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Holographic RG flows on curved manifolds and F-theorems.

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Introduction

• I will be addressing Unitary Lorentz invariant QFTs in this presentation.

• The Wilsonian paradigm builds the QFT landscape starting from CFTs (scale invariant theories).

• The rest of landscape is filled by the RG flows.

• It is well known that the Wilsonian RG is controlled by first order flow equations of the form

$$\frac{dg_i}{dt} = \beta_i(g_i) \quad , \quad t = \log \mu$$

• Although the basic rules of the flows are well known, the global structure of flows is known mostly for weakly coupled field theories.

• Despite more than 50 years of study, there are many aspects of QFT RG flows that are still not understood.

♠ It is not known if the end-points of RG flows in 4d are fixed points or include other exotic possibilities (limit circles or “chaotic" behavior)

♠ This is correlated with the potential symmetry of scale invariant theories:

♠ If a scale invariant theory is also conformally invariant then this excludes exotic possibilities.

• In 2d, every scale-invariant, relativistic, unitary QFT is also conformally invariant.

Todorov, Polchinski

♠ Although in 4d this has been analyzed recently, there are still loopholes in the argument.

El Showk+Rychkov+Nakayama, Luty+Polchinski+Rattazzi,

Bzowski+Skenderis, Dymarsky+Komargodksi+Schwimmer+Theisen+Farnsworth+Luty+Prilepina

The Goals

• Use holographic theories (large N, strong coupling) in order to investigate RG flows beyond weak coupling.

• Build an understanding of the general structure of holographic RG flows of QFTs on flat space.

• Build an understanding of the general structure of holographic RG flows of QFTs on curved spaces (spheres etc)

• Use this knowledge to revisit C and F-functions in 3 and more dimensions.

Holographic RG flows: the setup

• For simplicity and clarity I will consider the bulk theory to contain only the metric and a single scalar (Einstein-dilaton gravity), dual to the stress tensor $T_{\mu\nu}$ and a scalar operator O of a dual QFT.

• The two derivative action (after field redefinitions) is

$$S_{bulk} = M^{d-1} \int d^5x \sqrt{-g} \left[R - \frac{1}{2}(\partial\phi)^2 - V(\phi) \right] + S_{GH}$$

• We assume $V(\phi)$ is analytic everywhere except possibly at $\phi = \pm\infty$.

• We will consider the AdS regime: ($V < 0$ always) and look (in the beginning) for solutions with 4-dimensional Poincaré invariance.

$$ds^2 = du^2 + e^{2A(u)} dx_\mu dx^\mu \quad , \quad \phi(u)$$

• They correspond to the vacuum saddle point solutions.

• $\mu = \mu_0 \, e^{A(u)} \simeq \mu_0 \, e^{-\frac{u}{\ell}}$ and $A \simeq \log \mu$

• If $\phi(u)$ is not constant, the solution will correspond to the vacuum solution of an RG flow driven by the operator $O(x)$.

• The bulk Einstein and scalar equations become (d=4):

$$6\ddot{A}(u) + \dot{\phi}^2(u) = 0$$

$$12\dot{A}(u)^2 - \frac{1}{2}\dot{\phi}^2(u) + V(\phi) = 0$$

$$\left[\ddot{\phi} + d\dot{A}\dot{\phi} - V'(\phi) = 0\right]$$

• There are two independent equations and three integration constants: ϕ_0, ϕ_1 and A_0.

• In particular

$$\phi(u) = \phi_0 e^{(4-\Delta)\frac{u}{\ell}} + \cdots + \phi_1 e^{\Delta\frac{u}{\ell}} + \cdots = \phi_0 \mu^{4-\Delta} + \cdots + \langle O \rangle \mu^{\Delta} + \cdots$$

is the running QFT coupling, that contains also contributions from the vev of O.

• How can second order equations describe first order (Wilsonian) RG Flows?

The first order formalism

• The Einstein equations can be turned to first order equations by introducing the “superpotential" W:

$$\dot{A} = -\frac{1}{2(d-1)} W(\phi) \quad , \quad \dot{\phi} = W'(\phi) \quad , \quad dot = \frac{d}{du}$$

$$-\frac{d}{4(d-1)} W(\phi)^2 + \frac{1}{2} W(\phi)'^2 = V(\phi) \quad , \quad ' = \frac{d}{d\phi}$$

Boonstra+Skenderis+Townsend, De Wolfe+Freedman+Gubser+Karch, de Boer+Verlinde[2]

• These equations have the same number of integration constants. In particular there is a continuous one-parameter family of $W(\phi)$.

• Given a $W(\phi)$, $A(u)$ and $\phi(u)$ can be found by integrating the first order flow equations.

• The two integration constants ϕ_0 and A_0 will be interpreted as couplings of the dual QFT.

$$\frac{dg}{d\log\mu} = \beta(g) \qquad \to \qquad \frac{d\phi}{dA} = \frac{\dot\phi}{\dot A} = -2(d-1)\frac{W'(\phi)}{W(\phi)} \equiv \beta(\phi)$$

- The third integration constant, ϕ_1 is hidden in $W(\phi)$ and controls the *vev* of the operator O.

- Global regularity fixes $\phi_1 \sim \langle O \rangle$ to typically a unique value.

- Therefore:

RG flows are in one-to one correspondence with the solutions of the "superpotential equation".

$$-\frac{d}{4(d-1)}W(\phi)^2 + \frac{1}{2}W(\phi)'^2 = V(\phi)$$

• Regularity of the bulk solution fixes the W-equation integration constant (uniquely in generic cases).

• It therefore looks like: if the superpotential equation has a unique regular solution, then the holographic RG Flows look like Wilsonian first order RG Flows.

General properties of the superpotential

- Because of the symmetry $(W, u) \to (-W, -u)$ we can always take $W > 0$.

- The superpotential equation implies

$$W(\phi) \geq \sqrt{-\frac{4(d-1)}{d} V(\phi)} \equiv B(\phi) > 0$$

- The holographic "c-theorem" holds for all flows:

$$\frac{dW}{du} = \frac{dW}{d\phi}\,\frac{d\phi}{du} = W'^2 \geq 0$$

- The only singular flows are those that end up at $\phi \to \pm\infty$.

- All regular solutions to the equations are flows from an extremum of V to another extremum of V (for finite ϕ).

The extrema of V

• Solutions with constant scalar ϕ require them to be at an extremum of the potential, $V' = 0$.

• Therefore, extrema of the potential describe (holographic) CFTs.

• We will examine solutions for $W(\phi)$ near a maximum of V.

• We put the maximum at $\phi = 0$ and set $d = 4$.

$$V(\phi) = -\frac{1}{\ell^2}\left[12 - \frac{m^2\ell^2}{2}\phi^2 + \mathcal{O}(\phi^3)\right]$$

$$\Delta = 2 + \sqrt{4 + m^2\ell^2} \quad , \quad m^2\ell^2 \quad < 0 \quad , \quad 2 \leq \Delta \leq 4$$

• We set (locally) $\ell = 1$ from now on.

• The solution describes the region near a UV fixed point, upon a perturbation by a relevant operator of dimension $\Delta \leq 4$.

• The general structure of the solution for W has a “perturbative piece" (a power series in ϕ) and a non-perturbative piece (a trans-series in powers of $\phi^{\frac{4}{4-\Delta}}$)

$$W(\phi) = 6 + \frac{(4-\Delta)}{2}\phi^2 + \mathcal{O}(\phi^3) + C\phi^{\frac{4}{(4-\Delta)}}\left[1 + \mathcal{O}(\phi)\right] + \mathcal{O}(C^2\phi^{\frac{8}{(4-\Delta)}})$$

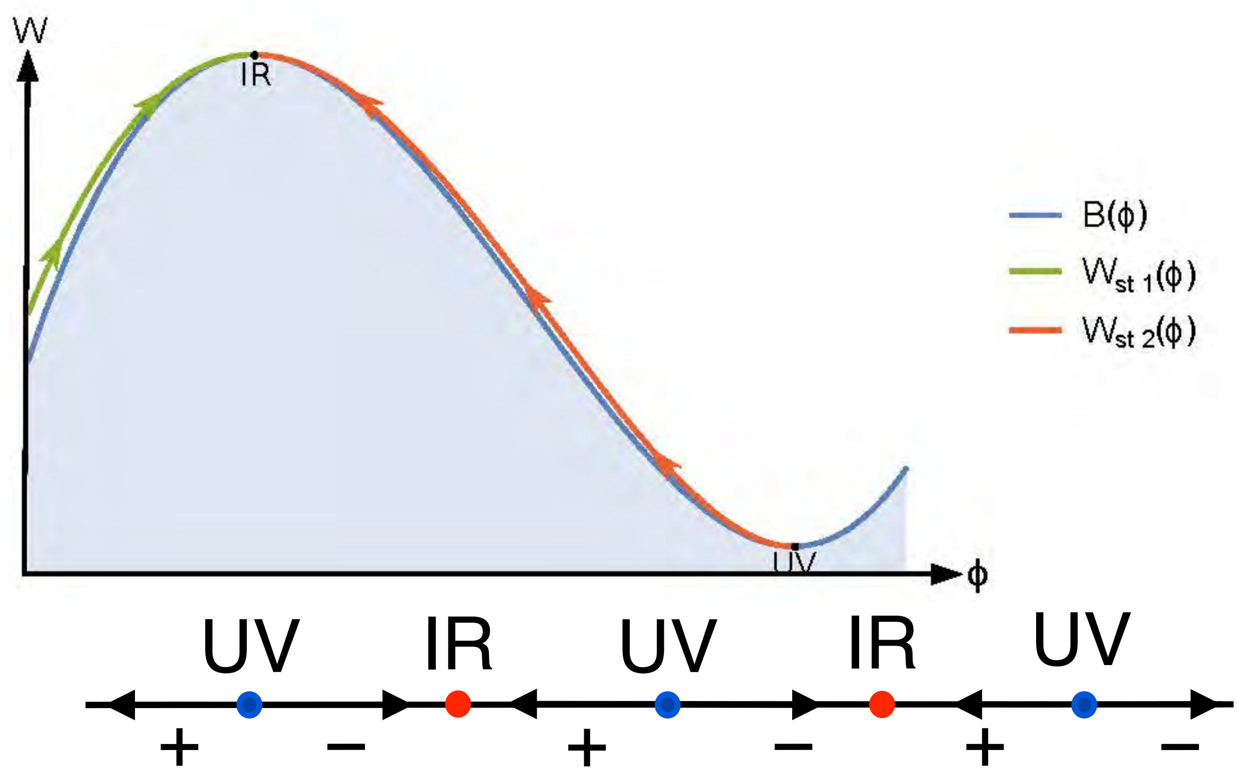
• C determines the vev: $\langle O \rangle \sim C\ \phi_0^{\frac{\Delta}{4-\Delta}}$.

$$\beta(\phi) = (\Delta - 4)\phi + \mathcal{O}(\phi^2) + \frac{4C}{4-\Delta}\phi^{\frac{\Delta}{4-\Delta}} + \cdots$$

• Maxima always describe UV CFTs. Minima generically describe IR CFTs.

The standard holographic RG flows

• The standard lore says that the maxima of the potential correspond to UV fixed points, the minima to IR fixed points, and the flow from a maximum is to the next minimum.



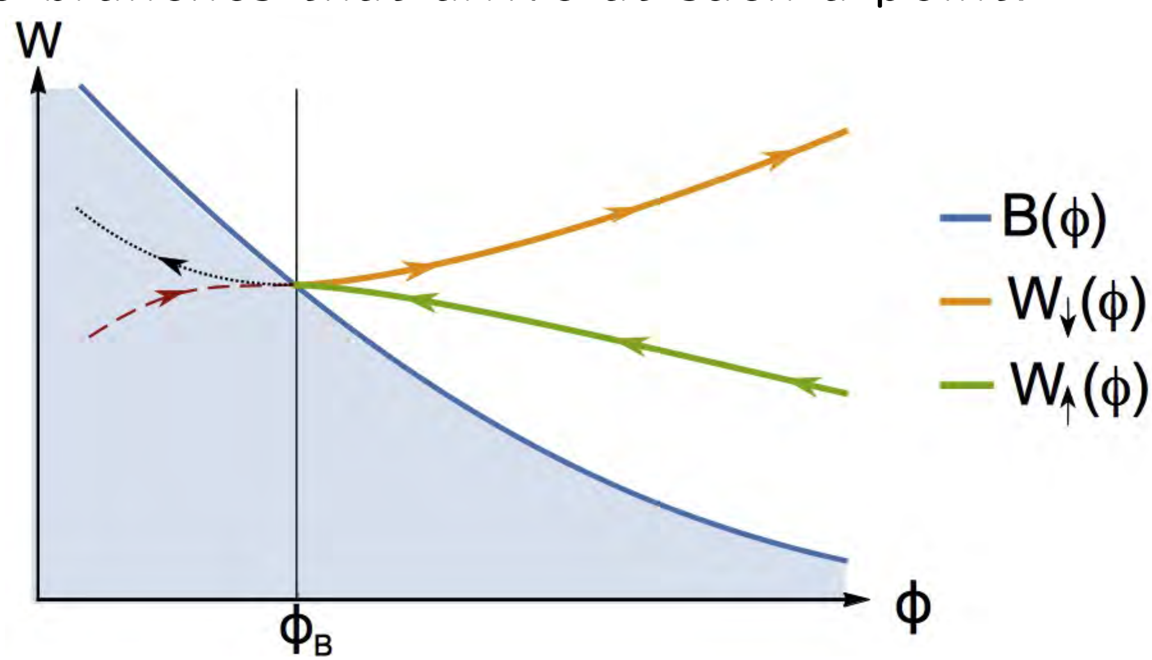
• The real story is a bit more complicated.

"Bounces"

- When W reaches the boundary region $B(\phi)$ at a generic point, it develops a generic non-analyticity.

$$W_{\pm}(\phi) = B(\phi_B) \pm (\phi - \phi_B)^{\frac{3}{2}} + \cdots$$

- There are two branches that arrive at such a point.



• Although W is not analytic at ϕ_B, the full solution (geometry+ϕ) is regular at the bounce.

• The only special thing that happens is that $\dot{\phi} = 0$ at the bounce.

• All bulk curvature invariants are regular at the bounce!

• The holographic β-function behaves as

$$\beta \equiv \frac{d\phi}{dA} = \pm\sqrt{-2d(d-1)\frac{V'(\phi_B)}{V(\phi_B)}(\phi - \phi_B) + \mathcal{O}(\phi - \phi_B)}$$

• The β-function is patch-wise defined. It has a branch cut at the position of the bounce.

• It vanishes at the bounce without the flow stopping there. This is non-perturbative behavior.

• Such behavior was conjectured that could lead to limit cycles without violation of the C-theorem.

Curtright+Zachos

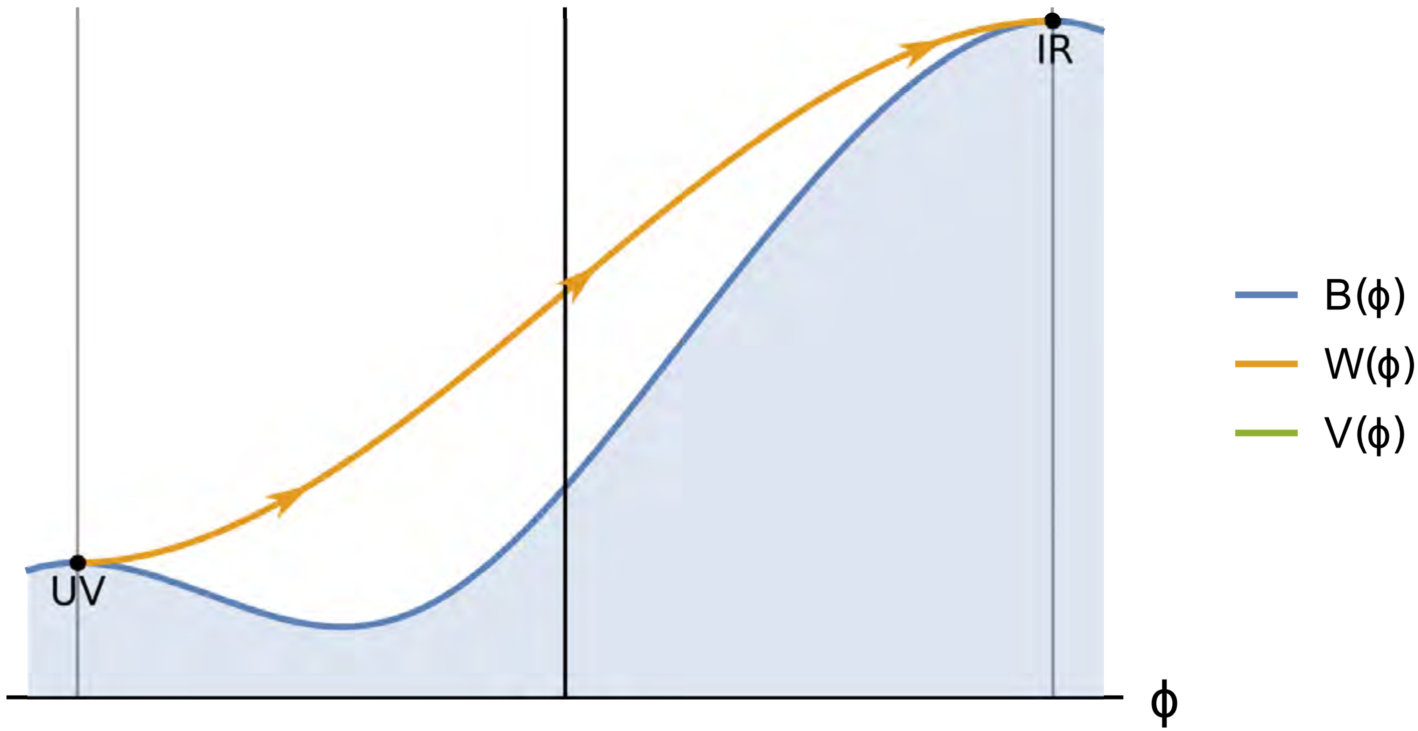
• Indeed, here W always increases despite the presence of the bounce, in agreement with the holographic C-theorem.

• In field theory terms, a coupling changes flow-direction at a bounce.

• It is however easy to show, that although a flow can go back and forth a few times, limit cycles cannot happen in theories with holographic duals.

Exotica

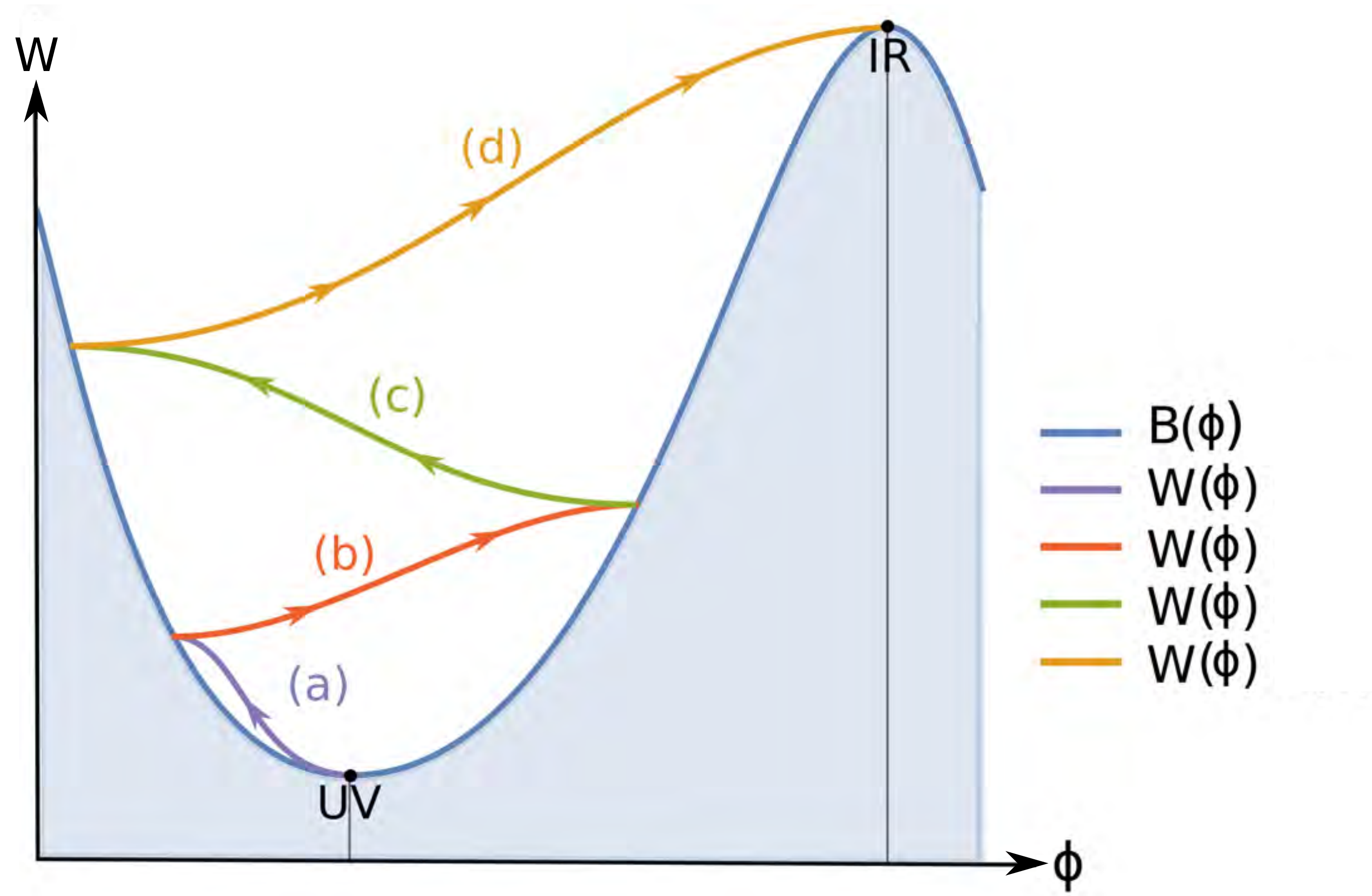
- Vev flow between two minima of the potential



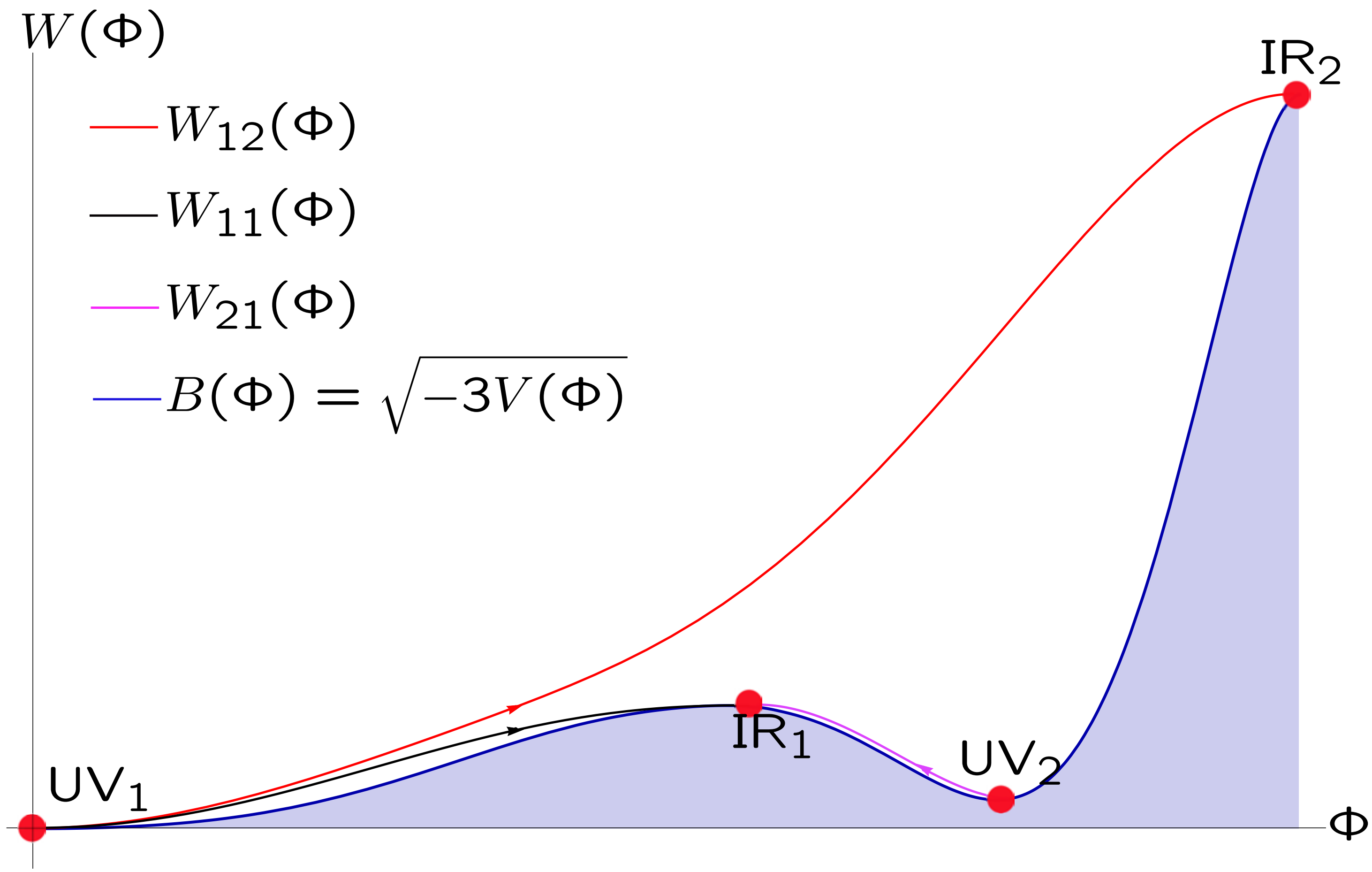
- Exists only for special potentials. It is a flow driven by the vev of an irrelevant operator. There is always a moduli space in such a case.

- A analogous phenomenon happens in N=1 sQCD (Baryonic Branch).
Seiberg, Aharony

Regular multibounce flows



Skipping fixed points



Summary

• Holographic RG flows, mostly look like QFT RG flows, but,

♠ The holographic β-function contains non-analytic (non-perturbative) contributions.

♠ There are RG flows which have β-functions with branch cuts and the flows change direction without stopping.

♠ The (holographic) C-theorem is still valid.

♠ There are flows that skip nearby fixed-points.

Quantum field theories on curved manifolds

• There are many reasons to be interested in QFTs over curved manifolds:

♠ Compact manifolds like S^n are important to regularize massless/CFTs in the IR.

♠ QFT on deSitter manifolds is interesting due to the fact we live in a patch of (almost) de Sitter.

♠ As we shall see, holography predicts that a QFT on the static patch of de Sitter has a partition function that is thermal.

♠ The induced effective gravitational action as a function of curvature can serve as a Hartle-Hawking wave-function for three-metrics.

• AdS/CFT can provide concrete quantitative wave-functions that can depend on cosmological constant and the 3-geometry.

Hartle+Hertog

♠ Curvature, although UV-irrelevant, is IR relevant and can change importantly the IR structure of a given theory.

• We find examples of quantum phase transitions driven by the S^4 curvature.

Ghosh+Kiritsis+Nitti+Witkowski

• We find examples of moduli spaces that exist only at finite curvature.

Ghosh+Kiritsis+Nitti+Witkowski

♠ It will also turn out to be a useful tool in analysing sphere partition functions and their relationship to $\mathcal{F}$-theorems.

♠ Finally it can be used to provide a concrete check on claims of backreaction on the cosmological constant, beyond perturbation theory.

Mazur+Mottola, Tsamis+Woodard, Ghosh+Kiritsis+Nitti+Witkowski

The setup

- The holographic ansatz for the ground-state solution is

$$ds^2 = du^2 + e^{2A(u)} \zeta_{\mu\nu}\, dx^\mu dx^\nu \quad , \quad \phi(u)$$

- $\zeta_{\mu\nu}$ is proportional to the boundary metric: we will take it to be of constant curvature.

- This includes the maximally symmetric manifolds sphere (S^d), de Sitter (dS_d) or Euclidean/Minkowski AdS_d.

- Therefore we consider a strongly-coupled QFT on S^d, dS_d, AdS_d.

