_{6}$ | $-24_{1}$ | $-34_{6}$ | $-64_{1}$ |
| | $0_{14}$ | $-7_{6}$ | $-16_{18}$ | $-16_{2}$ | $-28_{24}$ | $-48_{55}$ |
| | $8_{1}$ | $-3_{2}$ | $-12_{16}$ | $-12_{16}$ | $-24_{8}$ | $-40_{106}$ |
| | | $-1_{22}$ | $-8_{60}$ | $-8_{23}$ | $-22_{76}$ | $-36_{256}$ |
| | | $1_{22}$ | $-4_{42}$ | $-4_{16}$ | $-20_{40}$ | $-32_{810}$ |
| | | $3_{2}$ | $0_{228}$ | $0_{140}$ | $-18_{14}$ | $-28_{256}$ |
| | | $7_{6}$ | $4_{42}$ | $4_{16}$ | $-16_{152}$ | $-24_{3250}$ |
| | | $13_{2}$ | $8_{60}$ | $8_{23}$ | $-14_{168}$ | $-20_{1024}$ |
| | | | $12_{16}$ | $12_{16}$ | $-12_{40}$ | $-16_{4985}$ |
| | | | $16_{18}$ | $16_{2}$ | $-10_{170}$ | $-12_{3072}$ |
| | | | $20_{6}$ | $24_{1}$ | $-8_{240}$ | $-8_{8932}$ |
| | | | | | $-6_{194}$ | $-4_{3584}$ |
| | | | | | $-4_{384}$ | $0_{12874}$ |
| | | | | | $-2_{270}$ | $4_{3584}$ |
| | | | | | $0_{248}$ | $8_{8932}$ |
| | | | | | $2_{640}$ | $12_{3072}$ |
| | | | | | $4_{384}$ | $16_{4985}$ |
| | | | | | $6_{76}$ | $20_{1024}$ |
| | | | | | $8_{312}$ | $24_{3250}$ |
| | | | | | $10_{216}$ | $28_{256}$ |
| | | | | | $14_{32}$ | $32_{810}$ |
| | | | | | $16_{128}$ | $36_{256}$ |
| | | | | | $18_{168}$ | $40_{106}$ |
| | | | | | $20_{64}$ | $48_{55}$ |
| | | | | | $26_{10}$ | $64_{1}$ |
| | | | | | $28_{24}$ | |
| | | | | | $30_{6}$ | |
| | | | | | $38_{2}$ | |

# Singlets in the Matrix Model

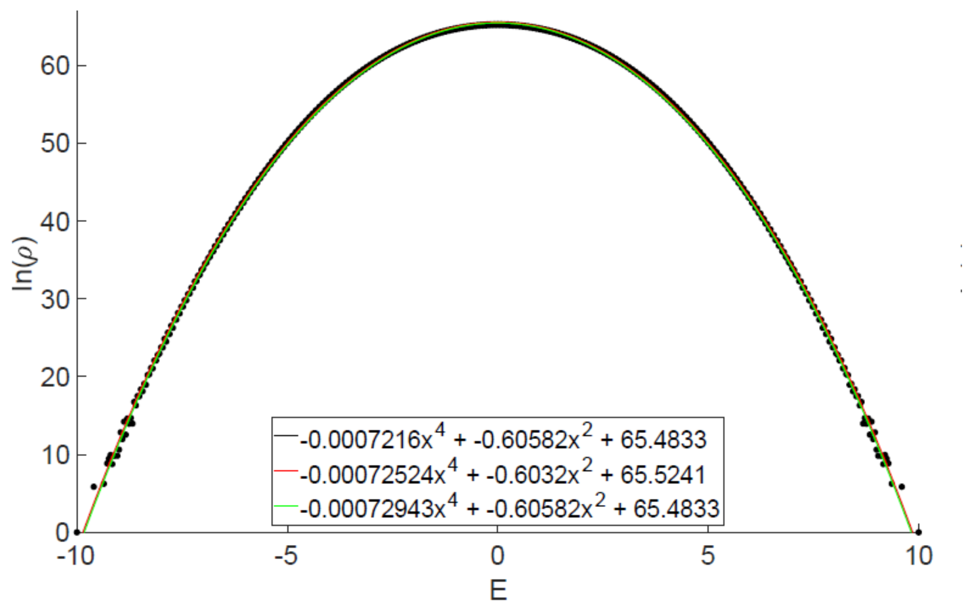
- Their number grows, but much more slowly than in the $O(N)^3$ model.