• We take the bulk theory to be the same as before

$$S_{bulk} = M^{d-1} \int d^{d+1}x \, \sqrt{-g} \left[R - \frac{1}{2}(\partial\phi)^2 - V(\phi) \right] + S_{GH}$$

• Now there are two parameters (couplings) for the solution: ϕ_0 and R_{UV}. They combine in a single dimensionless parameter:

$$\mathcal{R} \quad \equiv \quad \frac{R_{\rm UV}}{\phi_0^{\frac{2}{d-\Delta}}}$$

• $\mathcal{R} \to 0$ will probe the full original theory except a small IR region.

• $\mathcal{R} \to \infty$ will explore only the UV of the original theory.

• Therefore by varying $\mathcal{R}$ we have an invariant/well-defined dimensionless number that tracks the UV flow from the UV to the IR.

The first order RG flows

- We can again write two first order flow equations:

$$\dot{A} = -\frac{1}{2(d-1)}W(\Phi) \quad , \quad \dot{\Phi} = S(\Phi)$$

where the functions $W(\Phi)$, $S(\Phi)$ satisfy 2 first order non-linear equations.

- The two dimensionless integration constants that enter W, S, I will call $C, \mathcal{R}$. The first will be related to the vev of O dual to ϕ. $\mathcal{R}$ is related to the the curvature of the boundary metric.

- We obtain the connection to observables

$$\mathcal{R} = R_{UV}\, |\Phi_0|^{-2/(d-\Delta)} \quad , \quad \langle O\rangle(\mathcal{R}) = \frac{2d}{(d-\Delta)} C(\mathcal{R}) |\Phi_0|^{\frac{\Delta}{(d-\Delta)}}$$

- $\mathcal{R} > 0$ describes S^d and dS_d. $\mathcal{R} < 0$ describes AdS_d.

The IR limits

- When $R_{UV} = 0$ the IR end-poids are minima of $V(\Phi)$.

- When $R_{UV} \neq 0$, the IR end points cannot be minima of $V(\Phi)$.

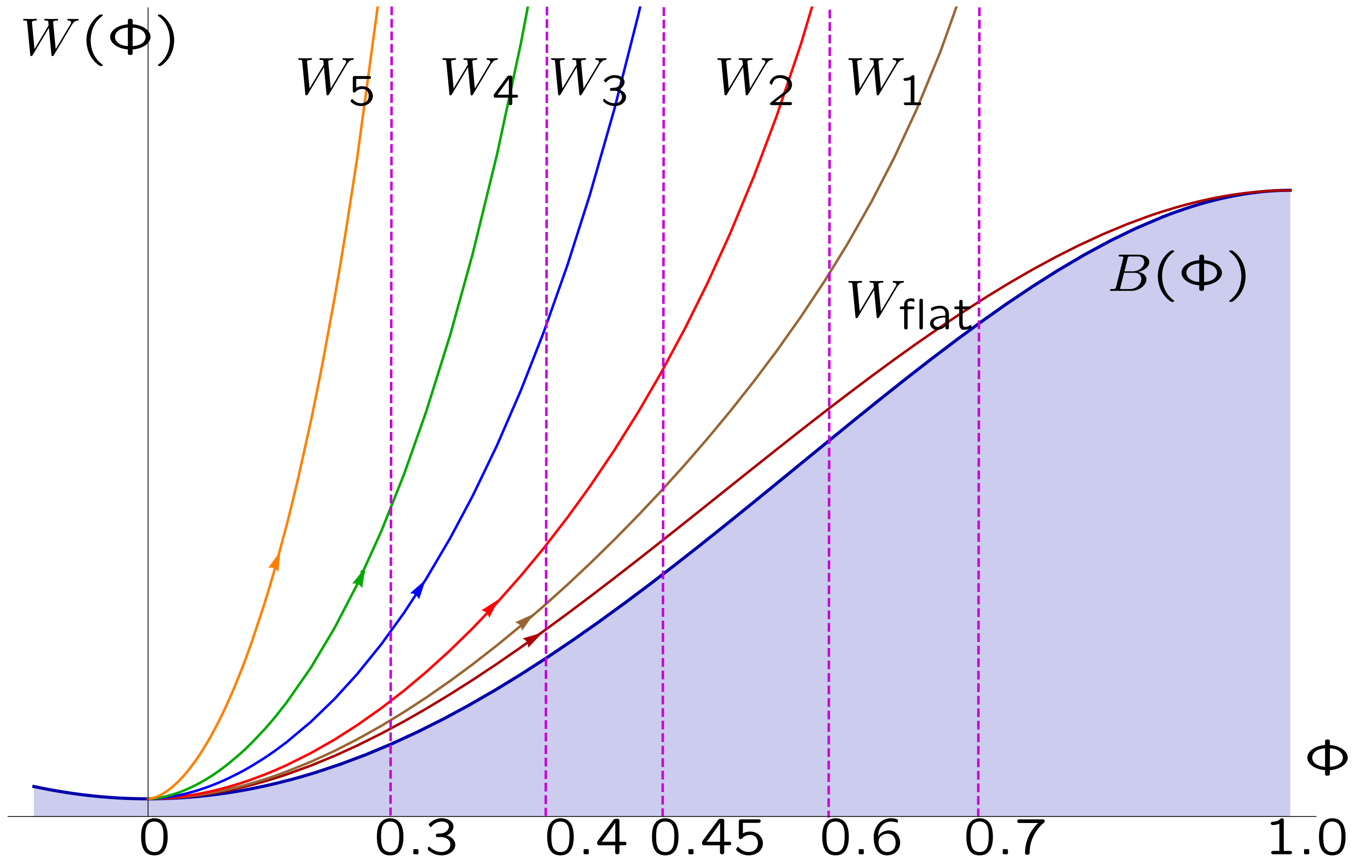
- The flow can end at any Φ_0, $V'(\Phi_0) \neq 0$, and there we have a regular horizon (similar to the Poincaré horizon).

- Generically for each end-point Φ_0 we have a unique solution.

- Solving the equations towards the UV, we obtain the parameters of the REGULAR flow $\mathcal{R}$ and $C(\mathcal{R})$ as a function of Φ_0.

- We can therefore take Φ_0 as the independent dimensionless parameter of the theory.

The vanilla flows at finite curvature



The on-shell action

• Once we understand the structure of flows, we must calculate the on-shell action for such flows.

♠ It is $S_{on-shell}$ that contains all the quantitative information that is important for the many applications.

• A direct calculation using the equations of motion gives:

$$F = 2M_p^{d-1}\; V_d \left[\; (d-1)\left[e^{dA}\;\dot{A}\right]_{\mathrm{UV}} + \frac{R_{UV}}{4}\int_{\mathrm{IR}}^{\mathrm{UV}} du\, e^{(d-2)A}\right],$$

• This is valid for the theory on S^d and it gives the partition function on S^d.

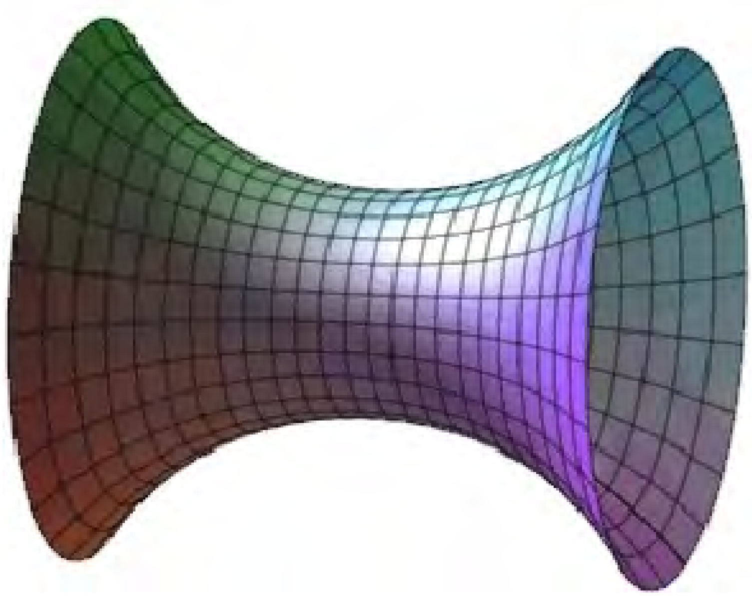
• For the theory on dS_d, F has a minus sign.

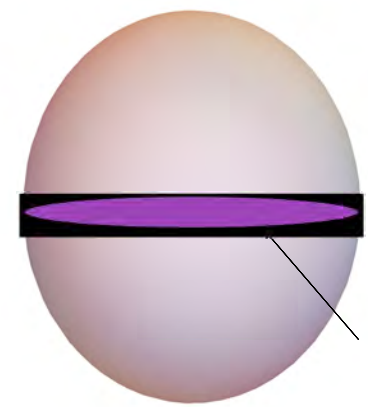
• The first term is there in the case of the theory in flat space.

Thermodynamics in de Sitter and (entanglement) entropy

• Consider a QFT_d on a d-dimensional deSitter space in global coordinates (where it is a changing S^{d-1} sphere):

$$ds^2 = -dt^2 + \alpha^2 \cosh^2(t/\alpha)(d\theta^2 + \sin^2\theta \; d\Omega^2_{d-2})$$





• Consider the entanglement entropy in that theory between two spatial hemispheres that have S^2 as boundary.

• The EE of the two hemispheres can be computed holographically using the Ryu-Takayanagi formula. The result is*,

$$S_{EE} = M_P^{d-1} \frac{2^{\frac{d(d-1)}{\alpha^2}}}{4} \; V_d \int_{\text{UV}}^{\text{IR}} du \, e^{(d-2)A(u)} \, .$$

Ben-Ami+Carmi+Smolkin

• This is precisely the second term that enters the curved on-shell action.

$$F = 2M_p^{d-1} V_d \left[(d-1) \left[e^{dA} \dot{A} \right]_{\text{UV}} + \frac{R}{4} \int_{\text{IR}}^{\text{UV}} du \, e^{(d-2)A} \right] ,$$

• The first term has also a thermodynamic interpretation: we change coordinates on the de Sitter slices and go to static patch coordinates.

Casini+Huerta+Myers

$$ds^2 = du^2 + e^{2A(u)} \left[-\left(1 - \frac{r^2}{\alpha^2}\right) d\tau^2 + \left(1 - \frac{r^2}{\alpha^2}\right)^{-1} dr^2 + r^2 d\Omega_{d-2}^2 \right] \, .$$

where α is the de Sitter radius and $0 < r < \alpha$.

• Now there is a bulk horizon at $r = \alpha$. The Bekenstein-Hawking entropy can be calculated and it is equal to the dS entanglement entropy, S_{EE}.

• The associated temperature to this horizon is constant (and fixed)

$$T = \frac{1}{2\pi\alpha}$$

• A similar computation of the "energy" U gives

$$\beta U = 2(d-1)M_P^{d-1}\ V_d \left[e^{dA(u)}\dot{A}(u)\right]_{UV}.$$

• This is the first term in the dS partition function of the (holographic) QFT$_d$.

• Putting everything together, we obtain a familiar thermodynamic formula

$$F = U - T\ S$$

for the de Sitter free-energy(partition function) and its S^d analytic continuation.

• For a CFT, the dS S_{EE}, is also the entanglement entropy for the S^{d-1} in flat space.

Casini+Huerta+Myers

• It is rather surprising that the partition function of a QFT on de Sitter space has a thermal interpretation.

$\mathcal{F}$-functions and $\mathcal{F}$-theorems

• I will call "global" C-theorem, the existence of a function, C on the space of CFTs that satisfies

$$C(CFT_{UV}) \quad > \quad C(CFT_{IR})$$

• I will call "local" C-theorem, the existence of a function $C(\log \mu)$ on the space of QFTs (a function of the RG flow parameter), that satisfies locally

$$\frac{dC}{du} < 0 \quad , \quad C(\mu = \infty) = C(CFT_{UV}) \quad , \quad C(\mu = 0) = C(CFT_{IR})$$

• A global F-function for 3d CFTs was proposed to be the renormalized "free energy" (or partition function) of a CFT on the 3-sphere.

Jafferis, Jafferis+Klebanov+Pufu+Safdi

• There is no general proof, but it has been checked in perturbative and supersymmetric examples.

• But the associated (renormalized) partition function fails to be a monotonic F-function along the flow.

Klebanov+Pufu+Safdi, Taylor+Woodhead

• An interpolating F-function satisfying the F-theorem was proposed to be the (appropriately renormalized) S^2 entanglement entropy in flat space.

Myers+Sinha, Myers+Casini+Huerta, Liu+Mezzei

• There is a general proof that in 3d it is always monotonic (but the proof cannot be extended to 5d).

,Casini+Huerta

• As we have seen, the partition function of the sphere contains a part that is related to entanglement entropy.

• We therefore concluded that de Sitter entanglement entropy and the S^3 partition function are tightly connected.

• Now that we have complete control of the holographic sphere partition function, we will use it to define variants of the F-function.

New $\mathcal{F}$-functions

• To obtain a “local" $\mathcal{F}$-function we must have a function $\mathcal{F}(\mathcal{R})$, with $\mathcal{R}$ some parameter along the flow, which exhibits the following properties:

♠ At the fixed points of the flow, the function $\mathcal{F}(\mathcal{R})$ takes the values $\mathcal{F}_{\rm UV}$ and $\mathcal{F}_{\rm IR}$ respectively that are given by the “global" F-function.

♠ The function $\mathcal{F}(\mathcal{R})$ evolves monotonically along the flow,

$$\frac{d}{d\mathcal{R}}\mathcal{F}(\mathcal{R}) \leq 0\,,$$

• We will use $\mathcal{R}$ as an interpolating variable between

$$IR: \;\; \mathcal{R} \to 0 \qquad \text{and} \qquad UV: \mathcal{R} \to \infty$$

and demand

1. $\mathcal{F}$ must be UV and IR finite.

2. It must also satisfy:

$$\lim_{\mathcal{R}\to\infty} \mathcal{F}(\mathcal{R}) \equiv \mathcal{F}_{UV} = 8\pi^2(M\ell_{UV})^2$$

$$\lim_{\mathcal{R}\to 0} \mathcal{F}(\mathcal{R}) \equiv \mathcal{F}_{IR} = 8\pi^2(M\ell_{IR})^2$$

$$\frac{d\mathcal{F}}{d\mathcal{R}} \geq 0$$

• The sphere free energy is a function of $\mathcal{R}$ and a UV cutoff Λ.

• It is UV divergent as $\Lambda \to \infty$. The detailed structure of the general UV divergences are known.

• It is IR divergent as $\mathcal{R} \to 0$. The detailed structure of the general IR divergences we determined.

Ghosh+Kiritsis+Nitti+Witkowski

• The subtraction of UV divergences is standard and the renormalized partition function of a generic QFT on S^3 depends on two arbitrary scheme dependent constants.

• There are four **simple** distinct ways of subtracting the IR divergences. When this is done, the resulting $\mathcal{F}$ functions are scheme independent ($\mathcal{F}_{1,2,3,4}$).

• We can construct also two distinct F-functions starting directly from the de Sitter entanglement entropy ($\mathcal{F}_{5,6}$).

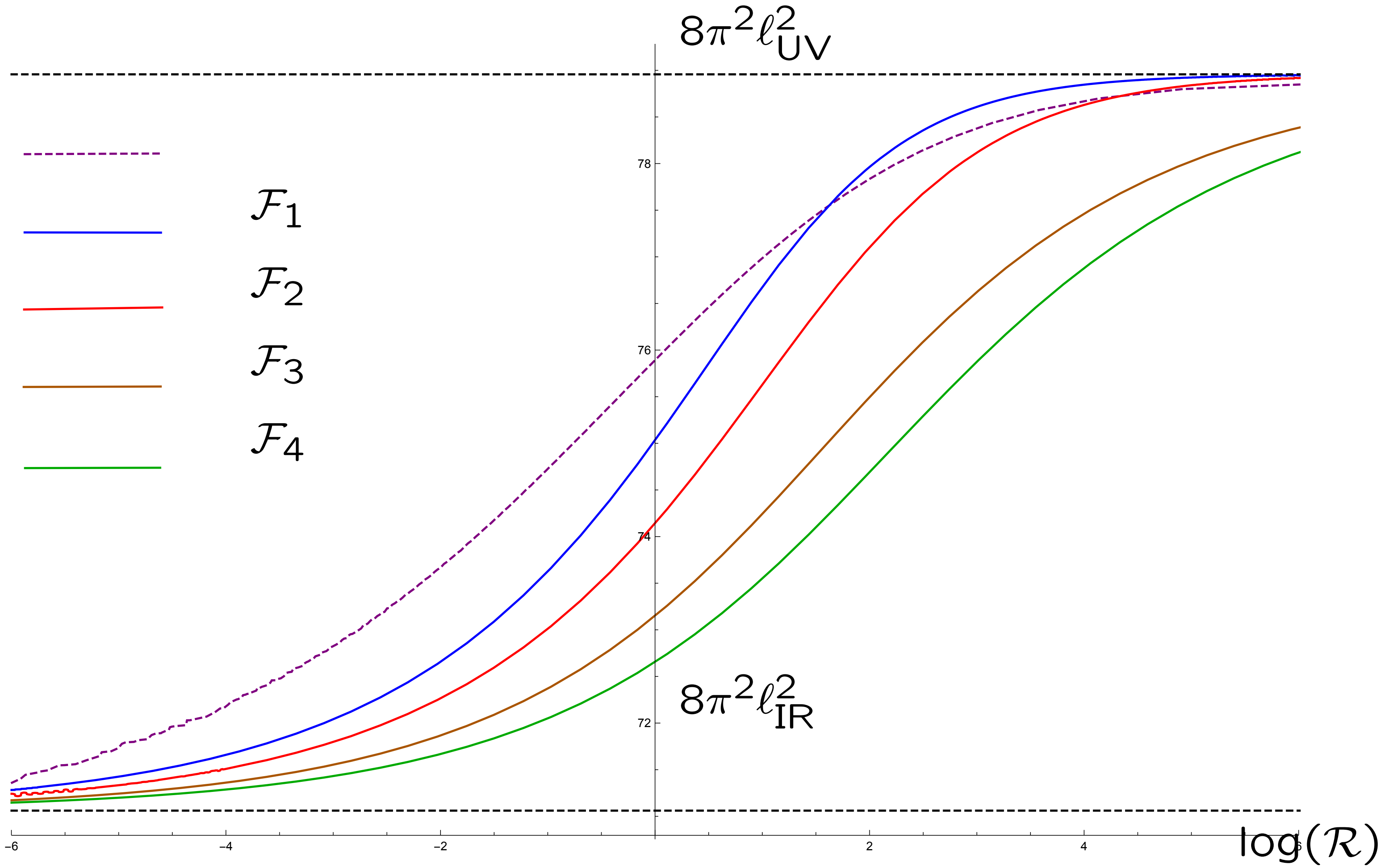
• It turns out that

$$\mathcal{F}_5 = \mathcal{F}_1 \quad , \quad \mathcal{F}_6 = \mathcal{F}_3$$

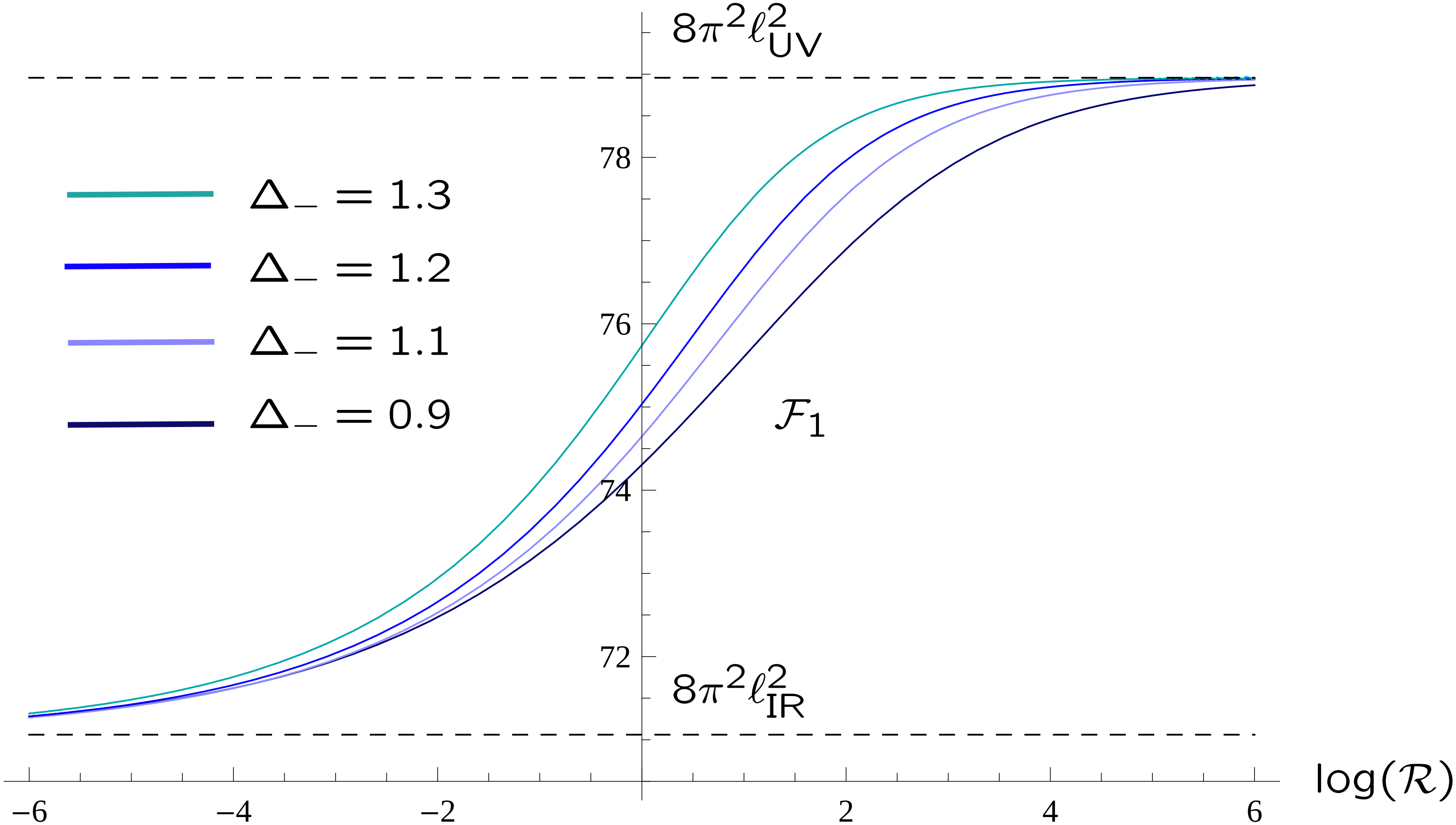
• We must also supplement these functions with the prescription that when $\boldsymbol{\Delta} < \frac{d}{2}$, then instead of the partition function we must use the effective action (ie. its Legendre transform).

• All $\mathcal{F}_{1,2,3,4}$ passed many checks both in holography and standard perturbation theory

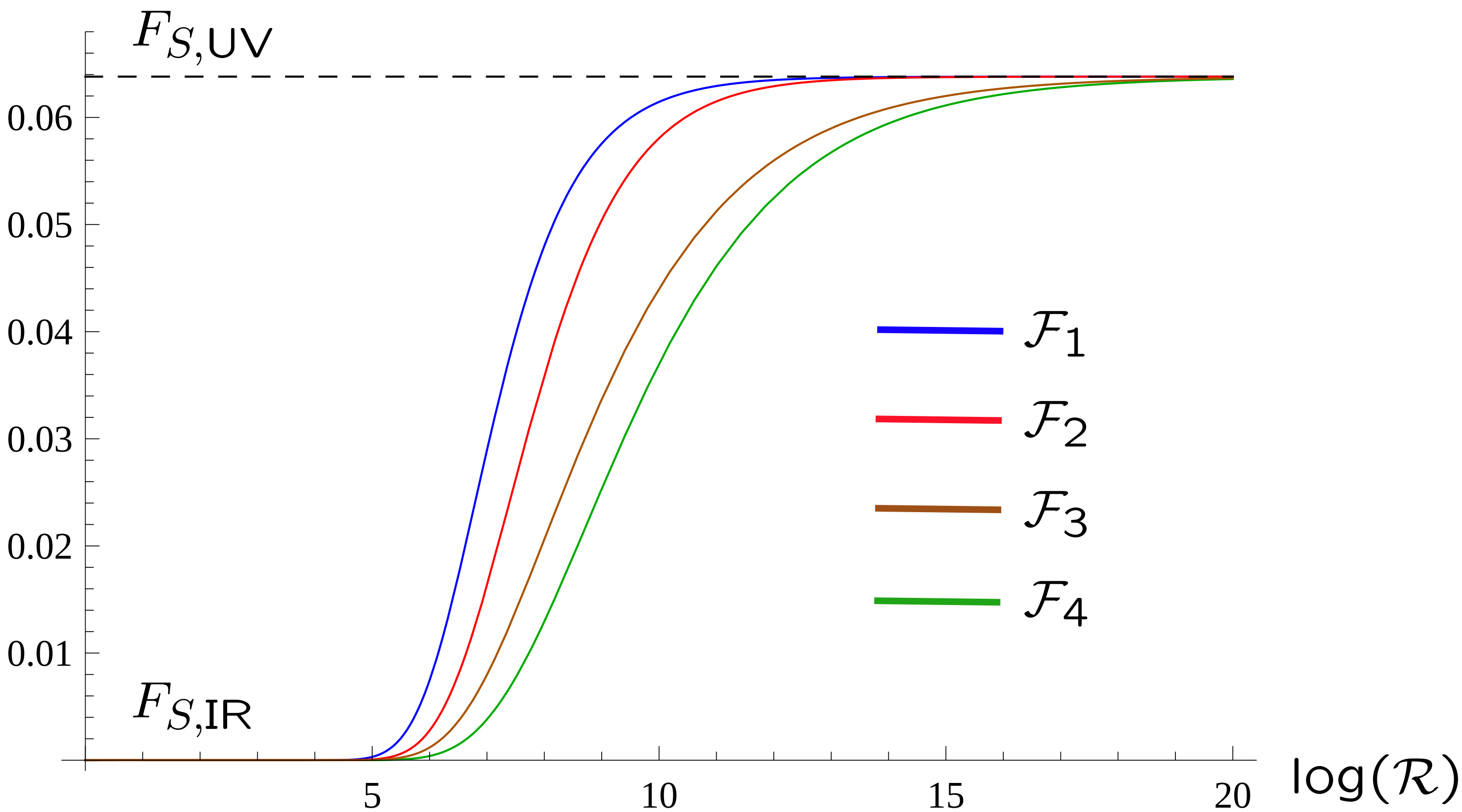
♠ All $\mathcal{F}_{1,2,3,4}$ are monotonic in many numerical holographic examples we analyzed when $\Delta > \frac{3}{2}$.



$\mathcal{F}_{1,2,3,4}$ vs. $\log(\mathcal{R})$ for a holographic model with Mex Hat potential and $\Delta_- = 1.2$.



$\mathcal{F}_1$ vs. $\log(\mathcal{R})$ for a holographic model with $\Delta_- = 0.9$ (dark blue), 1.1, (light blue), 1.2 (blue) and 1.3 (cyan).



Legendre-transformed $\mathcal{F}_{1,2,3,4}$ for a theory of a free massive boson on S^3.

♠ There is no general proof of monotonicity so far.

Outlook

• The space of holographic RG flows is richer than perturbative RG flows and allows several exotic possibilities.

• Some show radical departures from standard perturbative intuition and should be studied further.

• Possible synergies with exact methods of studying gauge theories (like lattice techniques) will be useful.

• The black holes associated with exotic RG flows have been analyzed and exhibit many of the phenomena that also appear in the finite curvature case. *Gursoy+Kiritsis+Nitti+Silva-Pimenda, Attems+Bea+Casalderrey-Solana+Mateos+Triana+Zilhao*

• The definition and study of $\mathcal{F}$ and C-functions is still an open problem in several cases/dimensions.

• The tools we developed seem very useful in order to understand the genericity of Coleman-de Lucia transitions in the AdS regime, with surprising conclusions.

Bibliography

Ongoing work with:

Francesco Nitti, Lukas Witkowski, Jewel Ghosh (APC, Paris)

Published work in:

- ArXiv:2003.09435 ArXiv:1901.04546
- ArXiv:1810.12318 ArXiv:1805.01769
- ArXiv:1711.08462 ArXiv:1611.05493

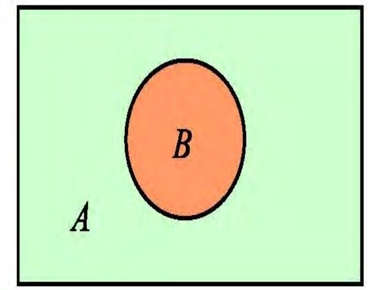
Based on earlier work:

- with Francesco Nitti and Wenliang Li ArXiv:1401.0888
- with Vassilis Niarchos ArXiv:1205.6205

Detour: entanglement entropy

• Consider a (3+1)-dimensional QFT on a space R×M.

• Consider a fixed time slice, $t = t_0$, and a 3d region B, bounded by a closed two-dimensional surface S, separating M into two parts, B and A=M-B.



• One can consider integrating our the QFT degrees of freedom in A, in order to obtain a density matrix, ρ_B, describing the degrees of freedom of region B.

• We can then compute the von Neumann entropy of ρ_B:

$$S_B \equiv Tr[\, \rho_B \log(\rho_B) \,]$$

• S_B is known as the entanglement entropy of region B. It contains information on the entanglement between the QFT degrees of freedom in B and those in $A = M - B$.

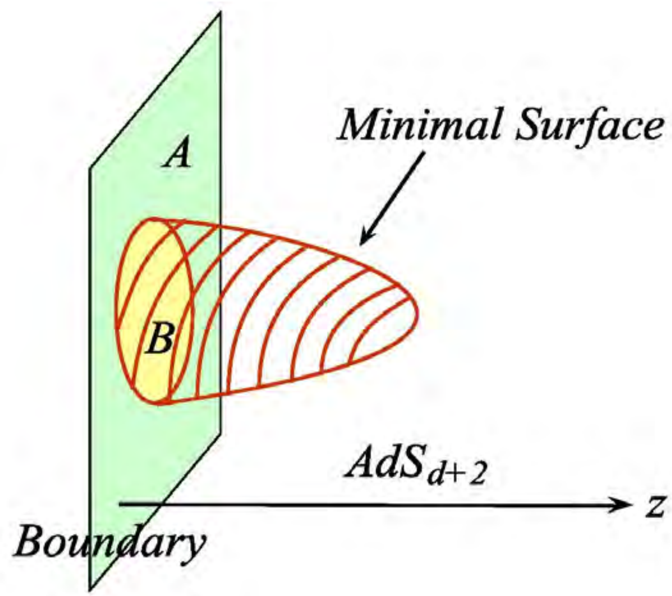
• In a local QFT it has a leading UV divergence that is proportional to the area of S.

• The entanglement entropy is very difficult to compute even in free QFTs.

• In Holographic QFTs there is a rather simple holographic formula for the entanglement entropy

Ryu+Takayanagi

$$S_B = \frac{Minimal \quad Area_B}{4G_5}$$



Holographic QFTs

• Large N (adjoint) quantum field theories are generically dual to string theories.

't Hooft

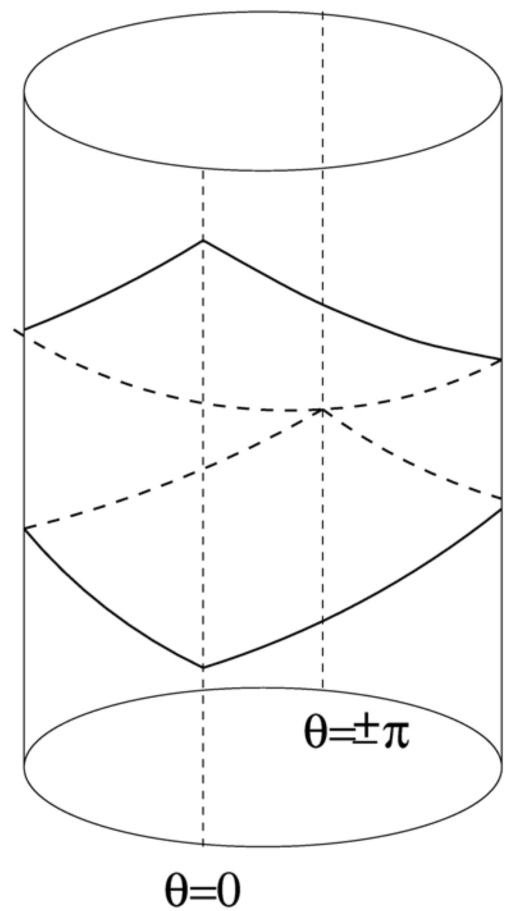
• Via the prototype example in 4d, namely N=4 Super YM theory dual to IIB string theory on $AdS_5 \times S^5$, we have understood that:

♠ The string theory has extra dimensions, one of which (inside AdS_5) is the holographic direction.

♠ When the coupling of the QFT is large, the string is stiff, and one can approximate the string theory by (super) gravity.

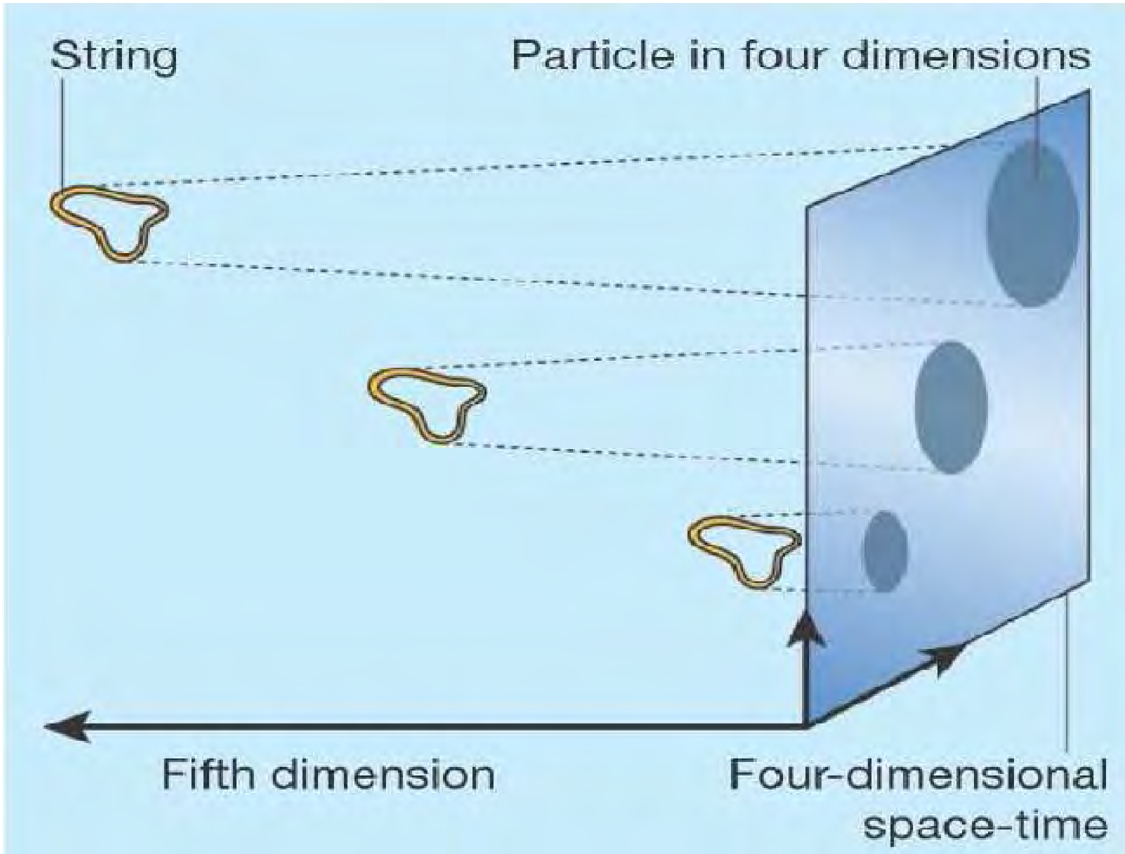
♠ This setup can be extended to many more theories that are connected via RG flows, giving in general a QFT/gravity correspondence (aka holographic correspondence).

♠ AdS space is a space with infinite volume and a boundary $R \times S^3$.



♠ The radial direction is describing the RG scale of the dual QFT.

♠ When the spatial sphere has large volume, then the boundary is isomorphic to Minkowski space.



$$ds^2 = du^2 + e^{-2\frac{u}{\ell}} \eta_{\mu\nu} dx^\mu dx^\nu$$

- The AdS boundary is at $u \to -\infty$

The holographic dictionary

• For every bulk (gravity) field, there corresponds a dual (single trace) operator in the QFT.

• In particular, the bulk metric $g_{\mu\nu}$ is dual to the QFT stress tensor $T_{\mu\nu}$.

• We will also use a bulk scalar field ϕ, dual to a scalar operator $O(x)$ in the QFT.

• We will use this scalar operator to perturb the UV CFT to generate an RG flow and therefore a (holographic) QFT.

• A classical solution of the bulk gravity equations with Dirichlet boundary conditions corresponds to a large-N saddle point of the dual QFT.

• For example the solution of the bulk scalar equation near the AdS boundary has the structure

$$\phi(u,x^\mu) = \phi_0(x^\mu)e^{(d-\Delta)\frac{u}{\ell}} + \cdots + \phi_1(x^\mu)e^{\Delta\frac{u}{\ell}} + \cdots \quad , \quad u \to -\infty$$

where $\phi_{0,1}(x^\mu)$ are the two independent boundary conditions for the scalar Laplacian equation.

♠ $\phi_0(x)$ is known as the “source" and corresponds to a source in the dual QFT:

$$S_{QFT} = S_{CFT_{UV}} + \int d^4x \ \phi_0(x) \ O(x)$$

♠ $\phi_1(x)$ is known as “the vev" because

$$\langle O(x) \rangle = 2(\Delta - 2) \ \phi_1(x)$$

♠ $\phi_1(x)$ is a functional of $\phi_0(x)$ for global regularity.

$$S_{\text{gravity}}(\phi(u,x))\Big|_{\text{on-shell}} = W_{\text{Schwinger}}(\phi_0(x))$$

$$e^{-W_{\text{Schwinger}}(\phi_0(x))} \equiv \langle e^{\int d^4x \ \phi_0(x) \ O(x)} \rangle$$

C-functions and C-theorems

• The concept of the C-function and C-theorem quantifies the naive expectation, that along an RG flow, we are losing degrees of freedom.

• It was first proven in 2d by Zamolodchikov.

• It states that there is a function along a flow, $C(g^i)$ such that:

(a) It is monotonically decreasing along RG Flows $\frac{dC}{d\log\mu} < 0$.

(b) It is extremal at the fixed points: $\left.\frac{dC}{d\log\mu}\right|_{g=g_*} = 0$

(c) The value at the fixed points is equal to the central charge c of the CFT_2: $C(g_*) = c$.

(d) In (near-CFT) perturbation theory the β-functions are gradients

$$\dot{g}^i = G^{ij}(g)\ \beta_j(g) = G^{ij}\frac{\partial C}{\partial g^j}$$

• It is not known, even in two dimensions, if (b) and (d) are correct beyond perturbation theory.

♠ It is a folk-theorem that the strong version of the c-theorem is expected to exclude limit cycles and other exotic behavior in Unitary Relativistic QFTs.

Zamolodchikov

• A potential loop-hole to this folk-theorem has been provided recently:

♠ If the β-functions have branch singularities away from the UV fixed point, then a limit cycle can be compatible with the strong version of the V-theorem.

Curtright+Zachos

• If this ever happens, it can only happen "beyond perturbation theory".

The C-function in 4 dimensions

• In 4 dimensions the analogue of the C-function is the a-coefficient of the conformal anomaly.

Cardy

• The global monotonicity $a(CFT_{UV}) > a(CFT_{IR})$ was proven recently.

Komargopodski+Schwimmer

• The strong form of the C-theorem was also proven in perturbation theory only.

Osborn, Jack+Osborn

• In 4d there are important subtleties: The β functions that enter $\dot{g}^i$ and ${T_\mu}^\mu$ are different and related by symmetry transformations.

Osborn, Fortin+Grinstein+Stergiou

$$\dot{g}_i = \beta_i(g) \quad , \quad T^\mu_\mu = \sum_i \tilde{\beta}_i(g)\ O_i + \text{curvature square terms}$$

• But at the fixed point they become the same, and vanish if an appropriate frame is chosen for the couplings.

• In 3 dimensions there is no conformal anomaly but there is an F-function.

• In five and six dimensions, no C/F-function is known so far.

F-functions

- It can be shown that

$$\mathcal{F}_6(\mathcal{R}) = \mathcal{F}_1(\mathcal{R}) \quad , \quad \mathcal{F}_5(\mathcal{R}) = \mathcal{F}_3(\mathcal{R})$$

$$F^{d=3}(\Lambda, \mathcal{R}) = -(M\ell)^2\tilde{\Omega}_3 \Big\{ \quad \mathcal{R}^{-3/2}\Big[4\Lambda^3\Big(1+\mathcal{O}\big(\Lambda^{-2\Delta_-}\big)\Big) + C(\mathcal{R})\Big]$$

$$+\mathcal{R}^{-1/2}\Big[\Lambda\Big(1+\mathcal{O}\big(\Lambda^{-2\Delta_-}\big)\Big) + B(\mathcal{R})\Big]\Big\} + \ldots ,$$

$$F^{d=3,\mathrm{ren}}(\mathcal{R}|B_{ct}, C_{ct}) = -(M\ell)^2\tilde{\Omega}_3\Big[\mathcal{R}^{-3/2}\Big(C(\mathcal{R}) - C_{ct}\Big) + \mathcal{R}^{-1/2}\Big(B(\mathcal{R}) - B_{ct}\Big)\Big] .$$

- We have

$$B(\mathcal{R}) = B_0 + B_1\mathcal{R} + \mathcal{O}(\mathcal{R}^2) - 8\pi^2\tilde{\Omega}_3^{-2}\frac{\ell_{\mathrm{IR}}^2}{\ell^2}\mathcal{R}^{1/2}\Big(1 + \mathcal{O}\big(\mathcal{R}^{-\Delta_-^{\mathrm{IR}}}\big)\Big)$$

$$C(\mathcal{R}) = C_0 + C_1\mathcal{R} + \mathcal{O}(\mathcal{R}^2) \quad , \quad \mathcal{R} \to 0$$

$$C(\mathcal{R}) = \mathcal{O}\left(\mathcal{R}^{3/2-\Delta_-}\right), B(\mathcal{R}) = -8\pi^2\tilde{\Omega}_3^{-2}\mathcal{R}^{1/2}\left(1 + \mathcal{O}\left(\mathcal{R}^{-\Delta_-}\right)\right) \quad , \quad \mathcal{R} \to \infty$$

See also Taylor+Woodhouse

• Using the above we can see that the $\mathcal{R} \to \infty$ limit of $F^{ren}(\mathcal{R})$ is finite and scheme independent

• We also obtain in the IR limit $\mathcal{R} \to 0$

$$F^{\rm ren} = -(M\ell)^2\tilde{\Omega}_3\Big(C_0 - C_{ct}\Big)\mathcal{R}^{-3/2} - (M\ell)^2\tilde{\Omega}_3\Big(B_0 + C_1 - B_{ct}\Big)\mathcal{R}^{-1/2} +$$

$$+8\pi^2(M\ell_{\rm IR})^2 + \mathcal{O}(\mathcal{R}^{-\Delta_-^{\rm IR}}) + \mathcal{O}(\mathcal{R}^{1/2})\,.$$

• It is generically IR divergent.

• There are two special values for the counterterms

$$B_{ct} = B_{ct,0} \equiv B_0 + C_1 \quad , \quad C_{ct} = C_{ct,0} \equiv C_0$$

• If chosen, the IR divergences cancel.

• We can also use the Liu-Mezzei method:

$$D_{3/2}\mathcal{R}^{-3/2} \equiv \left(\frac{2}{3}\mathcal{R}\frac{\partial}{\partial\mathcal{R}} + 1\right)\mathcal{R}^{-3/2} = 0$$

$$D_{1/2}\mathcal{R}^{-1/2} \equiv \left(2\mathcal{R}\frac{\partial}{\partial\mathcal{R}} + 1\right)\mathcal{R}^{-1/2} = 0$$

- There are four proposals using the free energy:

$$\mathcal{F}_1(\mathcal{R}) \equiv D_{1/2}\ D_{3/2}\ F(\Lambda, \mathcal{R})$$

$$\mathcal{F}_2(\mathcal{R}) \equiv D_{1/2}\ F^{\text{ren}}(\mathcal{R}|B_{ct}, C_{ct,0})$$

$$\mathcal{F}_3(\mathcal{R}) \equiv D_{3/2}\ F^{\text{ren}}(\mathcal{R}|B_{ct,0}, C_{ct}),$$

$$\mathcal{F}_4(\mathcal{R}) \equiv F^{\text{ren}}(\mathcal{R}|B_{ct,0}, C_{ct,0}).$$

- All of the above are "scheme independent".

- We can construct another two from the dS EE:

$$S_{EE}^{d=3,\text{ren}}(\mathcal{R}|\tilde{B}_{ct}) = (M\ell)^2 \tilde{\Omega}_3\, \mathcal{R}^{-1/2}\Big(B(\mathcal{R}) - \tilde{B}_{ct}\Big)\,,$$

- There are another two using the entanglement entropy

$$\mathcal{F}_5(\mathcal{R}) \equiv D_{1/2}\ S_{EE}(\Lambda, \mathcal{R})$$

$$\mathcal{F}_6(\mathcal{R}) = S_{EE}^{\text{ren}}(\mathcal{R}|B_{ct,0})$$

- Using the identity that links $B(\mathcal{R})$ and $C(\mathcal{R})$.

$$C'(\mathcal{R}) = \frac{1}{2}B(\mathcal{R}) - \mathcal{R}B'(\mathcal{R}).$$

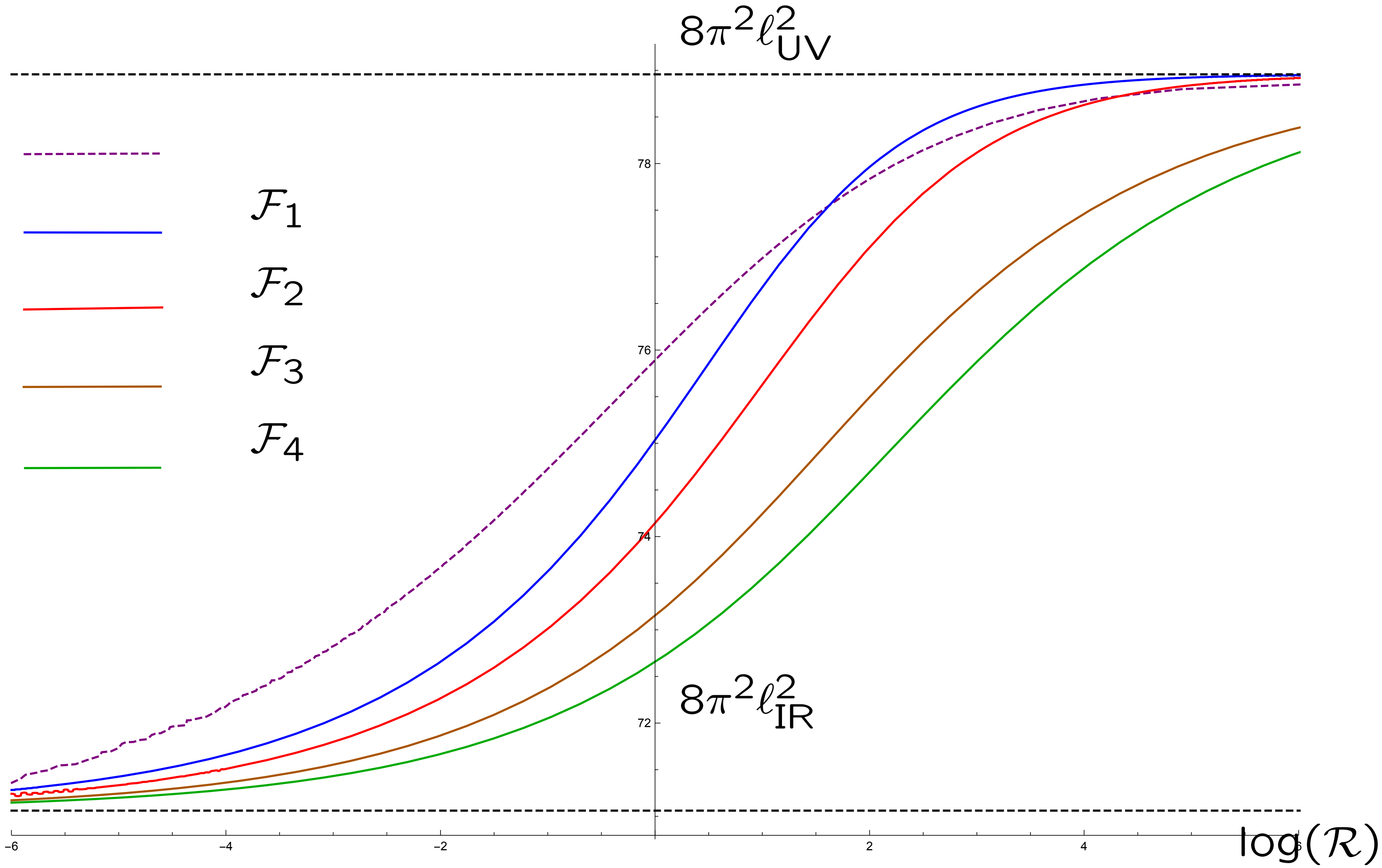
we can show that

$$\mathcal{F}_6(\mathcal{R}) = \mathcal{F}_1(\mathcal{R}) \quad , \quad \mathcal{F}_5(\mathcal{R}) = \mathcal{F}_3(\mathcal{R})$$

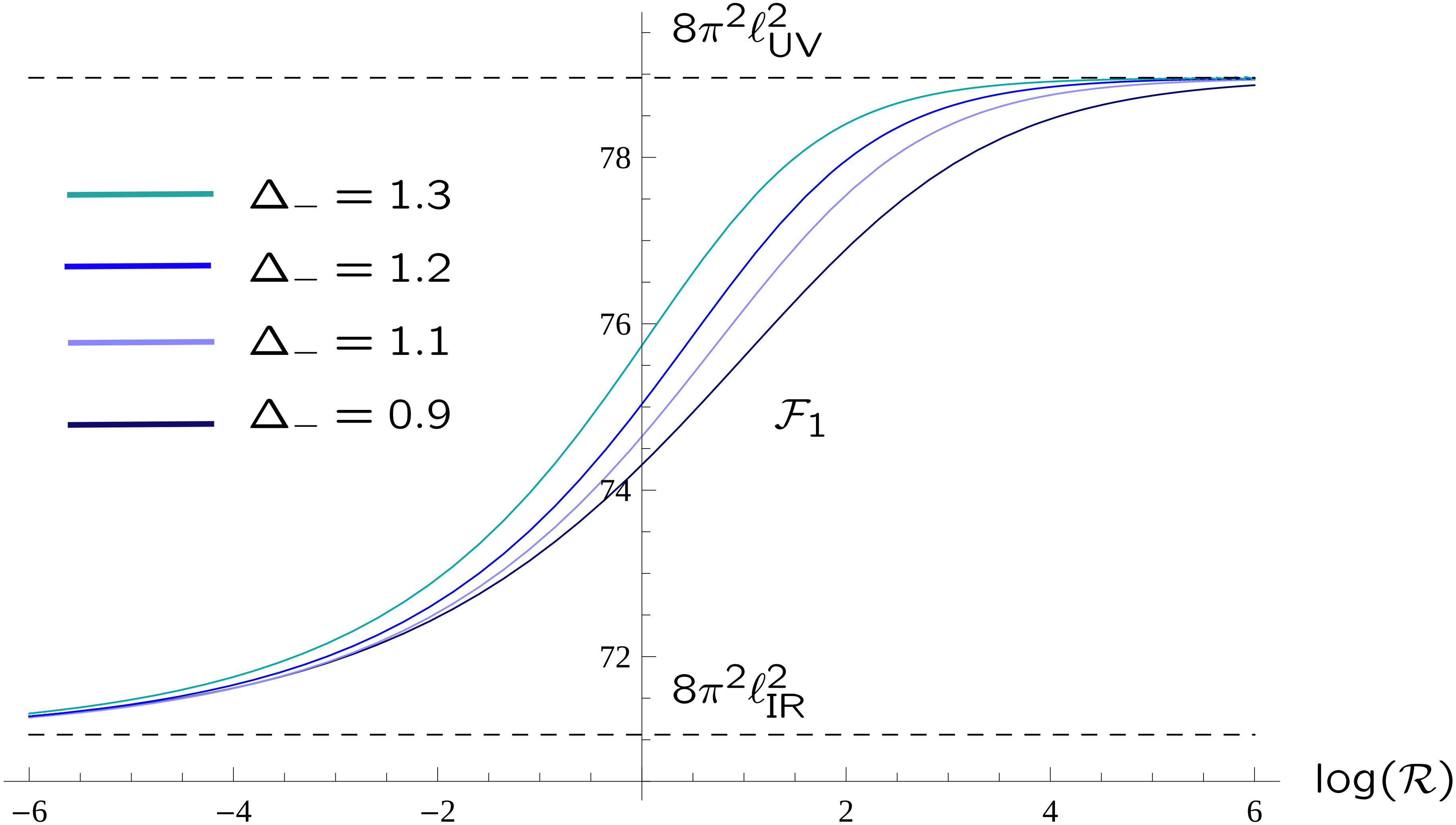
• It is interesting that renormalized EE and renormalized free-energy give the same answer in these cases.

• All $\mathcal{F}_{1,2,3,4}$ asymptote properly in the UV and IR limits.

♠ All $\mathcal{F}_{1,2,3,4}$ are monotonic in many numerical holographic examples we analyzed when $\Delta > \frac{3}{2}$.



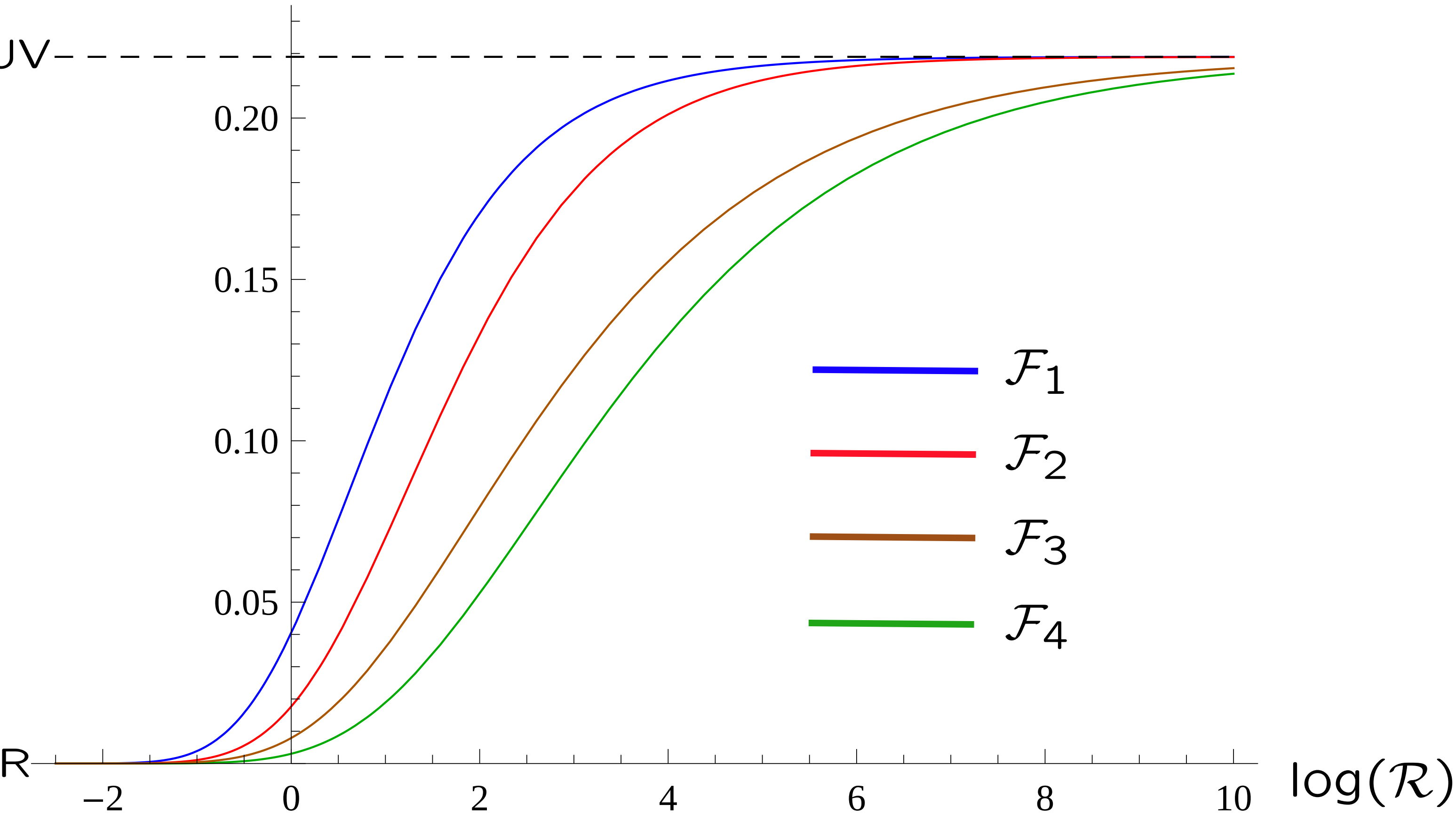
$\mathcal{F}_{1,2,3,4}$ vs. $\log(\mathcal{R})$ for a holographic model with Mex Hat potential and $\Delta_- = 1.2$.