| $(N_1, N_2)$ | # singlet states |
|---|---|
| (4,4) | 4 |
| (6,4) | 4 |
| (6,6) | 4 |
| (8,4) | 6 |
| (8,6) | 8 |
| (8,8) | 18 |
| (10,4) | 6 |
| (10,6) | 8 |
| (10,8) | 20 |
| (10,10) | 24 |

Table 3: Number of singlet states in the $O(N_1) \times O(N_2) \times O(2)$ model

# Full Density of States

- Approximately Gaussian in these Matrix Models. Here are the results for $N_1 = N_2 = 10$



# Fermionic Vector Models

- Only one rank becomes large, e.g. $O(N) \times O(2)^2$
- Vectorial large N limit with $gN = \lambda$ fixed.
- A related model with $O(N) \times SO(4)$ symmetry:

$$H_{O(N)\times SO(4)} = \frac{g}{2}\epsilon_{IJKL}\psi_{aI}\psi_{aJ}\psi_{a'K}\psi_{a'L}$$

- Exactly solvable Gaitan, IK, Pakrouski, Pallegar, Popov

$$E(Q_+, Q_-) = g\left[Q_+(Q_+ + 2) - Q_-(Q_- + 2)\right]$$

$$\deg(Q_+, Q_-) = \frac{(Q_+ + 1)^2(Q_- + 1)^2 N!(N+2)!}{\left(\frac{N-Q_+-Q_-}{2}\right)!\left(\frac{N+Q_+-Q_-+2}{2}\right)!\left(\frac{N-Q_++Q_-+2}{2}\right)!\left(\frac{N+Q_++Q_-+4}{2}\right)!}$$

# Large N limit

- Take $Q_\pm \sim \sqrt{N} \gg 1$ to keep $|E|/\lambda$ of order 1.
- Then $\dim(Q_+, Q_-) \approx 2^{2N} Q_+ Q_- \exp\left(-\frac{Q_+^2 + Q_-^2}{N}\right)$

- The integral for density of states

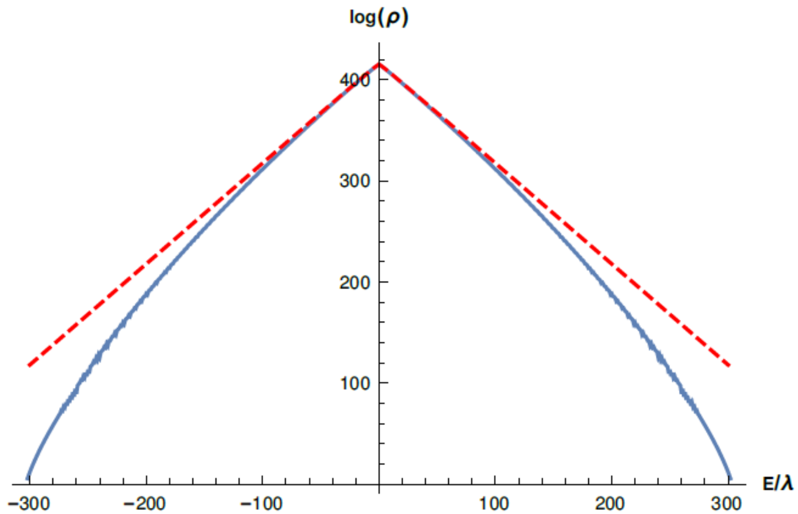
$$\rho(E) \sim \int_0^\infty dx_+ \int_0^\infty dx_- x_+^2 x_-^2 e^{-(x_+^2 + x_-^2)} \delta\left(E - \lambda\left(x_+^2 - x_-^2\right)\right)$$