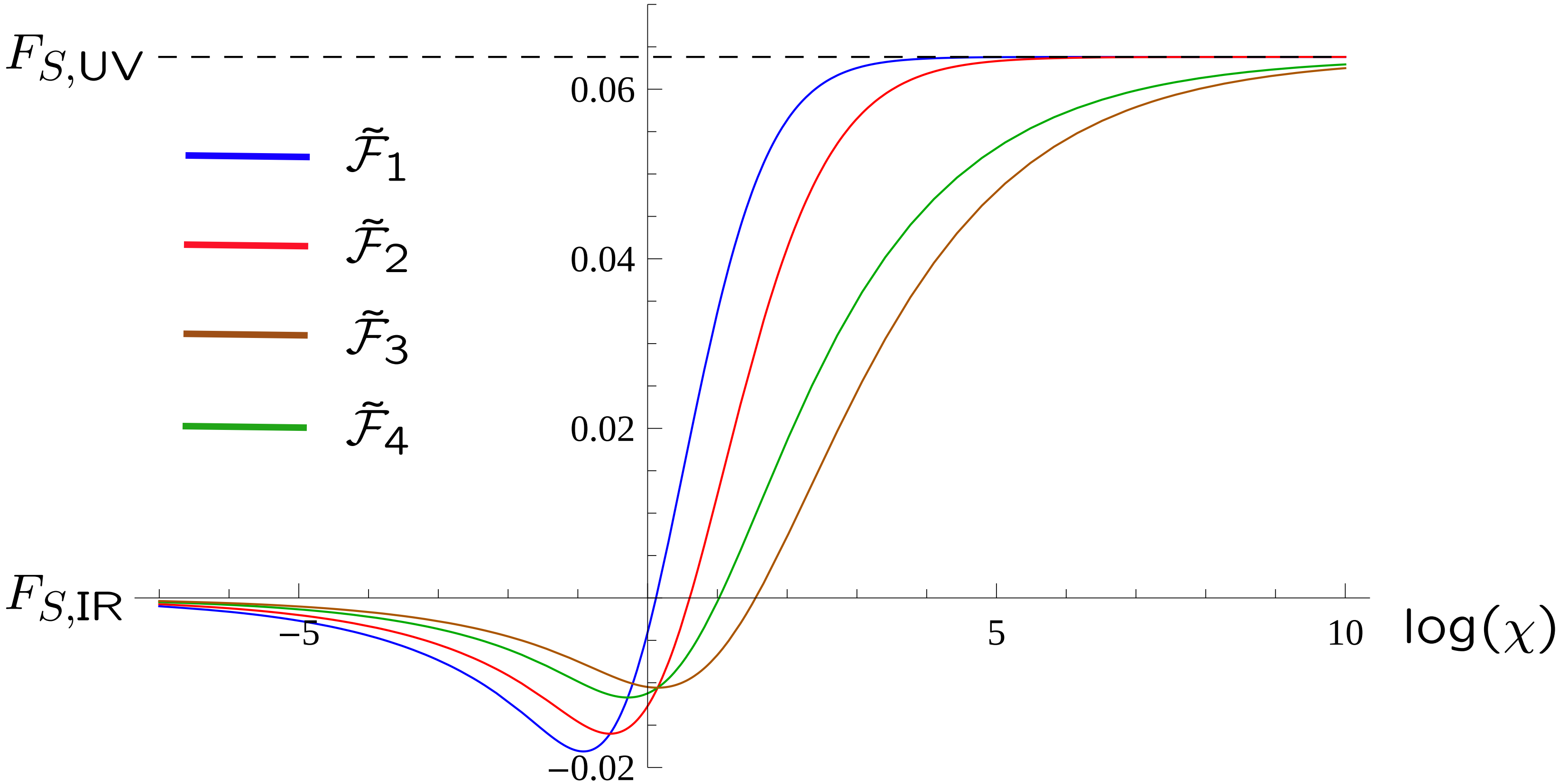


$\mathcal{F}_1$ vs. $\log(\mathcal{R})$ for a holographic model with $\Delta_- = 0.9$ (dark blue), 1.1, (light blue), 1.2 (blue) and 1.3 (cyan).

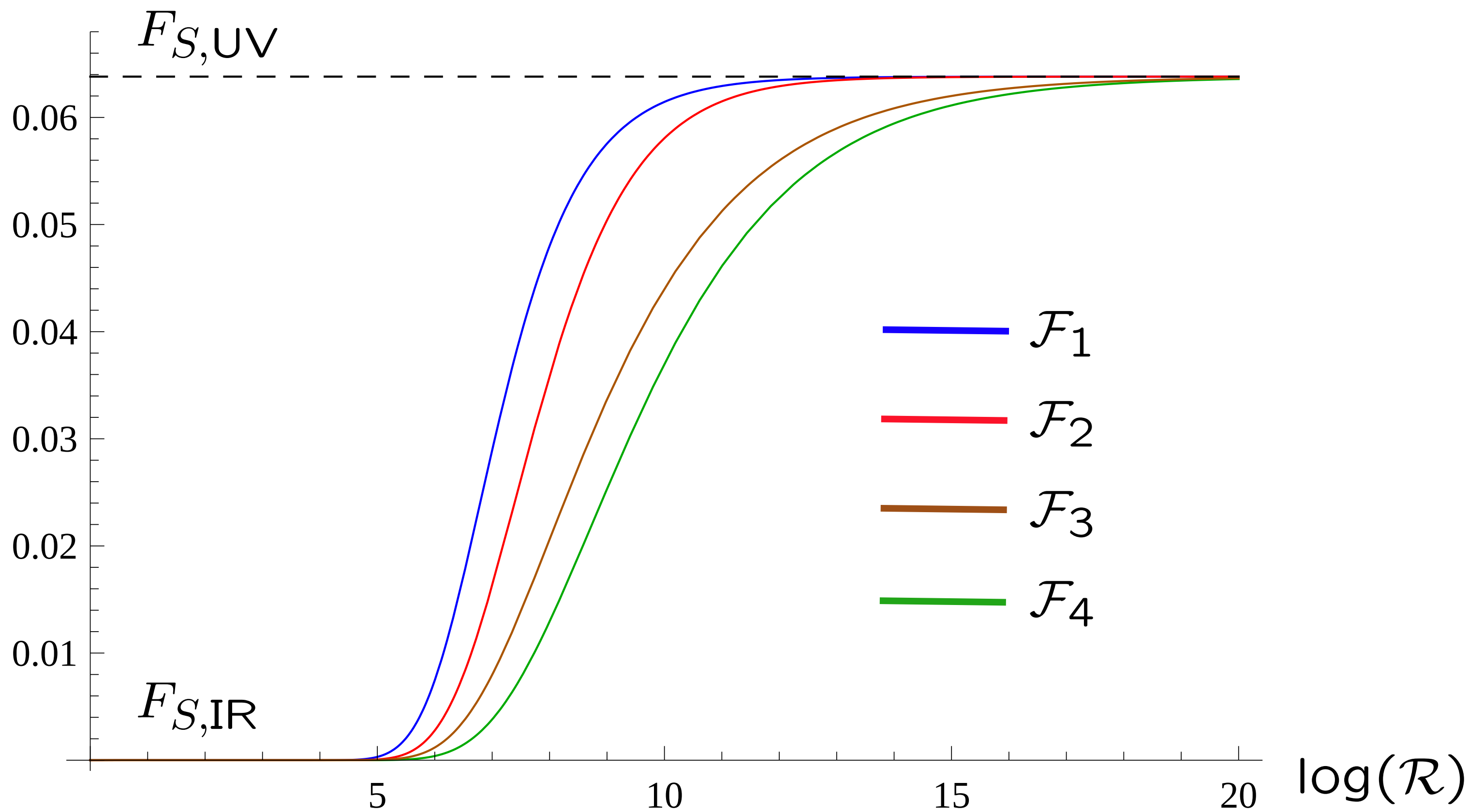
♠ In order for the proposal to work properly, when $\Delta < \frac{3}{2}$, $\mathcal{F}_{1,2,3,4}$ should be replaced by their Legendre transforms.

♠ This prescription also makes the free theories (the massive fermion and boson) to be monotonic as well.





$\tilde{\mathcal{F}}_{1,2,3,4}$ for a theory of a free massive scalar on S^3.



Legendre-transformed $\mathcal{F}_{1,2,3,4}$ for a theory of a free massive boson on S^3.

♠ We have no general proof of monotonicity so far.

Renormalization in d=3

- To define the finite on-shell action we must study the structure of divergences and then subtract them.

Skenderis+Henningson, Papadimitriou+Skenderis, Papadimitriou

$$F^{d=3}(\Lambda,\mathcal{R}) = -(M\ell)^2\tilde{\Omega}_3\Big\{ \quad \mathcal{R}^{-3/2}\Big[4\Lambda^3\Big(1+\mathcal{O}\Big(\Lambda^{-2\Delta_-}\Big)\Big)+C(\mathcal{R})\Big]$$

$$+\mathcal{R}^{-1/2}\Big[\Lambda\Big(1+\mathcal{O}\Big(\Lambda^{-2\Delta_-}\Big)\Big)+B(\mathcal{R})\Big]\Big\}+\ldots,$$

- To remove the divergences in general we must subtract two counterterms

$$F^{(0)}_{ct} = M^{d-1}\int_{\mathrm{UV}} d^dx\sqrt{|\gamma|}\,W_{ct}(\Phi) \quad , \quad F^{(1)}_{ct} = M^{d-1}\int_{\mathrm{UV}} d^dx\sqrt{|\gamma|}\,R^{(\gamma)}U_{ct}(\Phi)$$

where

$$\frac{d}{4(d-1)}W_{ct}^2-\frac{1}{2}\big(W'_{ct}\big)^2=-V \quad , \quad W'_{ct}\,U'_{ct}-\frac{d-2}{2(d-1)}W_{ct}\,U_{ct}=-1\,.$$

- The functions W_{ct}, U_{ct} are determined by two constants C_{ct}, B_{ct}.

• Therefore the renormalized on-shell action is

$$F^{\rm ren}(\mathcal{R}|B_{ct},C_{ct},\ldots) = \lim_{\Lambda\to\infty}\left[F(\Lambda,\mathcal{R}) + \sum_{n=0}^{n_{\rm max}} F_{ct}^{(n)}\right]$$

• In d=3 we obtain

$$F^{d=3,{\rm ren}}(\mathcal{R}|B_{ct},C_{ct}) = -(M\ell)^2\tilde{\Omega}_3\Big[\mathcal{R}^{-3/2}\Big(C(\mathcal{R})-C_{ct}\Big)+\mathcal{R}^{-1/2}\Big(B(\mathcal{R})-B_{ct}\Big)\Big].$$

• This is the (scheme-dependent) renormalized on-shell action on S^3.

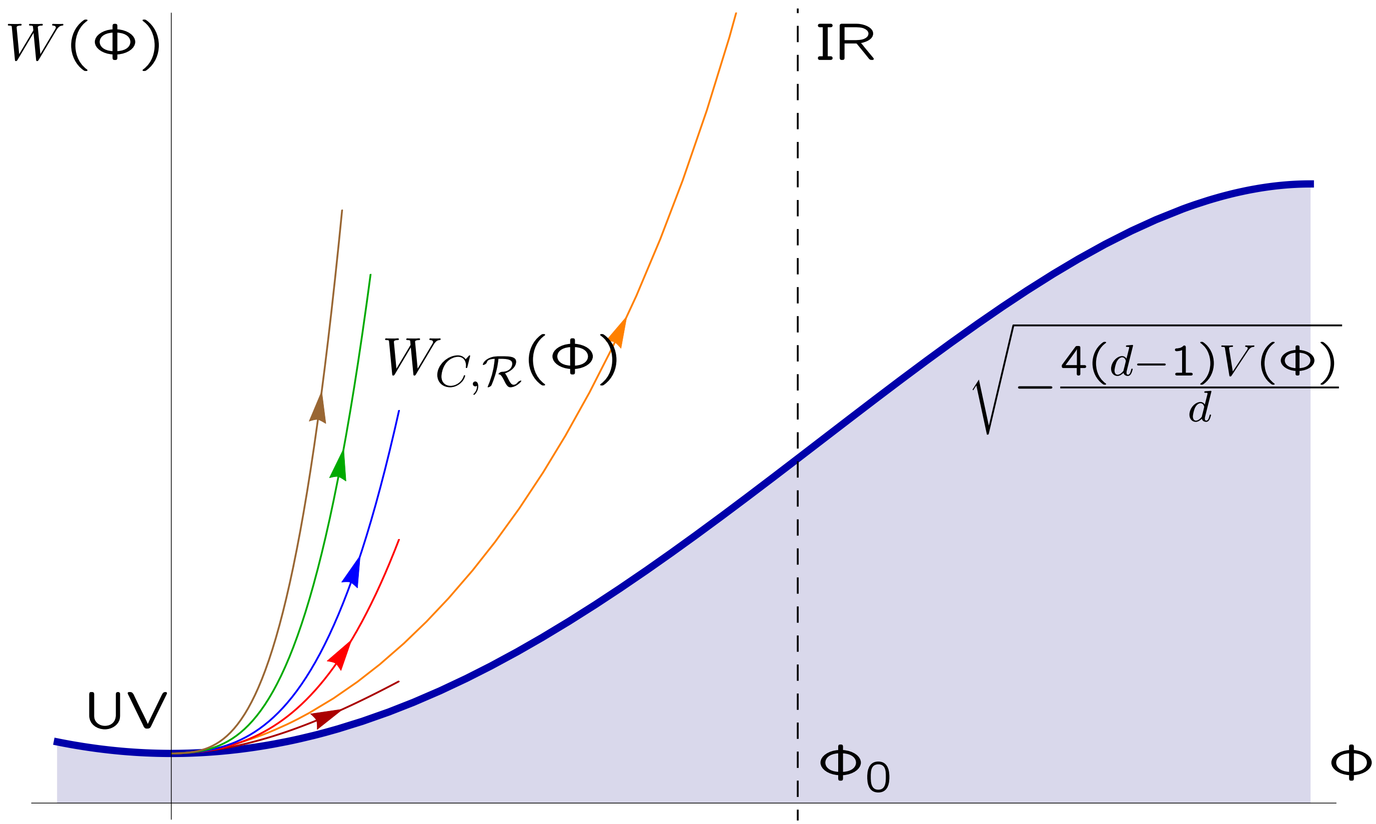
• It depends on two calculable functions $C(\mathcal{R})$ and $B(\mathcal{R})$ and two arbitrary renormalization constants C_{ct}, B_{ct}.

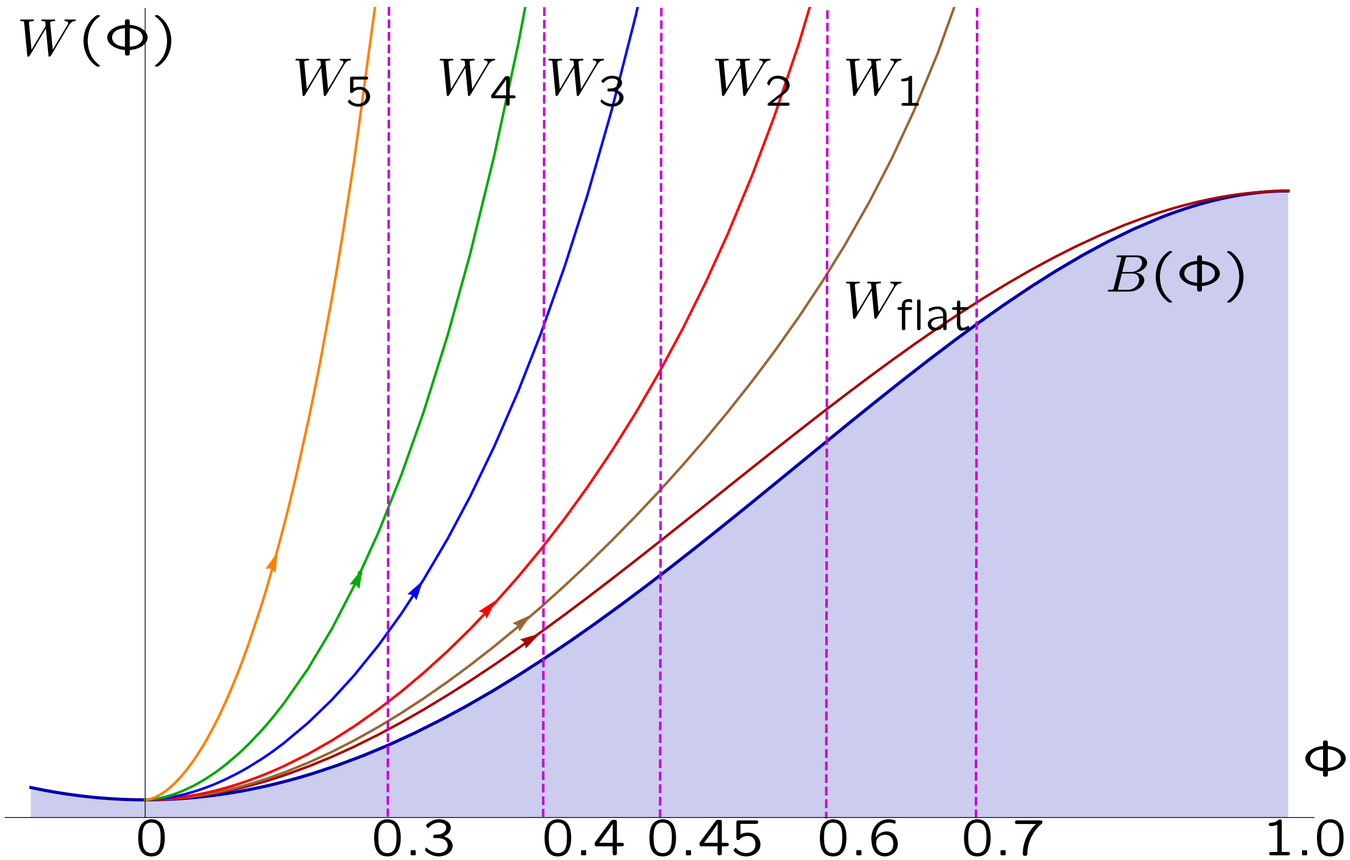
• It has two sources of IR divergences:

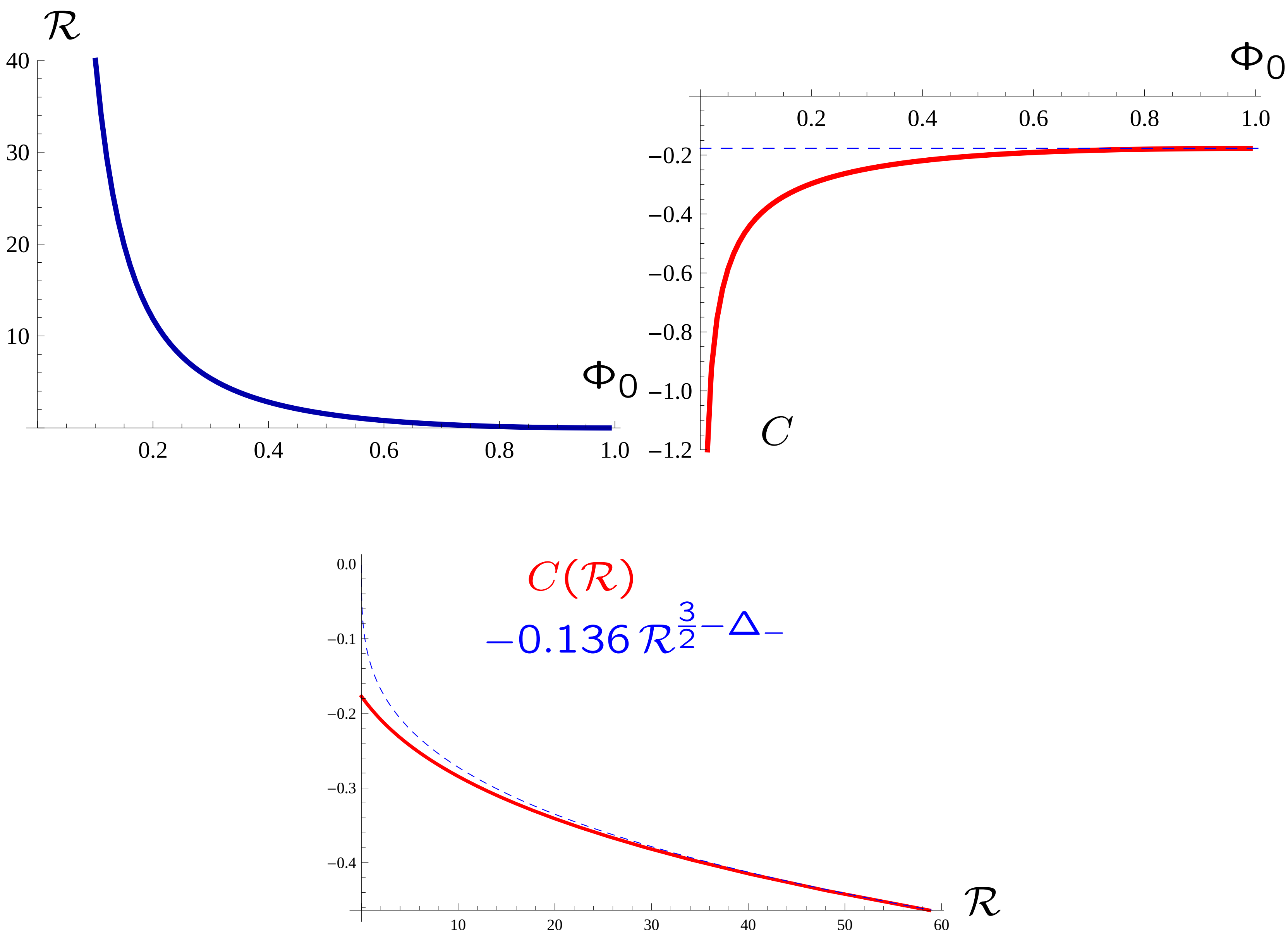
♠ $\mathcal{R}^{-3/2}$ is the expected volume divergence.

♠ $\mathcal{R}^{-1/2}$ is a subleading linear divergence.

The vanilla flows at finite curvature, II




$W(\Phi)$
W_5
W_4
W_3
W_2
W_1
W_{flat}
$B(\Phi)$
Φ
0
0.3
0.4
0.45
0.6
0.7
1.0


$\mathcal{R}$
Φ_0
Φ_0
C
$C(\mathcal{R})$
$-0.136\,\mathcal{R}^{\frac{3}{2}-\Delta_-}$
$\mathcal{R}$

The IR limits, II

- When $R_{UV} = 0$ the IR end-poids are minima of $V(\Phi)$.

- When $R_{UV} \neq 0$, the IR end points cannot be minima of $V(\Phi)$.

- The flow can end at any Φ_0, $V'(\Phi_0) \neq 0$, as

$$W(\Phi) = \frac{W_0}{\sqrt{|\Phi - \Phi_0|}} + \mathcal{O}\Big(|\Phi - \Phi_0|^0\Big) \quad , \quad S(\Phi) = S_0\sqrt{|\Phi - \Phi_0|} + \mathcal{O}\Big(|\Phi - \Phi_0|\Big)$$

with

$$S_0{}^2 = \tfrac{2|V'(\Phi_0)|}{d+1} \quad , \quad W_0 = (d-1)S_0$$

• At $\Phi = \Phi_0$,

$$T \simeq \frac{d}{4}\frac{W_0 S_0}{|\Phi - \Phi_0|} \to \infty \quad \text{as} \quad \Phi \to \Phi_0$$

• We have a regular horizon (similar to the Poincaré horizon).

• Generically for each Φ_0 we have a unique solution.

• Solving the equations towards the UV, we obtain the parameters of the REGULAR flow $\mathcal{R}$ and $C(\mathcal{R})$ as a function of Φ_0.

• We can therefore take Φ_0 as the independent dimensionless parameter of the theory.

• It has the advantage, that there is a unique solution for each Φ_0.

The first order RG flows

- We have two first order flow equations:

$$\dot{A} = -\frac{1}{2(d-1)}W(\Phi) \quad , \quad \dot{\Phi} = S(\Phi)$$

where the functions $W(\Phi)$, $S(\Phi)$ satisfy 2 first order non-linear equations

$$\frac{d}{2(d-1)}W^2 + (d-1)S^2 - dSW' + 2V = 0 \quad , \quad SS' - \frac{d}{2(d-1)}SW - V' = 0$$

- The two dimensionless integration constants that enter W, S, I will call $C, \mathcal{R}$. The first will be related to the vev of O dual to ϕ. $\mathcal{R}$ is related to the the curvature of the boundary metric.

- We also define

$$T(\Phi) \equiv R\, e^{-2A} = \frac{d}{2}S(\Phi)(W'(\Phi) - S(\Phi))$$

- $T \sim R$, and therefore $T = 0$ in the flat case.

The interpretation of parameters

- The solutions have four parameters:

♠ Two (A_0, ϕ_-) come from integrating the flow equations:

$$\dot{A} \sim W \quad , \quad \dot{\Phi} \sim S$$

They are sources (generically):

- A_0 is the UV scale of length.

- ϕ_- is the UV coupling constant of O.

♠ The other two are in W, S. The expansion near a UV fixed point is ($\Phi \to 0$)

$$W(\Phi) = \frac{2(d-1)}{\ell} + \frac{\Delta_-}{2\ell}\Phi^2 + \mathcal{O}\left(\Phi^3\right) + \delta W \,, \qquad S(\Phi) = \frac{\Delta_-}{2\ell}\Phi + \mathcal{O}\left(\Phi^2\right) + \delta S$$

- The non-analytic terms are:

$$\delta W(\Phi) = \frac{\mathcal{R}}{d\ell}|\Phi|^{\frac{2}{\Delta_-}}\Big(1+\mathcal{O}(\Phi)+\mathcal{O}(|\Phi|^{2/\Delta_-}\mathcal{R})+\mathcal{O}(|\Phi|^{d/\Delta_-}C(\mathcal{R}))\Big)$$

$$+\frac{C(\mathcal{R})}{\ell}|\Phi|^{\frac{d}{\Delta_-}}\Big(1+\mathcal{O}(\Phi)+\mathcal{O}(|\Phi|^{2/\Delta_-}\mathcal{R})+\mathcal{O}(|\Phi|^{d/\Delta_-}C(\mathcal{R}))\Big)$$

$$\delta S(\Phi) = \frac{d}{\Delta_-}\frac{C(\mathcal{R})}{\ell}|\Phi|^{\frac{d}{\Delta_-}-1}\Big(1+\mathcal{O}(\Phi)+\mathcal{O}(|\Phi|^{2/\Delta_-}\mathcal{R})+\mathcal{O}(|\Phi|^{d/\Delta_-}C(\mathcal{R}))\Big)+$$

$$+\mathcal{O}\Big(|\Phi|^{2/\Delta_-+1}\mathcal{R}\Big)$$

$$T(\Phi) = \mathcal{R}|\phi|^{\frac{2}{\Delta_-}}+\cdots$$

- The expansions above give a precise definition of the function $C(\mathcal{R})$

- We obtain the connection to observables

$$\mathcal{R} = R|\Phi_-|^{-2/\Delta_-} \qquad , \qquad \langle O\rangle(\mathcal{R}) = \frac{d}{\Delta_-}C(\mathcal{R})|\Phi_-|^{\frac{\Delta_+}{\Delta_-}}$$

- $\mathcal{R} > 0$ describes S^d and dS_d. $\mathcal{R} < 0$ describes AdS_d.

- C_0 is the second integration constant.

$$C(\mathcal{R}) \underset{\mathcal{R}\to 0}{=} C_0 + C_1\mathcal{R} + \mathcal{O}\left(\mathcal{R}^2\right) + \mathcal{O}\left(\mathcal{R}^{3/2-\Delta_-^{\rm IR}}\right)$$

- The general structure near a maximum (UV) of the potential has the "resurgent" expansion

$$W(\phi) = \sum_{m,n,r\in Z_0^+} A_{m,n,r}\ (C\ \phi^{\frac{d}{\Delta_-}})^m\ (\mathcal{R}\ \phi^{\frac{2}{\Delta_-}})^n\ \phi^r$$

Detour: Curvature-dependent β-functions and geometric flows

- We can calculate from the first order formalism the curvature dependent (holographic) β-function

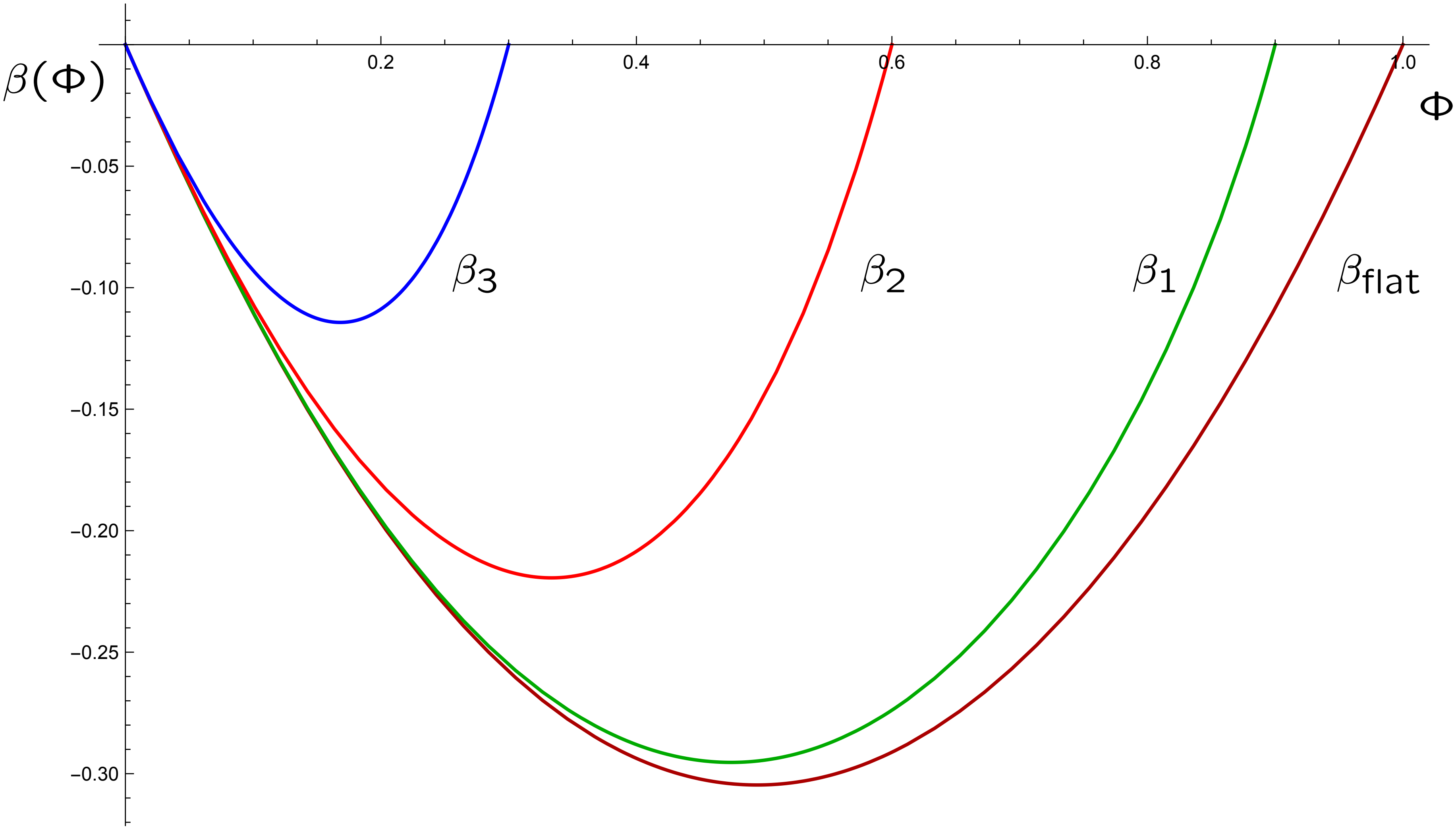
$$\beta(\Phi) \equiv \frac{d\Phi}{dA} = \frac{\dot{\phi}}{\dot{A}} = -2(d-1)\frac{S(\Phi)}{W(\Phi)}$$

- Near the UV

$$\beta(\Phi) = -\Delta_-\Phi + \mathcal{O}(\Phi^2) + \mathcal{O}\left(\mathcal{R}|\phi|^{1+\frac{2}{\Delta_-}}\right) + \cdots$$

- Near the IR (horizon)

$$\beta(\Phi) \sim (\Phi - \Phi_0)$$



$\beta(\Phi)$
0.2
0.4
0.6
0.8
1.0
Φ
−0.05
−0.10
−0.15
−0.20
−0.25
−0.30
β_3
β_2
β_1
β_{flat}

- The local RG takes couplings to weakly depend on x^μ.

Osborn

- The holographic RG can be generalized straightforwardly to the local RG

$$\dot\phi = W' - U'\ R + \frac{1}{2}\left(\frac{W}{W'}U'\right)'(\partial\phi)^2 + \left(\frac{W}{W'}U'\right)\Box\phi + \cdots$$

$$\dot\gamma_{\mu\nu} = -\frac{W}{d-1}\gamma_{\mu\nu} - \frac{1}{d-1}\left(U\ R + \frac{W}{2W'}U'(\partial\phi)^2\right)\gamma_{\mu\nu} +$$

$$+2U\ R_{\mu\nu} + \left(\frac{W}{W'}U' - 2U''\right)\partial_\mu\phi\partial_\nu\phi - 2U'\nabla_\mu\nabla_\nu\phi + \cdots$$

Papadimitriou, Kiritsis+Li+Nitti

- $U(\phi)$, $W(\phi)$ are solutions of

$$-\frac{d}{4(d-1)}W^2 + \frac{1}{2}W'^2 = V \quad , \quad W'\ U' - \frac{d-2}{2(d-1)}W\ U = 1$$

- Like in 2d σ-models we may use it to define "geometric" RG flows.

UV and IR divergences of F and S_{EE}

- The unrenormalized $F(\Lambda, \mathcal{R})$ and $S_{EE}(\Lambda, \mathcal{R})$.

♠ UV divergences $\Lambda \to \infty$:

$$F(\Lambda, \mathcal{R}) \quad : \quad \mathcal{R}^{-\frac{1}{2}}(\Lambda + \cdots) \ \text{ and } \ \mathcal{R}^{-\frac{3}{2}}(\Lambda^3 + \cdots)$$

$$S_{EE}(\Lambda, \mathcal{R}) \quad : \quad \mathcal{R}^{-\frac{1}{2}}(\Lambda + \cdots)$$

♠ IR divergences $\mathcal{R} \to 0$:

$$F(\Lambda, \mathcal{R}) \quad : \quad \mathcal{R}^{-\frac{1}{2}}\ (B_0 + C_1) \ \text{ and } \ \mathcal{R}^{-\frac{3}{2}}\ C_0$$

$$S_{EE}(\Lambda, \mathcal{R}) \quad : \quad \mathcal{R}^{-\frac{1}{2}}\ B_0$$

where

$$C(\mathcal{R}) \simeq C_0 + C_1 \mathcal{R} + \mathcal{O}(\mathcal{R}^2) \quad , \quad B(\mathcal{R}) \simeq B_0 + \mathcal{O}(\mathcal{R})$$

• The renormalized F and S_{EE}: only UR divergences, $\mathcal{R} \to 0$.

$$F^{\rm ren}(\mathcal{R}|B_{ct}, C_{ct}) \quad : \quad \mathcal{R}^{-\frac{1}{2}}(B_0 + C_1 - B_{ct}) \quad \text{and} \mathcal{R}^{-\frac{3}{2}}(C_0 - C_{ct})$$

$$S_{EE}^{\rm ren}(\mathcal{R}|\tilde{B}_{ct}, C_{ct}) \quad : \quad \mathcal{R}^{-\frac{1}{2}}(B_0 - \tilde{B}_{ct})$$

• We can remove UV divergences from unrenormalized functions by acting with

$$D_{3/2} \equiv \frac{2}{3}\frac{\partial}{\partial \mathcal{R}} + 1 \quad , \quad D_{1/2} \equiv 2\frac{\partial}{\partial \mathcal{R}} + 1 \quad , \quad D_{3/2}\mathcal{R}^{-\frac{3}{2}} = 0 \quad , \quad D_{1/2}\mathcal{R}^{-\frac{1}{2}} = 0$$

• We can remove IR divergences by choosing appropriately our scheme (subtractions)

$$B_{ct,0} = B(0) + C'(0) \quad , \quad C_{ct,0} = C(0) \quad , \quad \tilde{B}_{ct,0} = B(0)$$

$\mathcal{F}$-functions (II)

In terms of the two functions $B(\mathcal{R})$ and $C(\mathcal{R})$ the $\mathcal{F}$ functions can be written as

$$\frac{\mathcal{F}_1(\mathcal{R})}{(M\ell)^2\Omega_3} = -\ \frac{4}{3}\mathcal{R}^{\frac{1}{2}}(2B'(\mathcal{R}) + C''(\mathcal{R}) + \mathcal{R}\ B''(\mathcal{R}))$$

$$\frac{\mathcal{F}_2(\mathcal{R})}{(M\ell)^2\Omega_3} = -2\mathcal{R}^{-\frac{3}{2}}(-(C(\mathcal{R}) - C(0)) + \mathcal{R}C'(\mathcal{R}) + \mathcal{R}^2B'(\mathcal{R}))$$

$$\frac{\mathcal{F}_3(\mathcal{R})}{(M\ell)^2\Omega_3} = -\frac{4}{3}\mathcal{R}^{-\frac{1}{2}}(B(\mathcal{R}) + C'(\mathcal{R}) - B(0) - C'(0)) + \mathcal{R}B'(\mathcal{R}))$$

$$\frac{\mathcal{F}_4(\mathcal{R})}{(M\ell)^2\Omega_3} = -\mathcal{R}^{-\frac{3}{2}}(C(\mathcal{R}) - C(0)) + \mathcal{R}(B(\mathcal{R}) - B(0))$$

We also have the relation

$$C'(\mathcal{R}) = \frac{1}{2}B(\mathcal{R}) - \mathcal{R}B'(\mathcal{R}).$$

Holography and “Quantum" RG

• Enter holography as a means of probing strong coupling behavior.

• Holography provides a neat description of RG Flows.

• It also gives a natural a-function and the strong version of the a-theorem holds.

♠ But...the relevant equations that are converted into RG equations are second order!

• It is known for some time that the Hamilton-Jacobi formalism in holography gives first order RG-equations.

de Boer+Verlinde^2, Skenderis+Townsend, Gursoy+Kiritsis+Nitti, Papadimitriou, Kiritsis+Li+Nitti

• This would imply that (conceptually at least) holographic RG flows are very similar to (perturbative) QFT flows.

The extrema of V

The expansion of the potential near an extremum is

$$V(\phi) = -\frac{1}{\ell^2}\left[d(d-1) - \frac{m^2\ell^2}{2}\phi^2 + \mathcal{O}(\phi^3)\right]$$

$$\Delta_\pm = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2\ell^2} \quad ,$$

• The series solution of the superpotential is

$$W_\pm = 2(d-1) + \frac{\Delta_\pm}{2}\phi^2 + \cdots$$

• Near a maximum, W_- is part of a continuous family (parametrized by a vev)

• W_+ is an isolated solution.

• Near a minimum, regularity makes W_- unique.

• Near a minimum, W_+ describes a “UV fixed point"

The strategy

• Review of the holographic RG flows.

• Understanding the space of solutions.

• Standard RG flows start a maximum of the bulk potential and end at a nearby minimum.

• We find exotic holographic RG flows:

♠ “Bouncing flows": the β-function has branch cuts.

♠ “Skipping flows": the theory bypasses the next fixed point.

♠ “Irrelevant vev flows": the theory flows between two minima of the bulk potential.

• Outlook

Regularity

• One key point: out of all solutions W, typically one only gives rise to a regular bulk solution. (and more generally a discrete number*).

• All others have bulk singularities and are therefore unacceptable* (holographic) classical solutions.

• This reduces the number of (continuous) integration constants from 3 to 2.

• This has a natural interpretation in the dual QFT: the theory determines it possible vevs (we exclude flat directions).

• The remaining first order equations are now the first order RG equations for the coupling and the space-time volume.

• Now we can favorably compare with QFT RG Flows.

General properties of the superpotential

- From the superpotential equation we obtain a bound:

$$W(\phi)^2 = -\frac{4(d-1)}{d}V(\phi) + \frac{2(d-1)}{d}W'^2 \geq -\frac{4(d-1)}{d}V(\phi) \equiv B^2(\phi) > 0$$

- Because of the $(u, W) \to (-u, -W)$ symmetry we can fix the flow (and sign of W) so that we flow from $u = -\infty$ (UV) to $u = \infty$ (IR). This implies that:

$$W > 0 \quad \text{always} \quad \text{so} \quad W \geq B$$

- The holographic "a-theorem":

$$\frac{dW}{du} = \frac{dW}{d\phi}\,\frac{d\phi}{du} = W'^2 \geq 0$$

so that the a-function any decreasing function of W always decreases along the flow, ie. W is positive and increases.

• The inequality now can be written directly in terms of W:

$$W(\phi) \quad \geq \quad B(\phi) \quad \equiv \quad \sqrt{-\frac{4(d-1)}{d}V(\phi)}$$

• The maxima of V are minima of B and the minima of V are maxima of B.

• The bulk potential provides a lower boundary for W and therefore for the associated flows.

• Regularity of the flow=regularity of the curvature and other invariants of the bulk theory:

A flow is regular iff W, V remain finite during the flow.

• V aws assumed finite for ϕ finite. The same can be proven for W.

Therefore singular flows end up at $\phi \to \pm\infty$

.

Holographic RG Flows

• A QFT with a (relevant) scalar operator $O(x)$ that drives a flow, has two parameters: the scale factor of a flat metric, and the $O(x)$ coupling constant.

• These two parameters, generically correspond to the two integration constants of the first order bulk equations.

• Since ϕ is interpreted as a running coupling and A is the log of the RG energy scale, the holographic β-function is

$$\dot{A} = -\frac{1}{2(d-1)} W(\phi) \quad , \quad \dot{\phi} = W'(\phi)$$

$$\frac{d\phi}{dA} = -\frac{1}{2(d-1)} \frac{d}{d\phi} \log W(\phi) \equiv \beta(\phi) \sim \frac{1}{C}\frac{d}{d\phi} C(\phi)$$

• $C \sim 1/W^{d-1}$ is the (holographic) C-function for the flow.

Girardello+Petrini+Porrati+Zaffaroni, Freedman+Gubser+Pilch+Warner

• $W(\phi)$ is the non-derivative part of the Schwinger source functional of the dual QFT =on-shell bulk action.

de Boer+Verlinde[2]

$$S_{on-shell} = \int d^d x \sqrt{\gamma}\; W(\phi) + \cdots \Big|_{u \to u_{UV}}$$

• The renormalized action is given by

$$S_{renorm} = \int d^d x \sqrt{\gamma}\; (W(\phi) - W_{ct}(\phi)) + \cdots \Big|_{u \to u_{UV}} =$$

$$= constant \int d^d x \; e^{dA(u_0) - \frac{1}{2(d-1)} \int_{\phi_{UV}}^{\phi_0} d\tilde{\phi} \frac{W'}{W}} + \cdots$$

• The statement that $\frac{dS_{renorm}}{du_0} = 0$ is equivalent to the RG invariance of the renormalized Schwinger functional.

• It is also equivalent to the RG equation for ϕ.

• We can prove that

$$T_\mu{}^\mu = \beta(\phi)\; \langle O \rangle$$

• The Legendre transform of S_{renorm} is the (quantum) effective potential for the vev of the QFT operator O.

Detour: The local RG

- The holographic RG can be generalized straightforwardly to the local RG

$$\dot{\phi} = W' - f'\ R + \frac{1}{2}\left(\frac{W}{W'}f'\right)'(\partial\phi)^2 + \left(\frac{W}{W'}f'\right)\Box\phi + \cdots$$

$$\dot{\gamma}_{\mu\nu} = -\frac{W}{d-1}\gamma_{\mu\nu} - \frac{1}{d-1}\left(f\ R + \frac{W}{2W'}f'(\partial\phi)^2\right)\gamma_{\mu\nu} +$$

$$+2f\ R_{\mu\nu} + \left(\frac{W}{W'}f' - 2f''\right)\partial_\mu\phi\partial_\nu\phi - 2f'\nabla_\mu\nabla_\nu\phi + \cdots$$

Kiritsis+Li+Nitti

- $f(\phi)$, $W(\phi)$ are solutions of

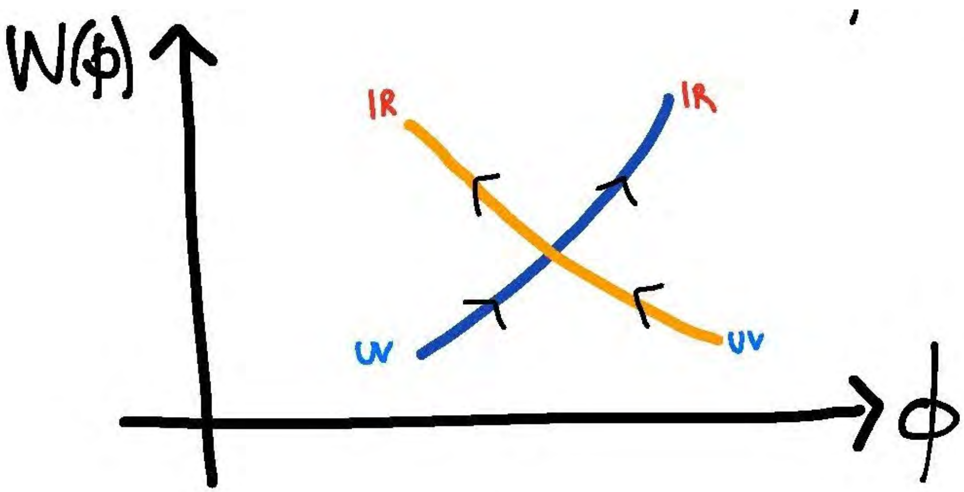
$$-\frac{d}{4(d-1)}W^2 + \frac{1}{2}W'^2 = V \quad , \quad W'\ f' - \frac{d-2}{2(d-1)}W\ f = 1$$

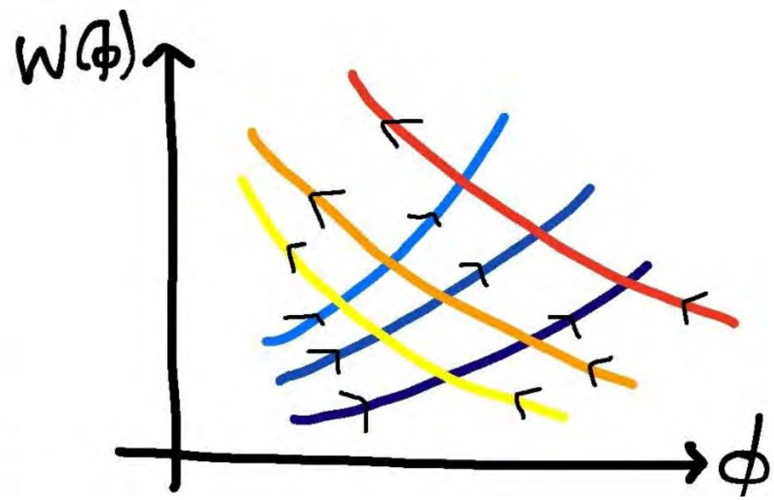
- Like in 2d σ-models we may use it to define "geometric" RG flows.

More flow rules

• At every point away from the $B(\phi)$ boundary ($W > B$) always two solutions pass:

$$W' = \pm\sqrt{2V + \frac{d}{2(d-1)}W^2} = \pm\sqrt{\frac{d}{2(d-1)}\left(W^2 - B^2\right)}$$





The critical points of W

- On the boundary $W = B$, we obtain $W' = 0$ and only one solution exists.

- The critical ($W' = 0$) points of W come in three kinds:

♠ $W = B$ at non-extremum of the potential (generic).

♠ Maxima of V (minima of B) (non-generic)

♠ Minima of V (maxima of B) (non-generic)

The BF bound

- The BF bound can be written as

$$\frac{4(d-1)}{d}\ \frac{V''(0)}{V(0)} \leq 1$$

- If a solution for W near $\phi = 0$ exists, then the BF bound is automatically satisfied as it can be written

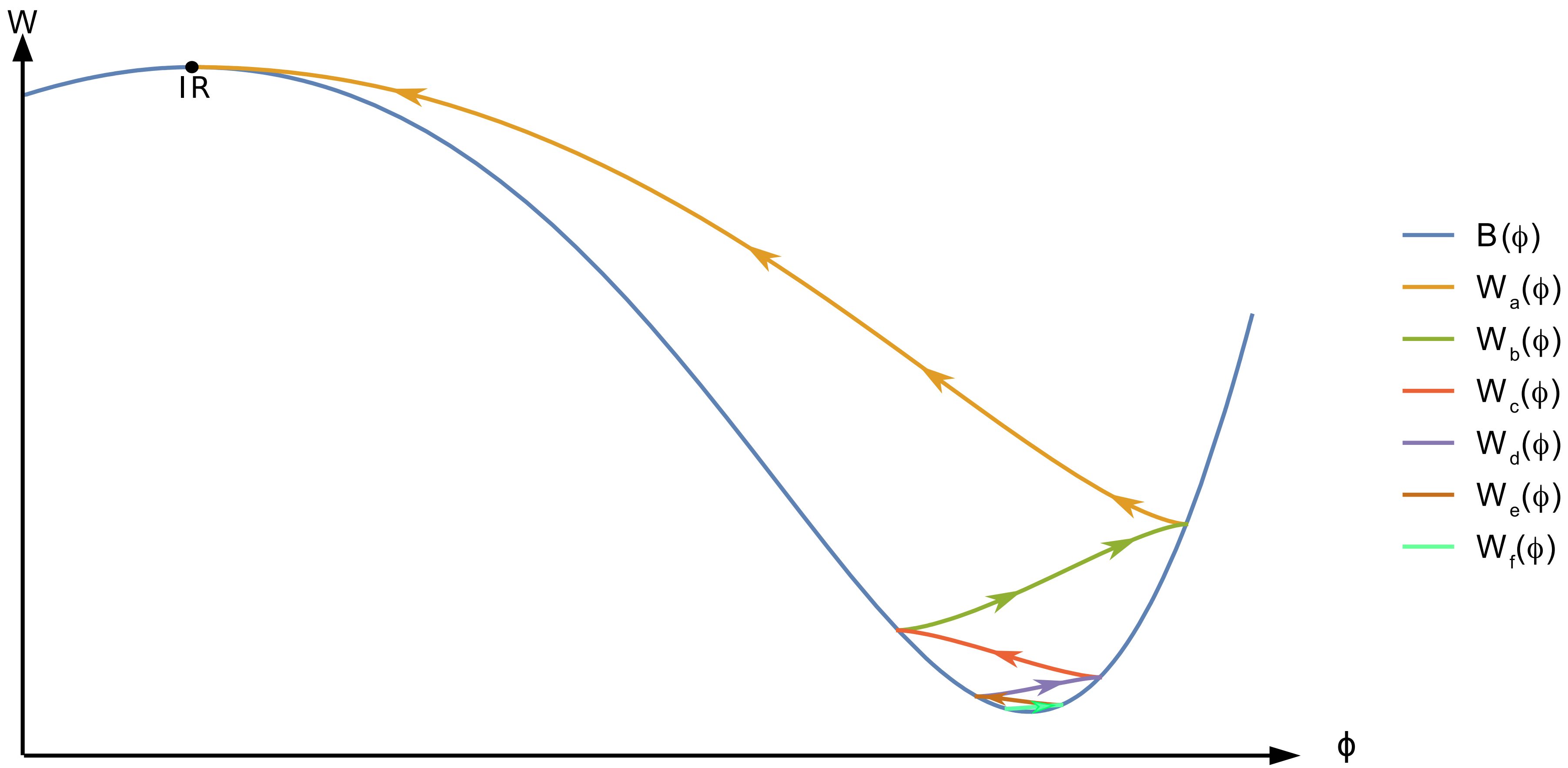
$$\left(\frac{4(d-1)}{d}\frac{W''(0)}{W(0)} - 1\right)^2 \geq 0$$

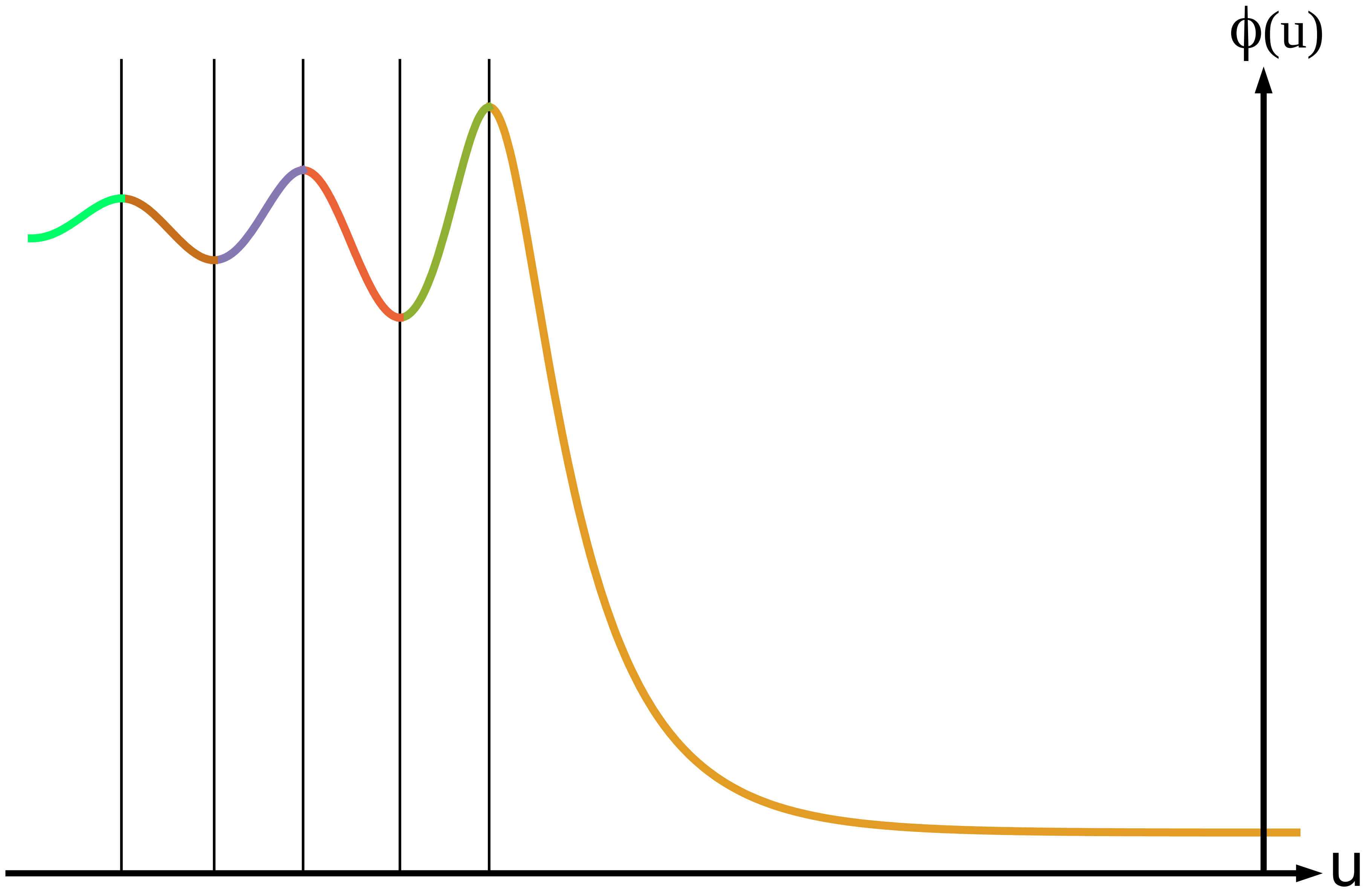
- When BF is violated, although there is no (real) W, there exists a UV-regular solution for the flow: $\phi(u)$, $A(u)$.