- Can be evaluated in closed form:

$$\rho(E) = 2^{2N} \frac{|E|}{\pi \lambda^2} K_1\left(\frac{|E|}{\lambda}\right)$$

# Hagedorn Temperature

- The complete spectrum includes all allowed values of the two SU(2) spins $Q_{\pm}/2$

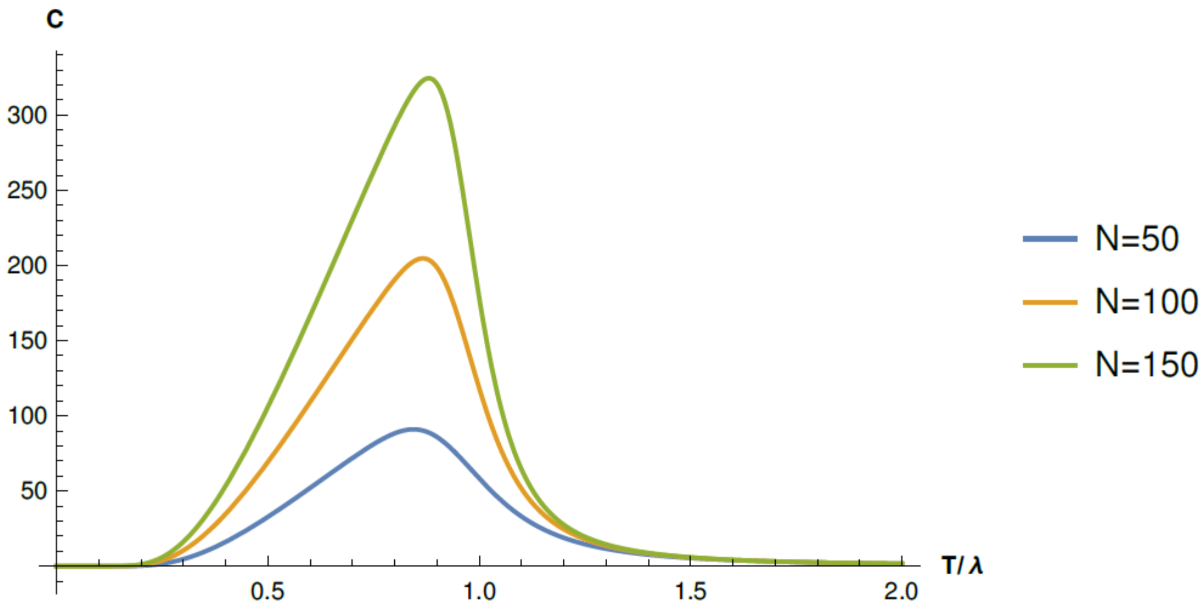



- At large N, $$\rho(E) = 2^{2N} \frac{|E|}{\pi\lambda^2} K_1\left(\frac{|E|}{\lambda}\right)$$

$$T_H = \lambda$$

$$C(T) = -T\frac{\partial^2 F}{\partial T^2} = \frac{3\lambda^2\left(T^2+\lambda^2\right)}{\left(T^2-\lambda^2\right)^2}$$

- Peak in specific heat seen at finite values of N




- Can think of $a = 1, \ldots N$ as 1-d lattice index.
- Non-local Hamiltonian with two complex fermions per site.
- Phase transition in the infinite volume limit.

# Conclusions

- The $O(N)^3$ fermionic tensor quantum mechanics seems to be the closest non-random counterpart of the basic SYK model for Majorana fermions.
- Finding the energy spectrum of the tensor QM is hard.
- Some Matrix and Vector Majorana Models are exactly solvable and exhibit interesting spectra.
- The Large N Fermionic Vector models exhibit a cusp in the density of states and Hagedorn transition. A dual description?