- This solution is unstable against linear perturbations (and corresponds to a non-unitary CFT).

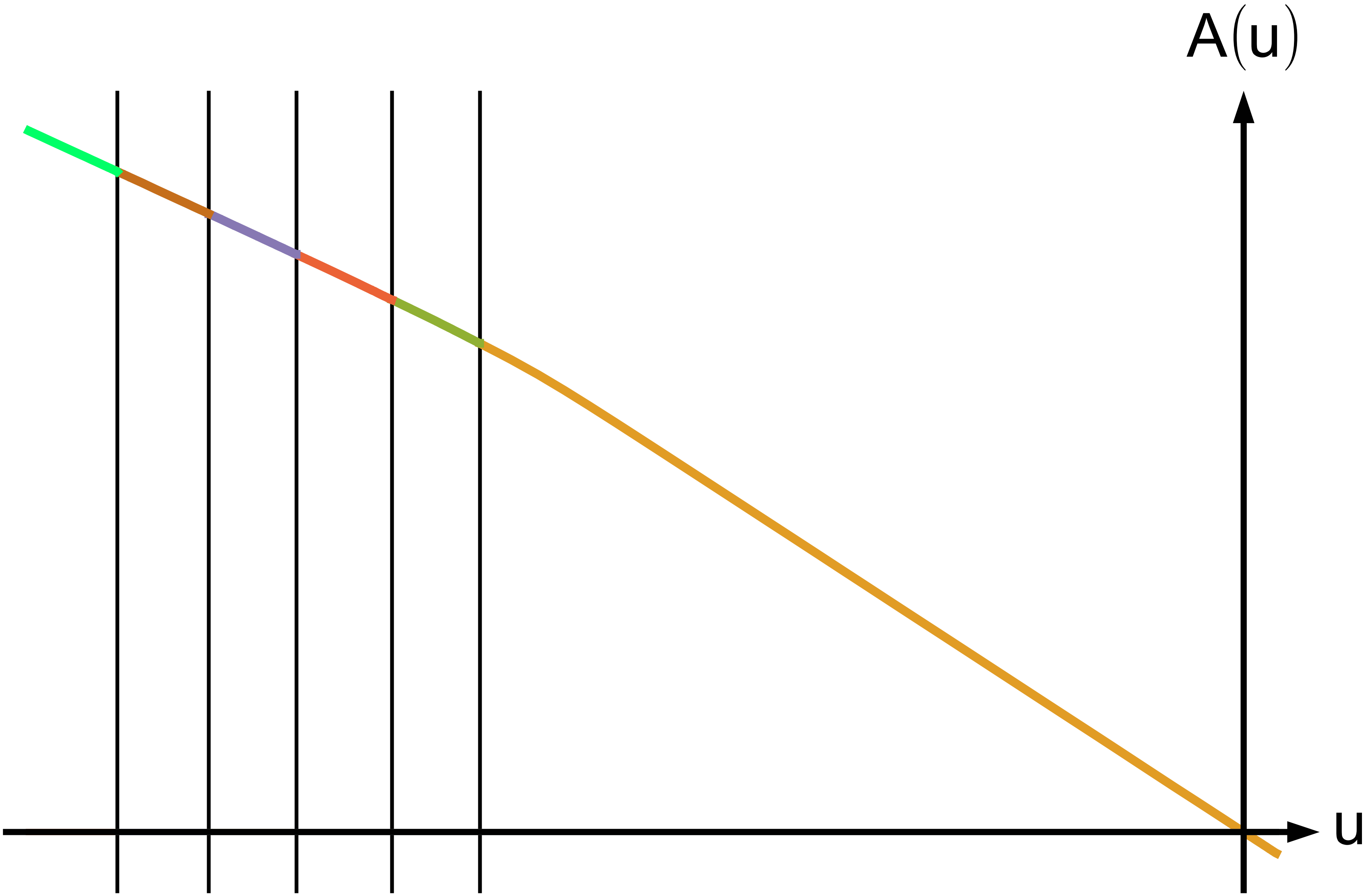
BF violating flows

- As mentioned there can be flows out of a BF-violating UV fixed point.
- No β-function description of such flows in the UV.
- Such flows have an infinite-cascade of bounces as one goes towards the UV.





φ(u)
u



• Although the flow is regular, it is unstable.

The extrema of V

- Solutions with constant scalar ϕ require them to be at an extremum of the potential, $V' = 0$.

- In that case, the metric is AdS_5 with symmetry $O(1,5)$.

- Therefore, extrema of the potential describe (holographic) CFTs.

- We will examine solutions for $W(\phi)$ near a maximum of V.

- We put the maximum at $\phi = 0$ and set $d = 4$.

$$V(\phi) = -\frac{1}{\ell^2}\left[12 - \frac{m^2\ell^2}{2}\phi^2 + \mathcal{O}(\phi^3)\right]$$

$$\Delta = 2 + \sqrt{4 + m^2\ell^2} \quad , \quad m^2\ell^2 \quad < 0 \quad , \quad 2 \le \Delta \le 4$$

• We set (locally) $\ell = 1$ from now on.

• The solution describes the region near a UV fixed point, upon a perturbation by a relevant operator of dimension $\Delta \leq 4$.

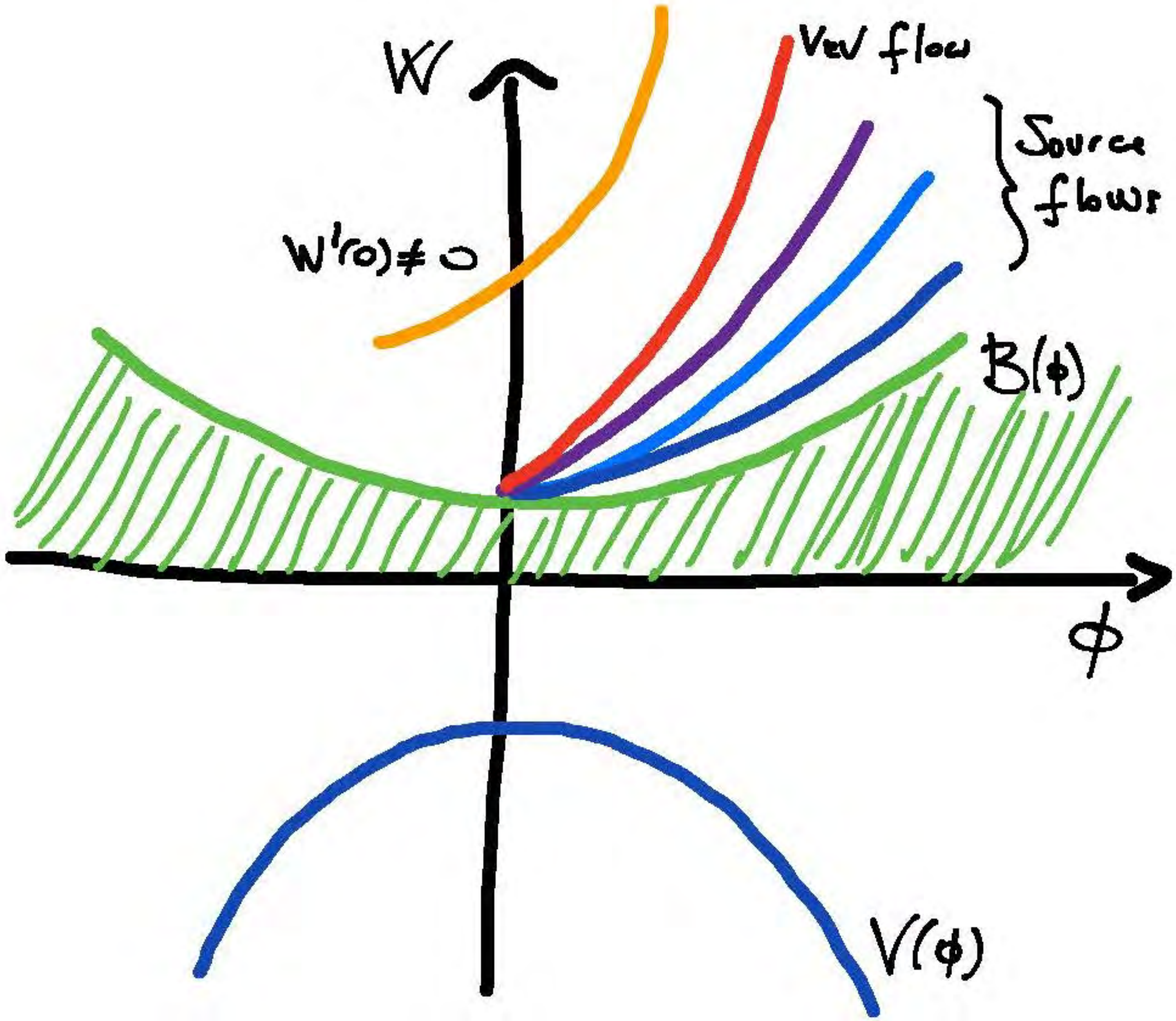
• The general structure of the solution for W has a "perturbative piece" (a power series in ϕ) and a non-perturbative piece (powers of $\phi^{\frac{4}{4-\Delta}}$)

$$W(\phi) = 6 + \frac{(4-\Delta)}{2}\phi^2 + \mathcal{O}(\phi^3) + C\phi^{\frac{4}{(4-\Delta)}}\left[1 + \mathcal{O}(\phi)\right] + \mathcal{O}(C^2\phi^{\frac{8}{(4-\Delta)}})$$

• C determines the vev: $\langle O \rangle \sim C\ \phi_0^{\frac{\Delta}{4-\Delta}}$.

$$\beta(\phi) = (\Delta - 4)\phi + \mathcal{O}(\phi^2) + \frac{4C}{4-\Delta}\phi^{\frac{\Delta}{4-\Delta}} + \cdots$$

• Maxima always describe UV CFTs. Minima generically describe IR CFTs.


W
Vev flow
Source flows
W'(0) ≠ 0
B(ϕ)
ϕ
V(ϕ)

The maxima of V

- We will examine solutions for W near a maximum of V.

- We put the maximum at $\phi = 0$.

$$V(\phi) = -\frac{1}{\ell^2}\left[d(d-1) - \frac{m^2\ell^2}{2}\phi^2 + \mathcal{O}(\phi^3)\right]$$

$$\Delta_\pm = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2\ell^2} \quad , \quad m^2\ell^2 \quad < 0 \quad , \quad \Delta_+ \geq \Delta_- \geq 0$$

- We set (locally) $\ell = 1$ from now on.

- If $W'(0) = 0$ there are two classes of solutions:

• A continuous family of solutions (the W_- family)

$$W_- = 2(d-1) + \frac{\Delta_-}{2}\phi^2 + \cdots + C\phi^{\frac{d}{\Delta_-}}[1+\cdots] + \mathcal{O}(C^2)$$

• The solution for ϕ and A corresponding to this, is the standard UV source flow:

$$\phi(u) = \alpha\, e^{\Delta_-\ u} + \cdots + \frac{\Delta_-}{d}\, C\, e^{\Delta_+\ u} + \cdots \quad , \quad e^A = e^{u-A_0} + \cdots \quad , \quad u \to -\infty$$

• the solution describes the UV region ($u \to -\infty$) with a perturbation by a relevant operator of dimension $\Delta_+ < d$.

• The source is α. It is not part of W.

• C determines the vev: $\langle O \rangle \sim C\ \alpha^{\frac{\Delta_+}{\Delta_-}}$.

• A single isolated solution W_+

$$W_+ = 2(d-1) + \frac{\Delta_+}{2}\phi^2 + \mathcal{O}(\phi^3) \quad , \quad \Delta_+ > \Delta_-$$

• The associated solution for ϕ, A is

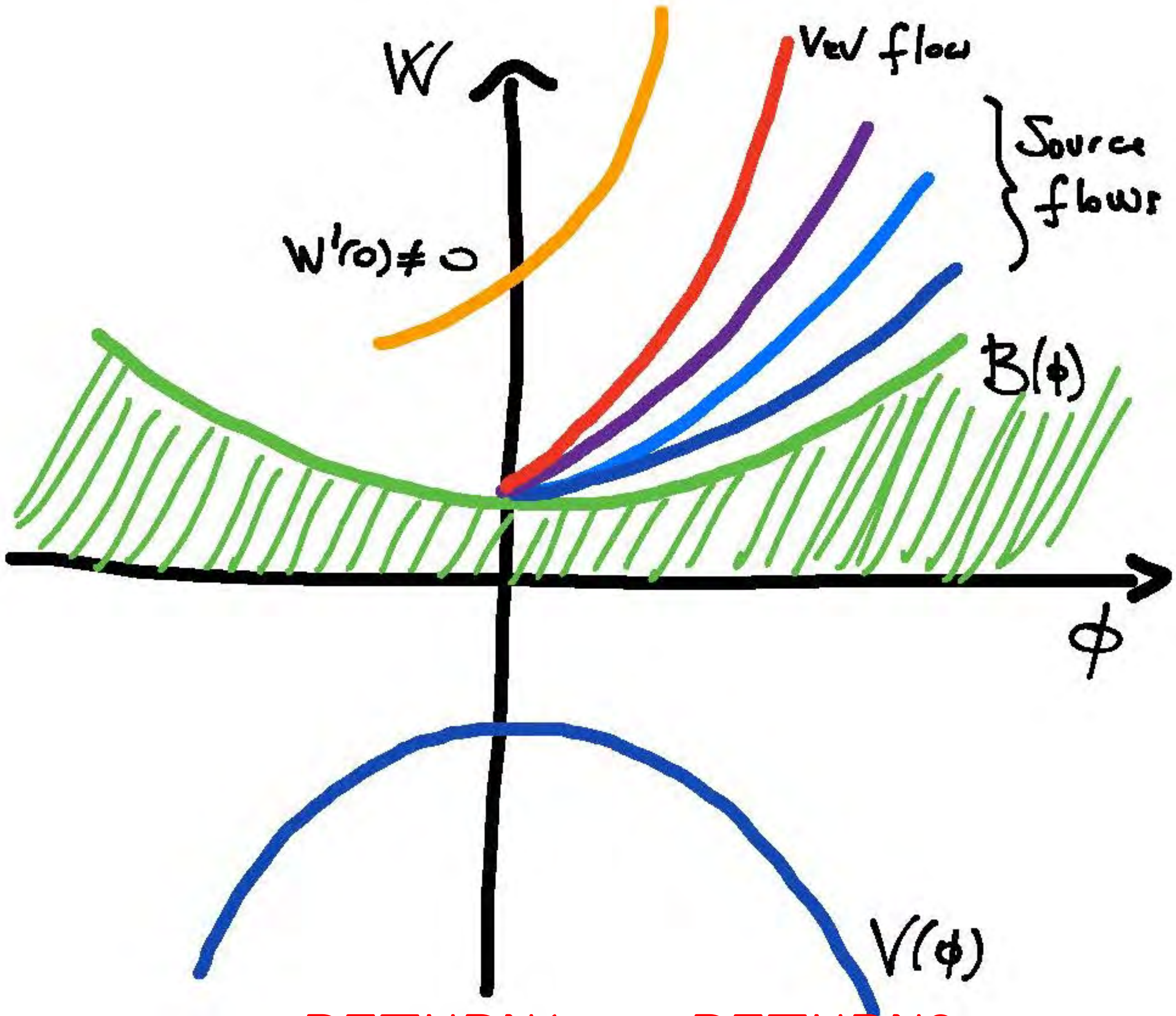
$$\phi(u) = \alpha \; e^{\Delta_+ \; u} + \cdots \quad , \quad e^A = e^{-u+A_0} + \cdots$$

• This is a vev flow ie. the source is zero.

$$\langle O \rangle = (2\Delta_+ - d) \; \alpha$$

• The value of the vev is NOT determined by the superpotential equation. This is a moduli space.

• The whole class of solutions exists both from the left of $\phi = 0$ and from the right.



RETURN1 RETURN2

The minima of V

- We expand the potential near the minimum:

$$V(\phi) = -\frac{1}{\ell^2}\left[d(d-1) - \frac{m^2\ell^2}{2}\phi^2 + \mathcal{O}(\phi^3)\right] \quad , \quad \Delta_\pm = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2\ell^2}$$

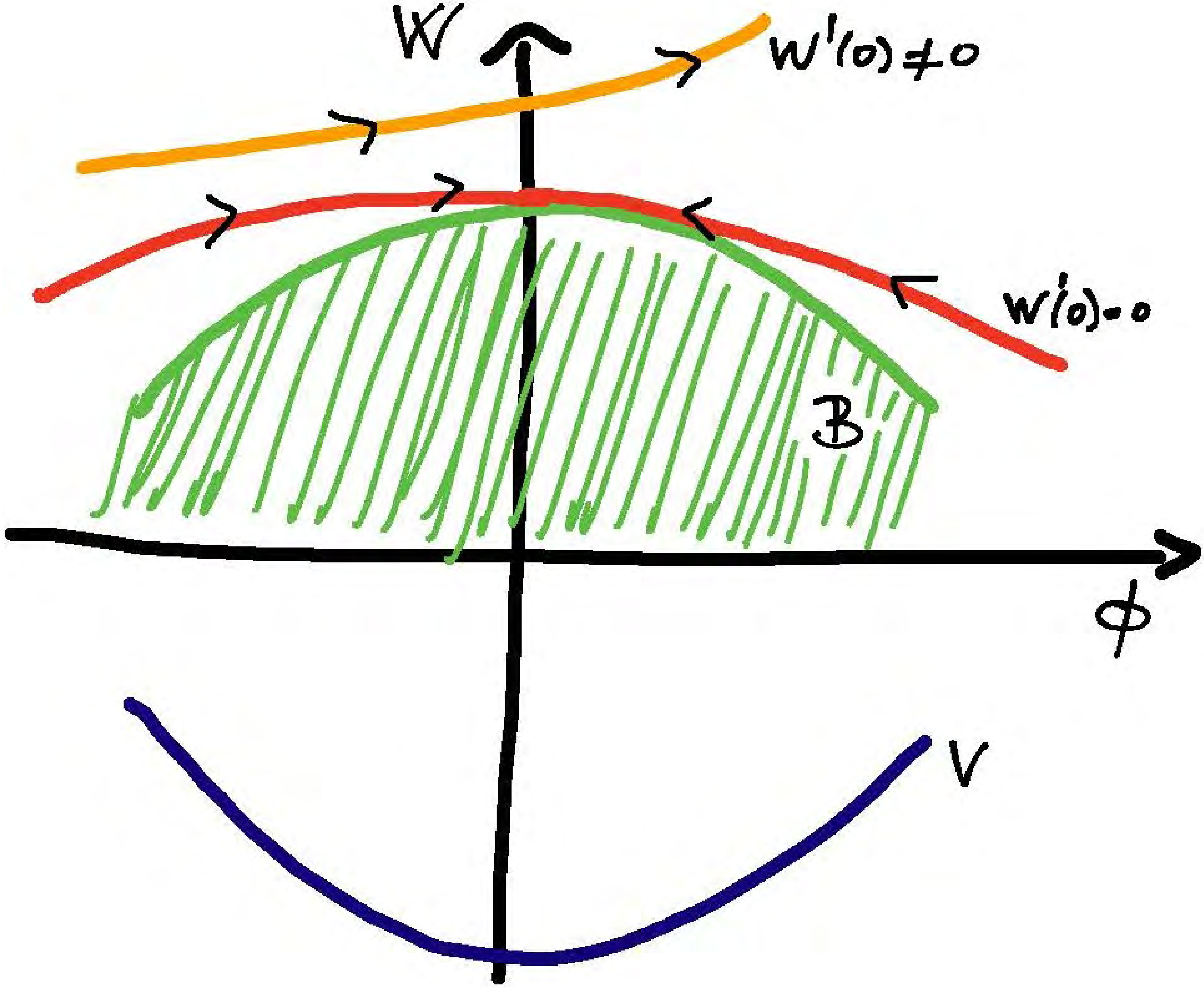
$$m^2 > 0 \quad , \quad \Delta_+ > 0 \quad , \quad \Delta_- < 0$$

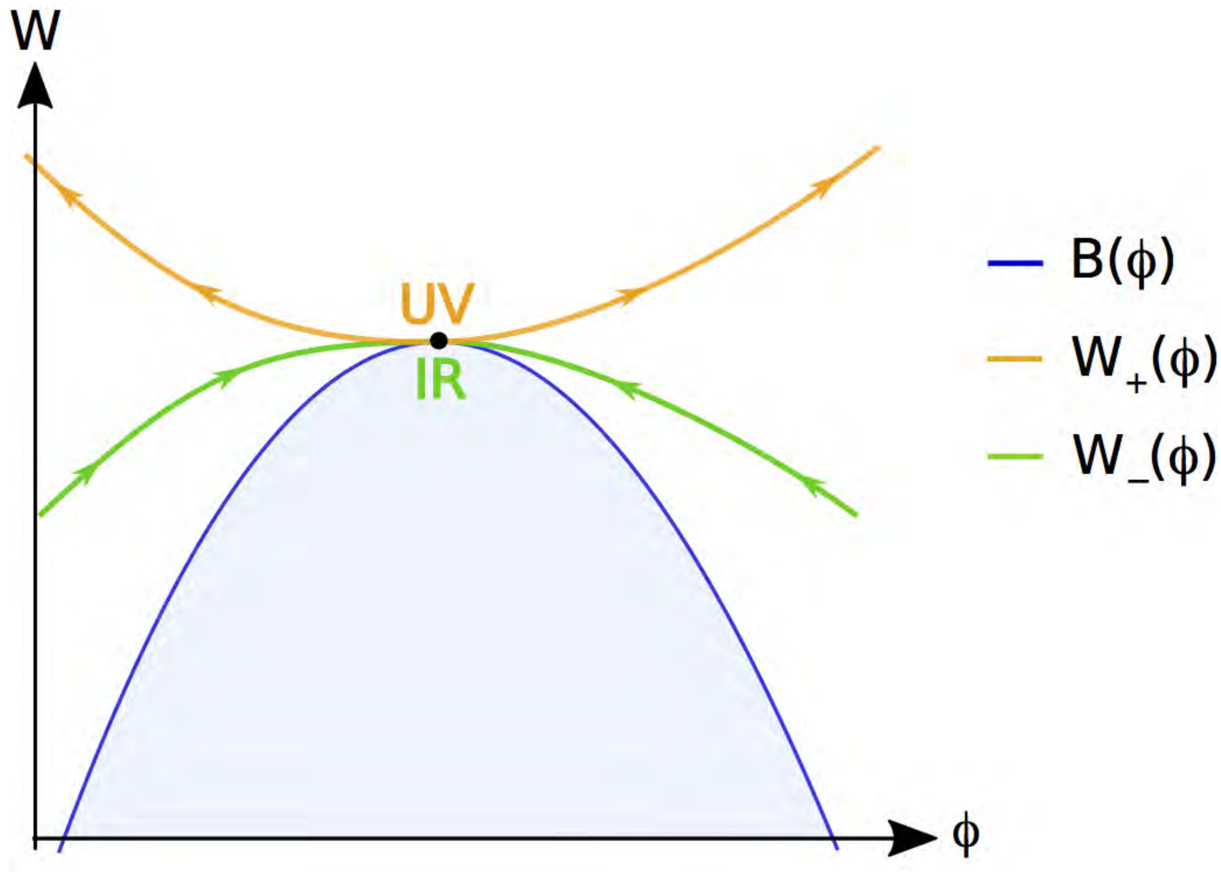
- There are two isolated solutions with $W'(0) = 0$.

$$W_\pm(\phi) = \frac{1}{\ell}\left[2(d-1) + \frac{\Delta_\pm}{2}\phi^2 + \mathcal{O}(\phi^3)\right],$$

- No continuous parameter here as it generates a singularity.

- Although the solutions look similar, their interpretation is very different. W_+ has a local minimum while W_- has a local maximum.


W
W'(0)≠0
W(0)=0
B
φ
V



• There is again a moduli space.

♠ A W_+ solution is globally regular only in special cases.

♠ Therefore a minimum of the potential can be either an IR fixed point or a UV fixed point.

The maxima of V

- We will examine solutions for W near a maximum of V.

- We put the maximum at $\phi = 0$.

- When $V'(0) = 0$, $W''(0)$ is finite.

$$V(\phi) = -\frac{1}{\ell^2}\left[d(d-1) - \frac{m^2\ell^2}{2}\phi^2 + \mathcal{O}(\phi^3)\right]$$

$$\Delta_{\pm} = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2\ell^2} \quad , \quad m^2\ell^2 \quad < 0 \quad , \quad \Delta_+ \geq \Delta_- \geq 0$$

- We set (locally) $\ell = 1$ from now on.

- If $W'(0) \neq 0$ there is one solution (per branch) off the critical curve,

- If $W'(0) = 0$ there are two classes of solutions:

• A continuous family of solutions (the W_- family)

$$W_- = 2(d-1) + \frac{\Delta_-}{2}\phi^2 + \cdots + C\phi^{\frac{d}{\Delta_-}}[1 + \cdots] + \mathcal{O}(C^2)$$

• The solution for ϕ and A corresponding to this, is the standard UV source flow:

$$\phi(u) = \alpha\, e^{\Delta_- \; u} + \cdots + \frac{\Delta_-}{d}\, C\, e^{\Delta_+ \; u} + \cdots \quad , \quad e^A = e^{u - A_0} + \cdots \quad , \quad u \to -\infty$$

• the solution describes the UV region ($u \to -\infty$) with a perturbation by a relevant operator of dimension $\Delta_+ < d$.

• The source is α. It is not part of W.

• C determines the vev: $\langle O \rangle \sim C\ \alpha^{\frac{\Delta_+}{\Delta_-}}$.

• The near-boundary AdS is an attractor of all these solutions.

• A single isolated solution W_+ also arriving at $W(0) = B(0)$

$$W_+ = 2(d-1) + \frac{\Delta_+}{2}\phi^2 + \mathcal{O}(\phi^3) \quad , \quad \Delta_+ > \Delta_-$$

• Always $W''_+ > W''_-$.

• The associated solution for ϕ, A is

$$\phi(u) = \alpha \; e^{\Delta_+ \; u} + \cdots \quad , \quad e^A = e^{-u+A_0} + \cdots$$

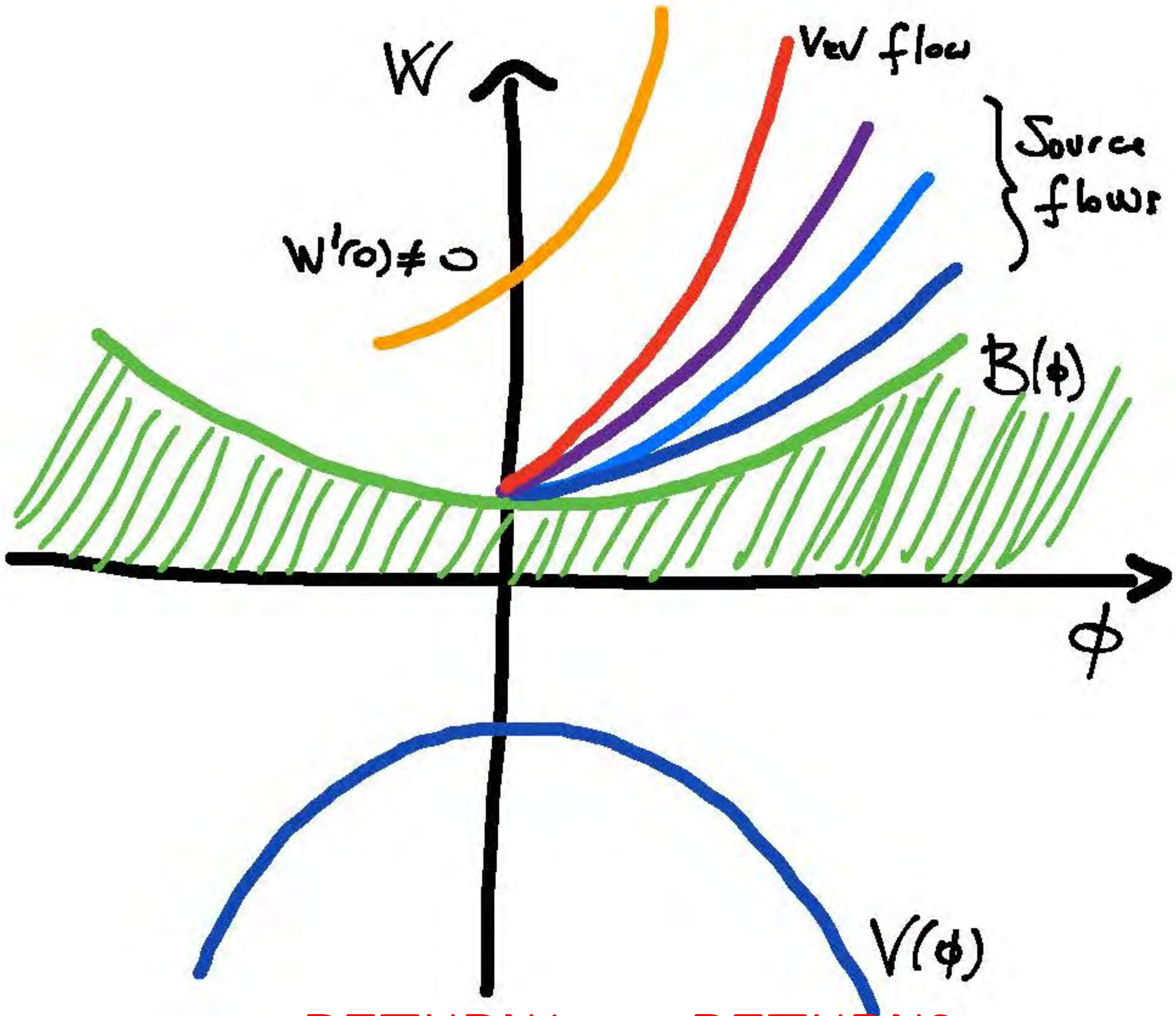
• This is a vev flow ie. the source is zero.

$$\langle O \rangle = (2\Delta_+ - d) \; \alpha$$

• The value of the vev is NOT determined by the superpotential equation.

• It can be reached in a appropriately defined limit $C \to \infty$ of the W_- family.

• The whole class of solutions exists both from the left of $\phi = 0$ and from the right.



RETURN1 RETURN2

The minima of V

- We expand the potential near the minimum:

$$V(\phi) = -\frac{1}{\ell^2}\left[d(d-1) - \frac{m^2\ell^2}{2}\phi^2 + \mathcal{O}(\phi^3)\right] \quad , \quad \Delta_\pm = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2\ell^2}$$

$$m^2 > 0 \quad , \quad \Delta_+ > 0 \quad , \quad \Delta_- < 0$$

- There are solutions with $W'(0) \neq 0$. These are solutions that do not stop at the minimum.

- There are two isolated solutions with $W'(0) = 0$.

$$W_\pm(\phi) = \frac{1}{\ell}\left[2(d-1) + \frac{\Delta_\pm}{2}\phi^2 + \mathcal{O}(\phi^3)\right],$$

- No continuous parameter here as it generates a singularity.

- Although the solutions look similar, their interpretation is very different. W_+ has a local minimum while W_- has a local maximum.

• The W_- solution:

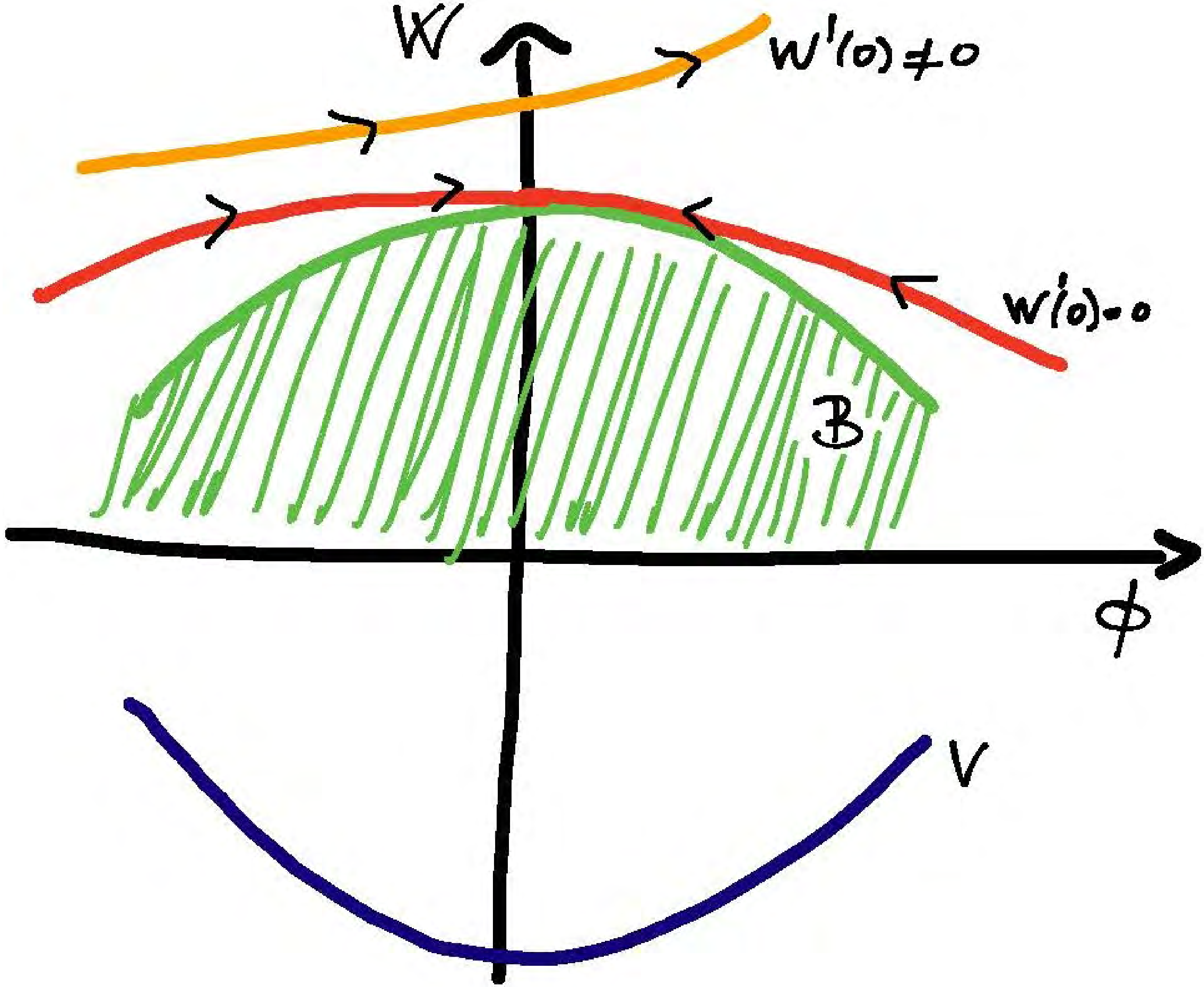
$$\phi(u) = \alpha\ e^{\Delta_-\ u} + \cdots \quad , \quad e^A = e^{-(u-u_0)} + \cdots .$$

• Since $\Delta_- < 0$, small ϕ corresponds to $u \to +\infty$ and $e^A \to 0$.

• This signal we are in the deep interior (IR) of AdS.

• The driving operator has (IR) dimension $\Delta_+ > d$ and a zero vev in the IR.

• Therefore W_- generates locally a flow that arrives at an IR fixed point.


W
W'(0) ≠ 0
W(0) = 0
B
φ
V

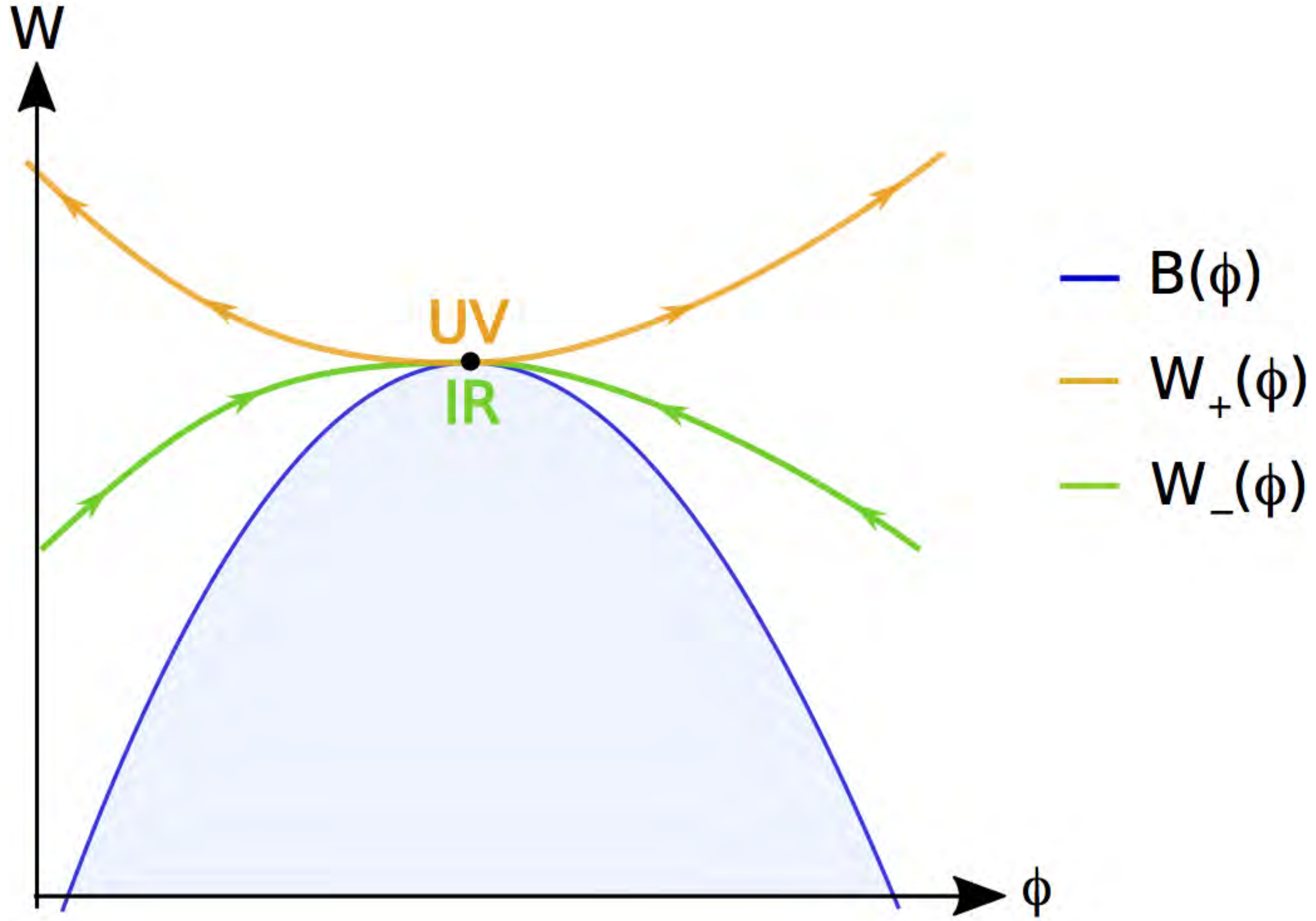
• The W_+ solution is:

$$\phi(u) = \alpha\ e^{\Delta_+\ u} + \cdots \quad , \quad e^A = e^{-(u-u_0)} + \cdots .$$

• Since $\Delta_+ > 0$ small ϕ corresponds to $u \to -\infty$ and $e^A \to +\infty$.

• This solution described the near-boundary (UV) region of a fixed point.

• This solution is driven by the vev of an operator with (UV) dimension $\Delta_+ > d$ (irrelevant).



♠ A minimum of the potential can be either an IR fixed point or a UV fixed point.

The first order formalism

- In this case the two first order flow equations are modified:

$$\dot{A} = -\frac{1}{2(d-1)}W(\phi) \quad , \quad \dot{\phi} = S(\phi)$$

$$\frac{d}{2(d-1)}W^2 + (d-1)S^2 - dSW' = -2V \quad , \quad SS' - \frac{d}{2(d-1)}WS = V'$$

- The two superpotential equations have two integration constants.

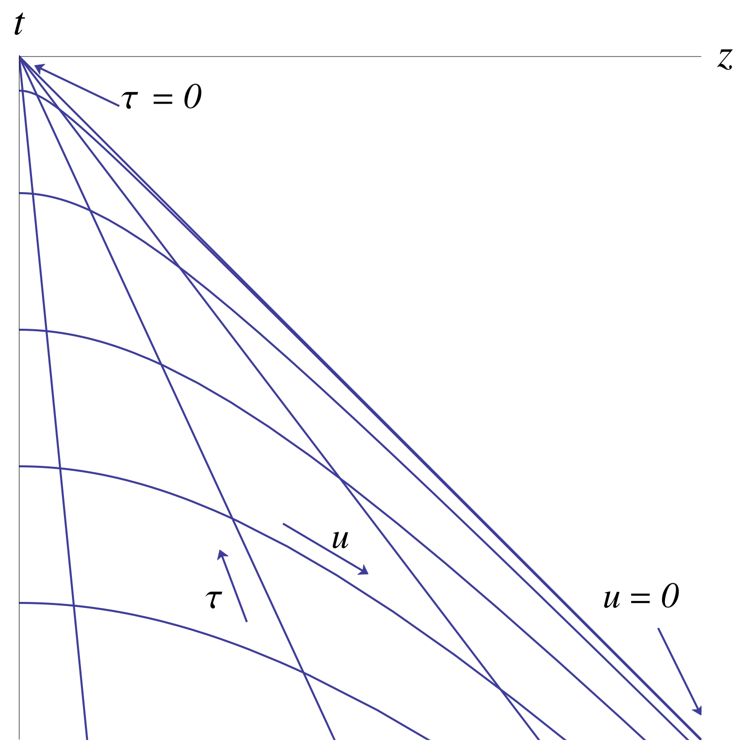
- One of them, C, is the vev of the scalar operator (as usual).

- The other is the dimensionless curvature, $\mathcal{R}$.

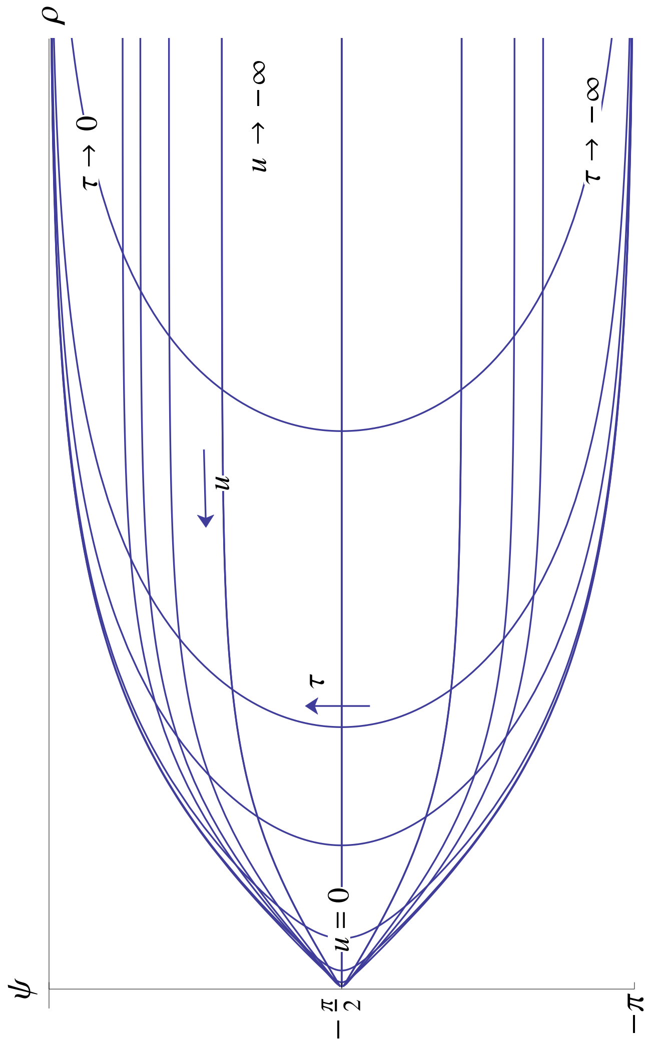
- The structure near an maximum (UV) of the potential has the "resurgent" expansion

$$W(\phi) = \sum_{m,n,r \in Z_0^+} A_{m,n,r} \; (C \; \phi^{\frac{d}{\Delta_-}})^m \; (\mathcal{R} \; \phi^{\frac{2}{\Delta_-}})^n \; \phi^r$$

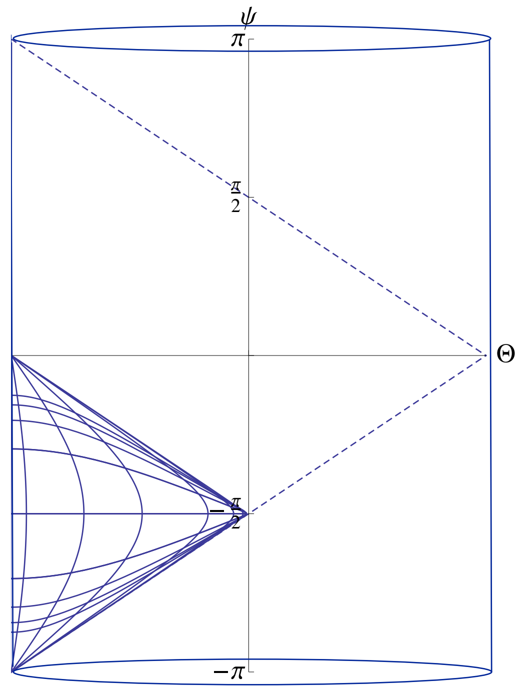
Coordinates



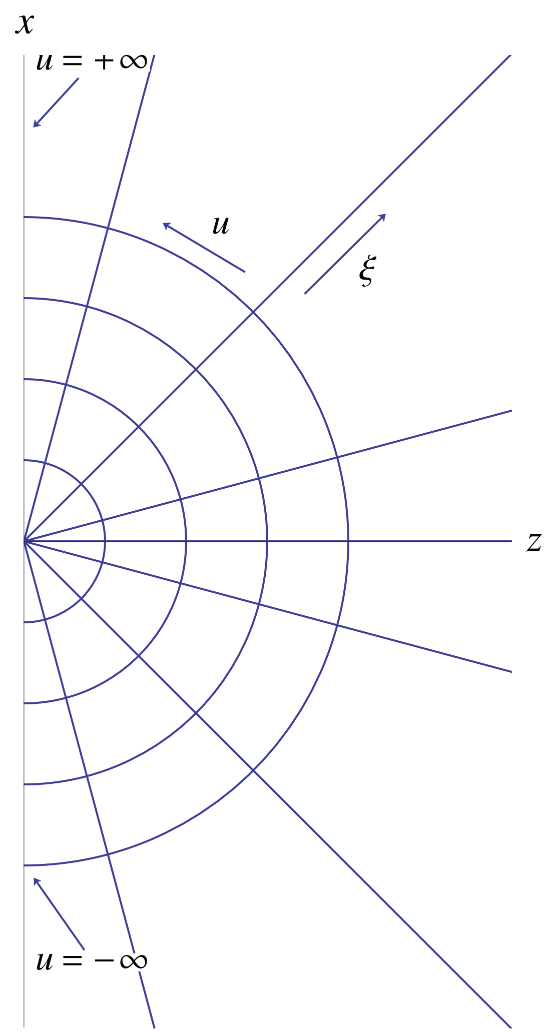
Relation between Poincaré coordinates (t, z) and dS-slicing coordinates (τ, u). Constant u curves are half straight lines all ending at the origin ($\tau \to 0^-$); Constant τ curves are branches of hyperbolas ending at $u = 0$ (null infinity on the $z = -t$ line). The boundary $z = 0$ corresponds to $u \to -\infty$.



Embedding of the dS patch in global coordinates. The flow endpoint $u = 0$ corresponds to the point $\rho = 0, \psi = -\pi/2$ in global coordinates. the AdS boundary is at $\rho = +\infty$ and it is reached along u as $u \to -\infty$, and along τ both as $\tau \to -\infty$ and as $\tau \to 0$.

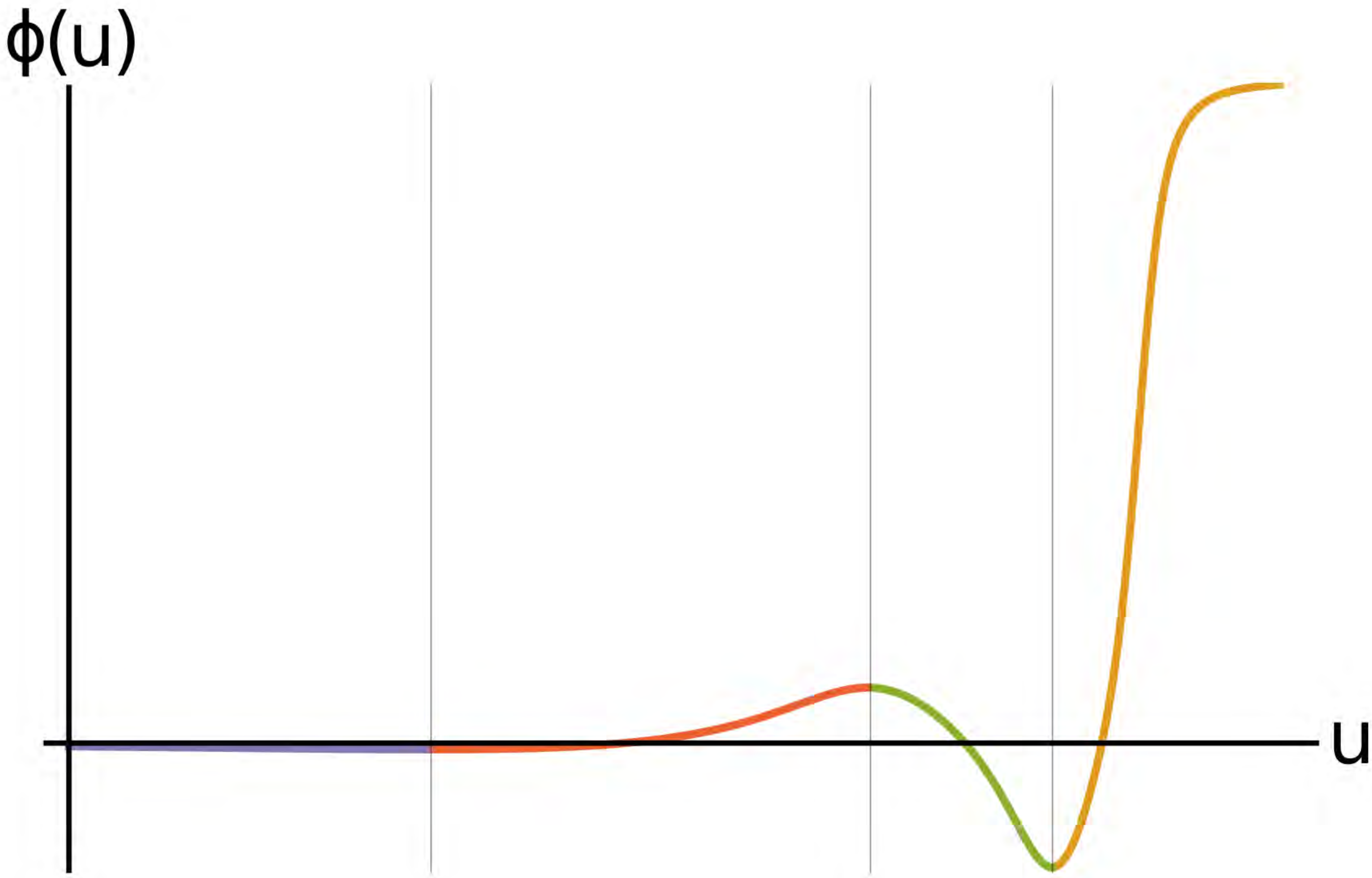


Embedding of the dS patch in global conformal coordinates, $\tan\Theta = \sinh\rho$, where each point is a $d-1$ sphere "filled" by Θ. The boundary is at $\Theta = \pi/2$. The dashed lines correspond to the Poincaré patch embedded in global conformal coordinates. The flow endpoint $u = 0$ is situated on the Poincaré horizon.



Relation between Poincaré coordinates (x, z) and AdS-slicing coordinates (ξ, u). Constant u curves are half straight lines all ending at the origin ($\xi \to 0^-$); Constant ξ curves are semicircle joining the two halves of the boundary at $u = \pm\infty$.

Bounces




A(u)
u


UV
IR

Curtright, Jin and Zachos gave an example of an RG Flow that is cyclic but respects the strong C-theorem

$$\beta_n(\phi) = (-1)^n \sqrt{1-\phi^2} \quad \to \quad \phi(A) = sin(A)$$

If we define the superpotential branches by $\beta_n = -2(d-1)W_n'/W_n$ we obtain

$$\log W_n = \frac{(2n+1)\pi + 2(-1)^n(\arcsin(\phi) + \phi\sqrt{1-\phi^2})}{8(d-1)}$$

and we can compute the potentials from $V = W'^2/2 - dW^2/4(d-1)$ to obtain $V_n(\phi)$.

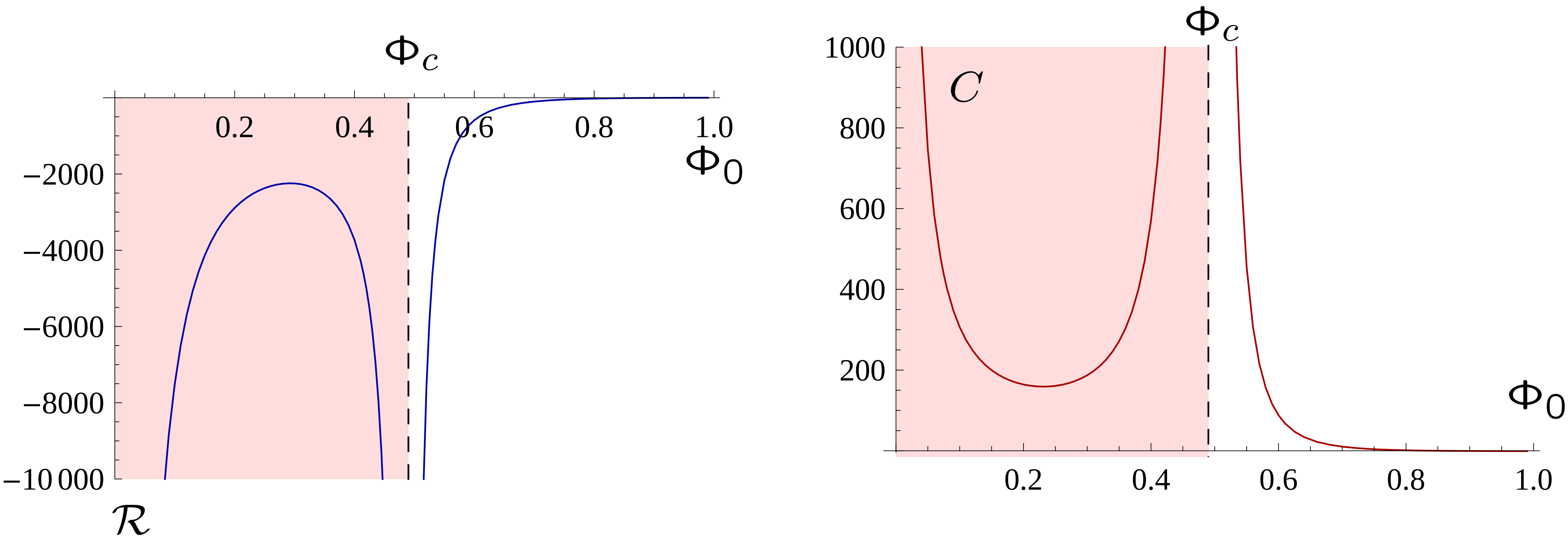
Such piece-wise potentials then satisfy

$$V_{n+2}(\phi) = e^{\frac{\pi}{2(d-1)}}\ V_n(\phi)$$

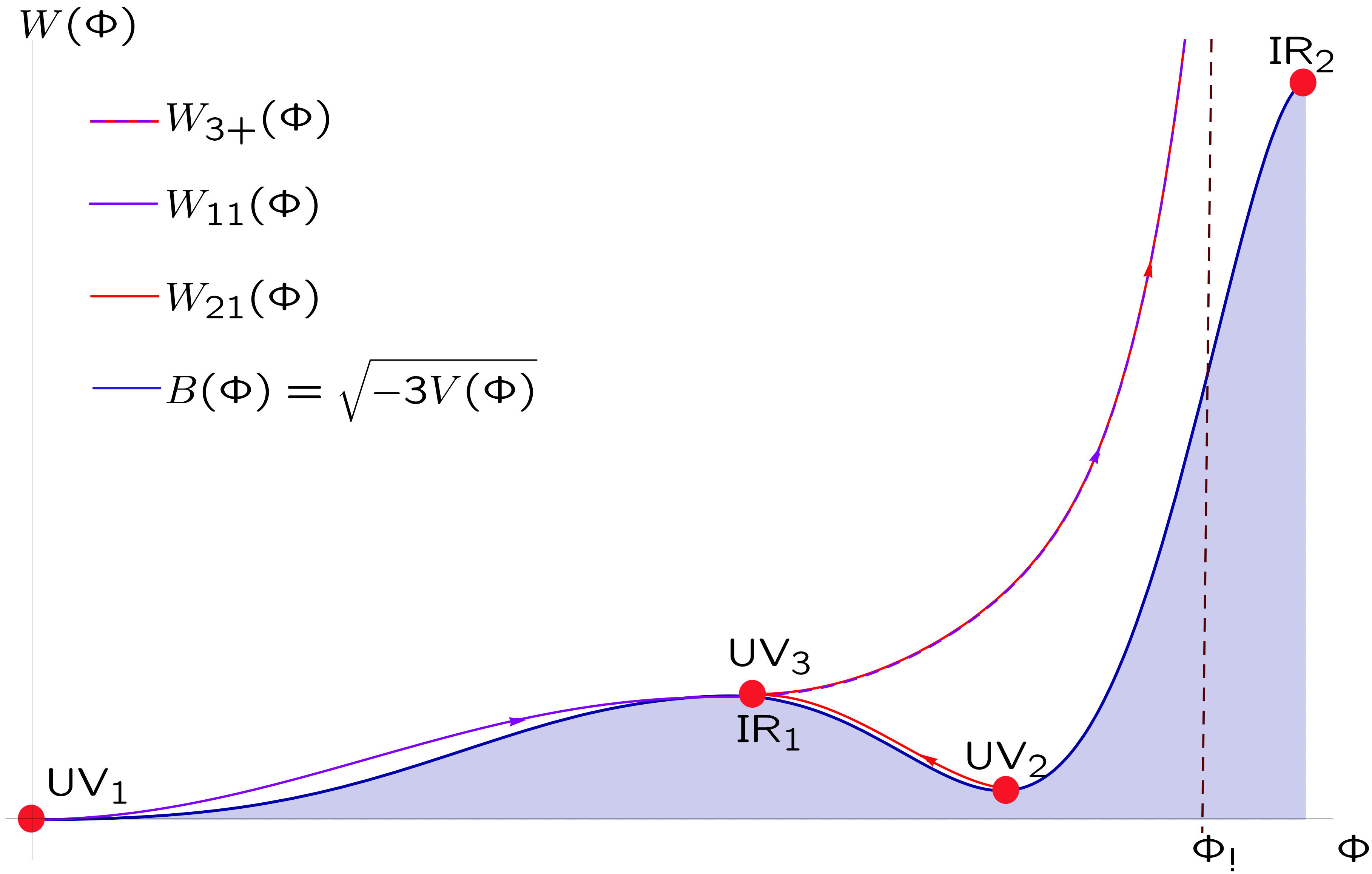
- No such potentials can arise in string theory.

- Holography can provide only “approximate" cycles.

Flows in AdS

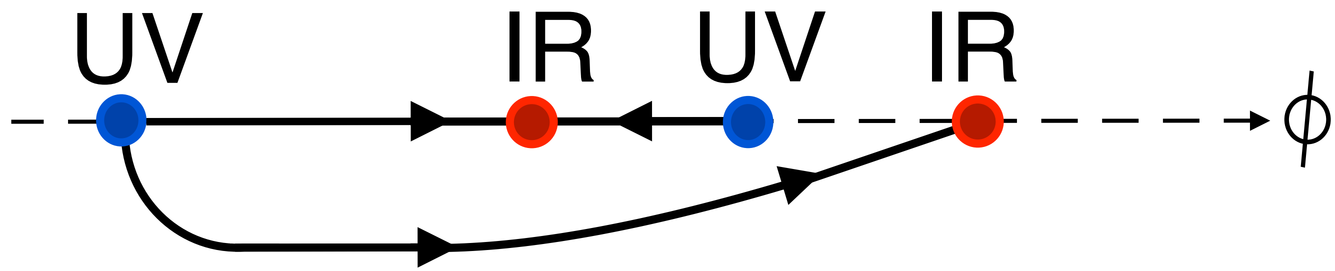


QFT on AdS_d: dimensionless curvature $\mathcal{R} = R^{(uv)}|\Phi_-|^{-2/\Delta_-}$ and dimensionless vev $C = \frac{\Delta_-}{d}\langle\mathcal{O}\rangle|\Phi_-|^{-\Delta_+/\Delta_-}$ vs. Φ_0 for the Mexican hat potential with $\Delta_- = 1.2$. Flows with turning points in the rose-colored region leave the UV fixed point at $\Phi = 0$ to the left before bouncing and finally ending at positive Φ_0. Flows with turning points in the white region are direct: They leave the UV fixed point at $\Phi = 0$ to the right and do not exhibit a reversal of direction. The flow with turning point Φ_c on the border between the bouncing/non-bouncing regime corresponds to a theory with vanishing source Φ_-. As a result, both $\mathcal{R}$ and C diverge at this point.



$W(\Phi)$
$W_{3+}(\Phi)$
$W_{11}(\Phi)$
$W_{21}(\Phi)$
$B(\Phi) = \sqrt{-3V(\Phi)}$
IR$_2$
UV$_3$
IR$_1$
UV$_2$
UV$_1$
$\Phi_!$
Φ

RG flows with IR endpoint $\Phi_0 \to \Phi_!$. When the endpoint Φ_0 approaches $\Phi_!$ flows from both UV$_1$ and UV$_2$ pass by closely to IR$_1$, passing through IR$_1$ exactly for $\Phi_0 = \Phi_!$. This is shown by the purple and red curves. Beyond IR$_1$ both these solutions coincide, which is denoted by the colored dashed curve. These have the following interpretation. The flows from UV$_1$ and UV$_2$ should not be continued beyond IR$_1$, which becomes the IR endpoint for the zero curvature flows W_{11} and W_{21}. The remaining branch (the colored dashed curve) is now an independent flow denoted by W_{3+}. This is a flow from a UV fixed point at a minimum of the potential (denoted by UV$_3$ above) to $\Phi_!$ and corresponds to a W_+ solution with fixed value $\mathcal{R} = R^{\text{uv}}|\Phi_+|^{-2/\Delta_+} \neq 0$. While flows from UV$_1$ and UV$_2$ can end arbitrarily close to $\Phi_!$, the endpoint $\Phi_0 = \Phi_!$ cannot be reached from UV$_1$ or UV$_2$.

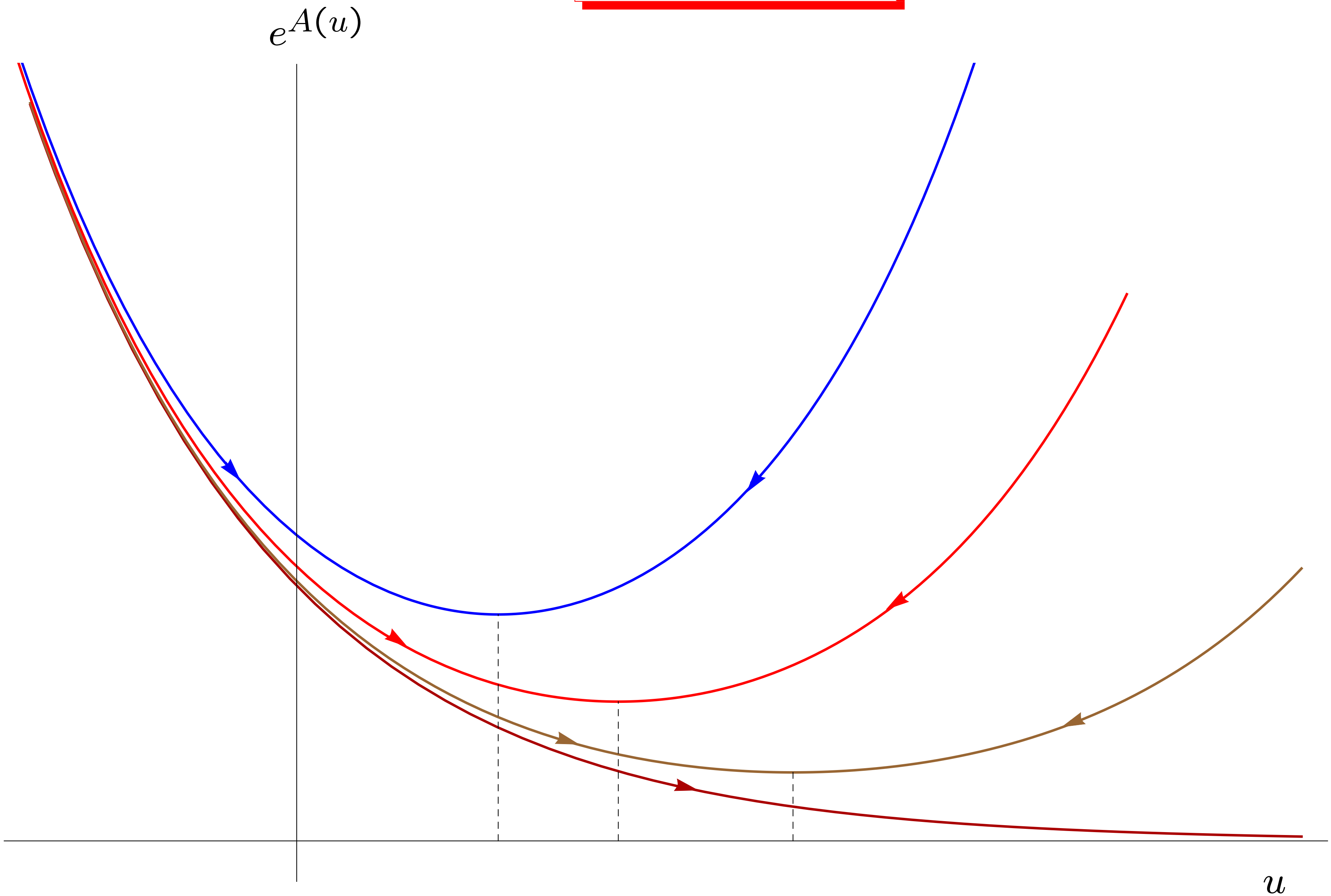


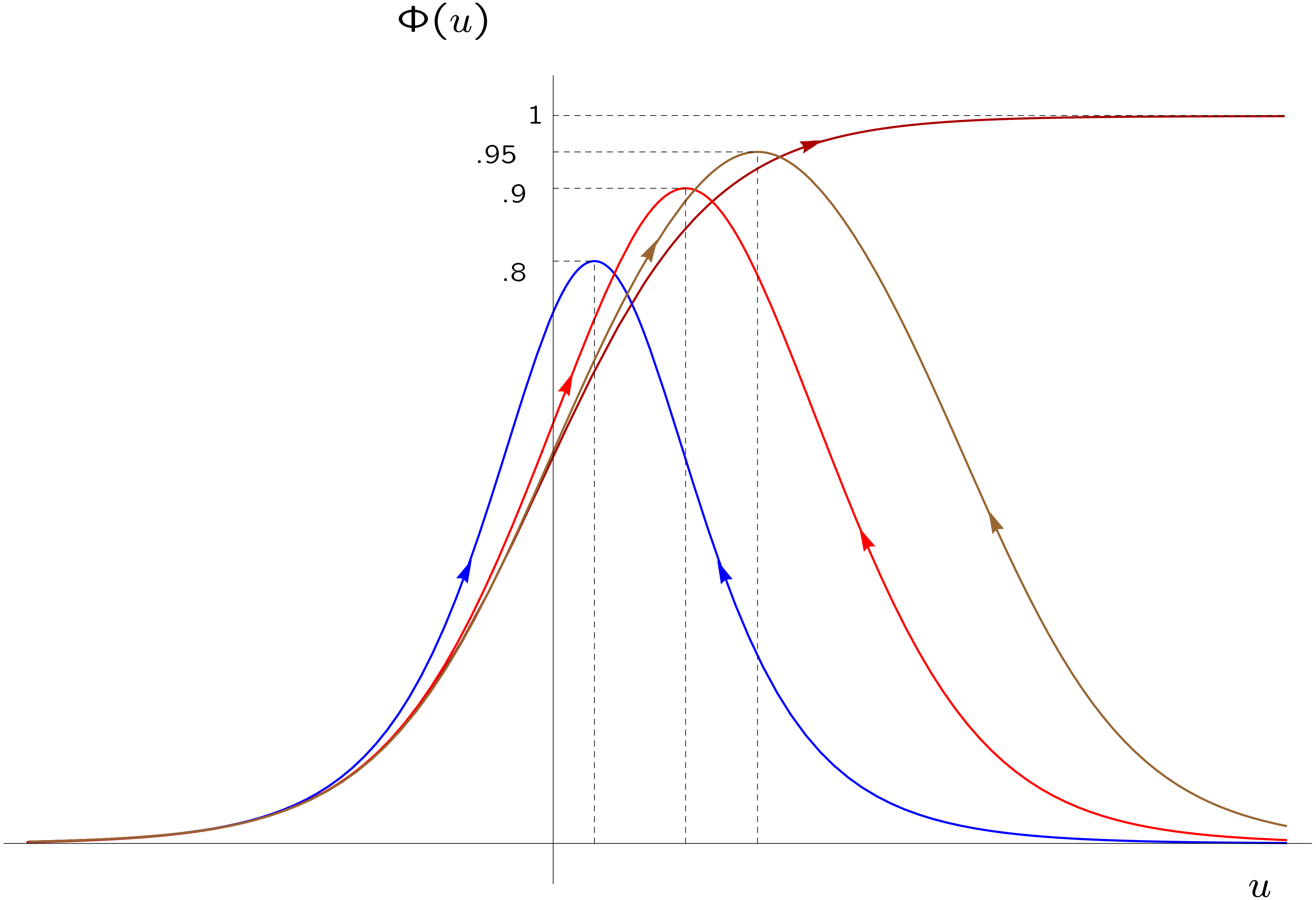
• It is not possible in this example to redefine the topology on the line so that the flow looks “normal"

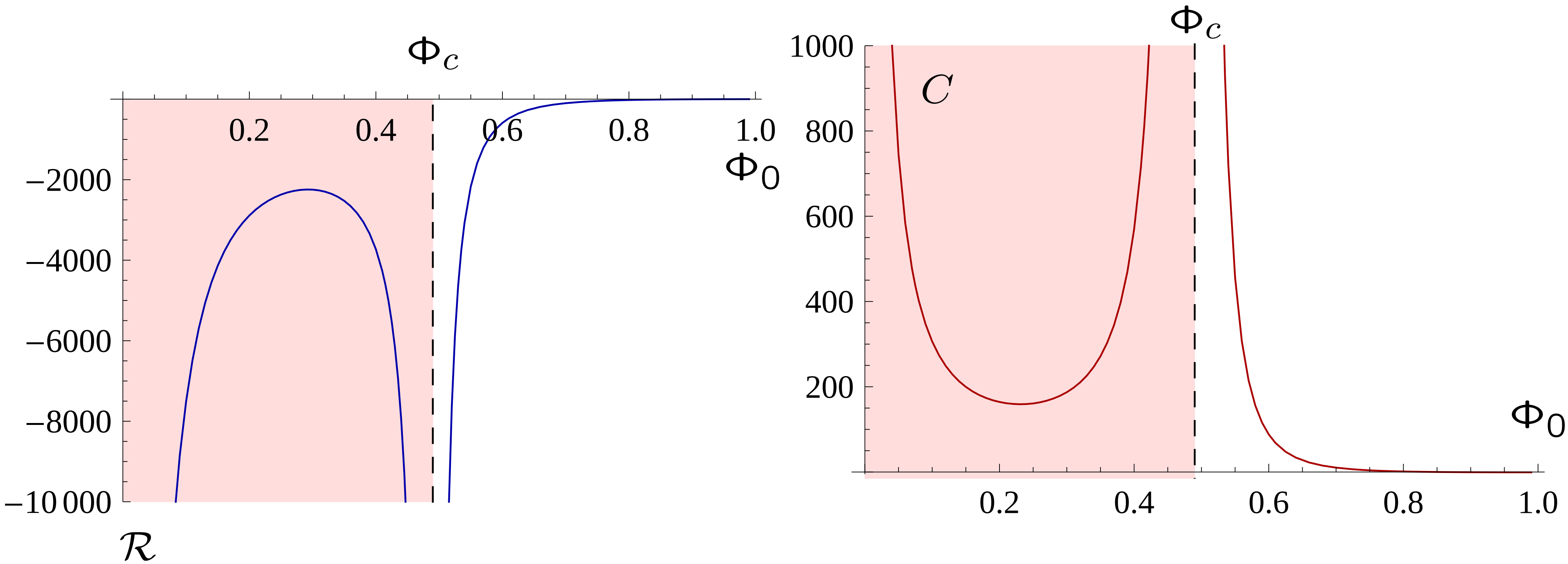
• The two flows $UV_1 \to IR_1$ and $UV_1 \to IR_2$ correspond to the same source but different vev’s.

• One can calculate the free-energy difference of these two flows: the one that arrives at the IR fixed point with lowest a, is the dominant one.

AdS flows




$\Phi(u)$
1
.95
.9
.8
u



Φ_c
0.2
0.4
0.6
0.8
1.0
Φ_0
−2000
−4000
−6000
−8000
−10 000
$\mathcal{R}$
Φ_c
1000
800
600
400
200
C
Φ_0
0.2
0.4
0.6
0.8
1.0

Renormalization in 3d

$$F_{d=3}(\Lambda,\mathcal{R}) = -(M\ell)^2\Omega_3\left[\mathcal{R}^{-\frac{3}{2}}\left(4\Lambda^3(1+\mathcal{O}(\Lambda^{-2\Delta_-}))+C(\mathcal{R})\right)`+\right.$$

$$\left.+\mathcal{R}^{-\frac{1}{2}}\left(\Lambda(1+\mathcal{O}(\Lambda^{-2\Delta_-}))+B(\mathcal{R}\right)+\cdots\right] \quad , \quad \Lambda\equiv\frac{e^{A(\epsilon)}}{\ell|\phi_0|^{\frac{1}{\Delta_-}}}$$

- $B(\mathcal{R}), C(\mathcal{R})$ are the vevs of O and a (part of a) derivative of the stress tensor.

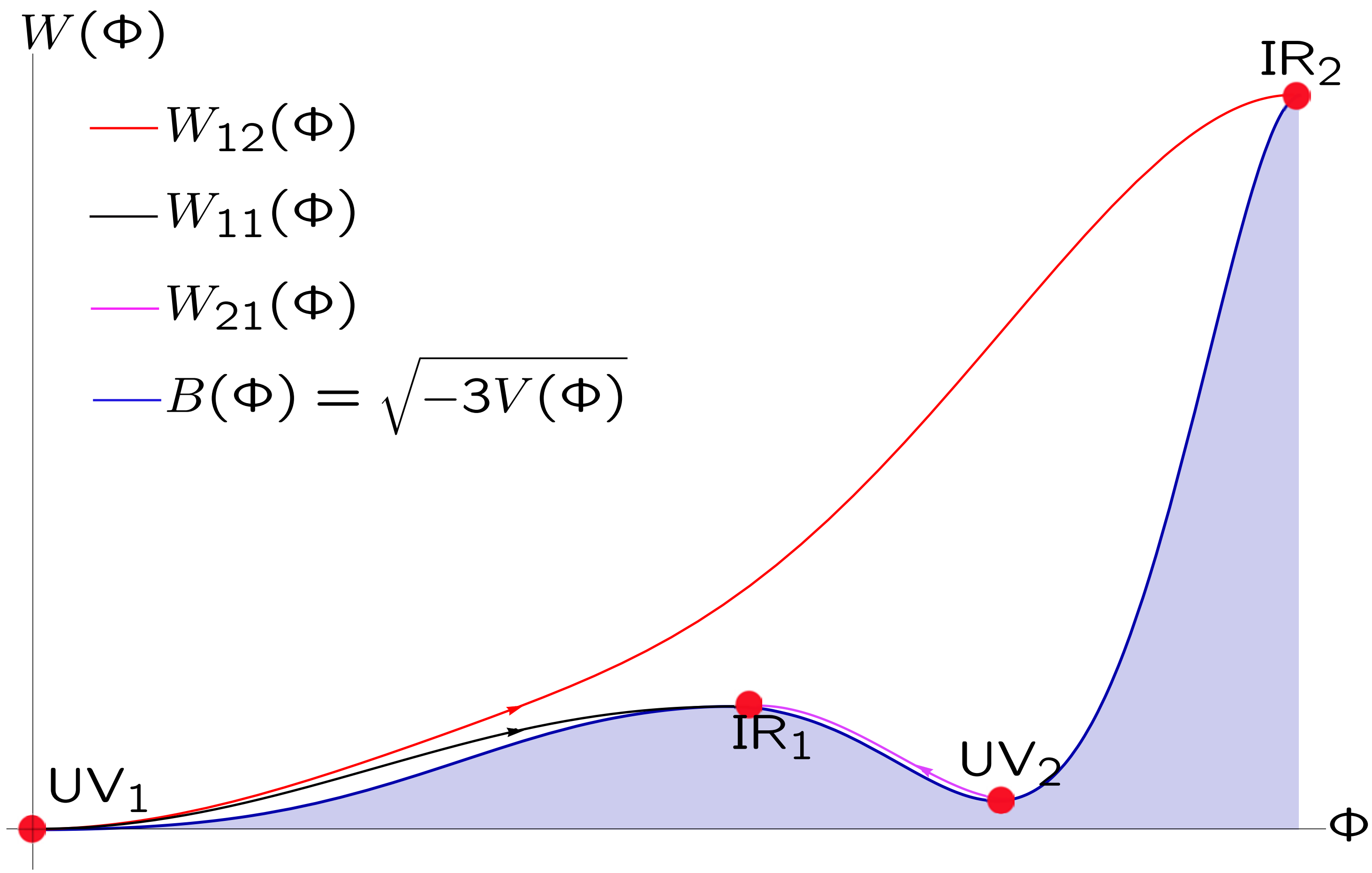
- We renormalize

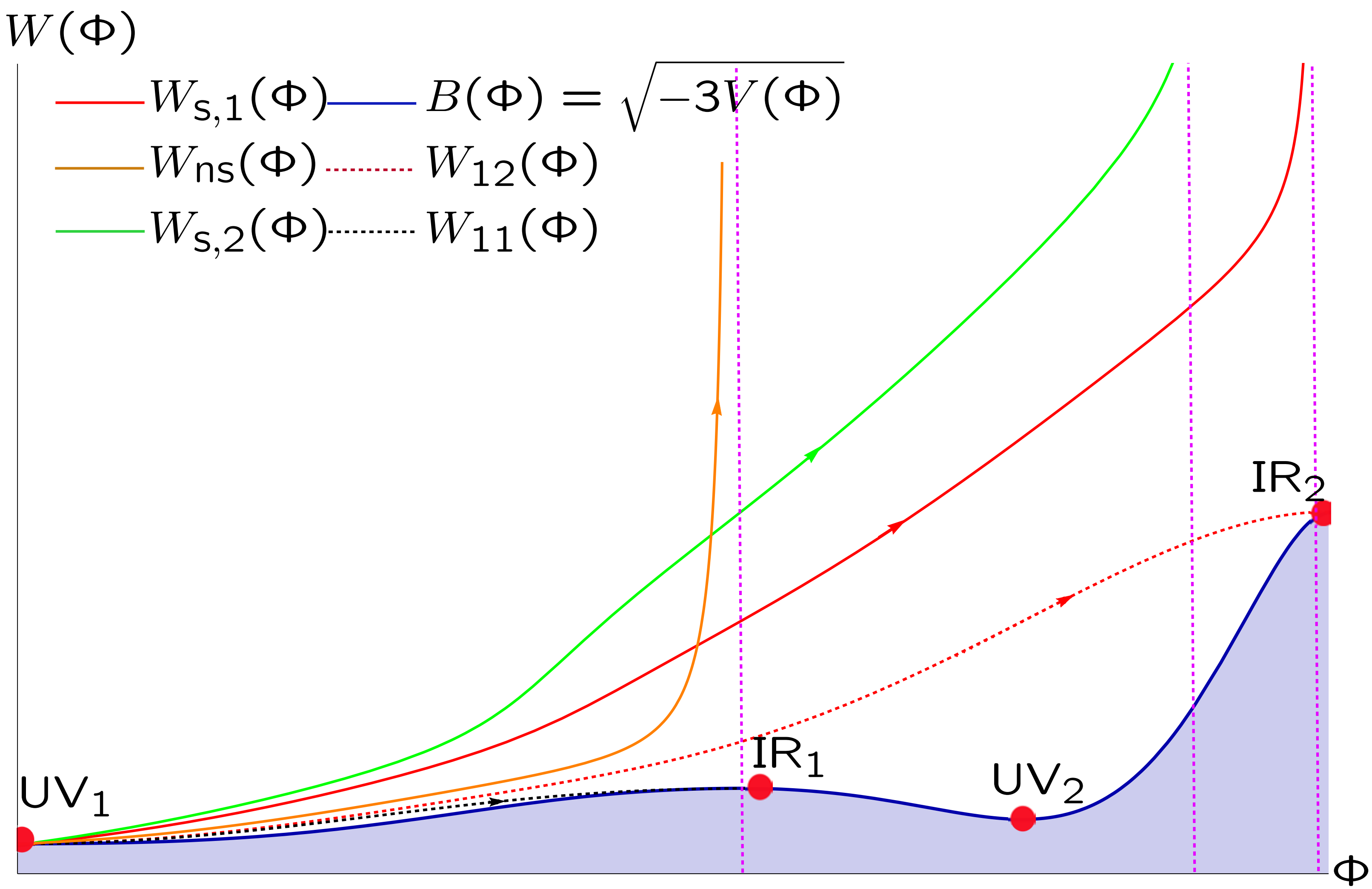
$$F^{\text{renorm}}_{d=3}(\mathcal{R}|B_{ct},C_{ct}) = -(M\ell)^2\Omega_3\left[\mathcal{R}^{-\frac{3}{2}}(C(\mathcal{R})-C_{ct})+\mathcal{R}^{-\frac{1}{2}}(B(\mathcal{R})-B_{ct})\right]$$

- Similarly the renormalized deSitter entanglement entropy is

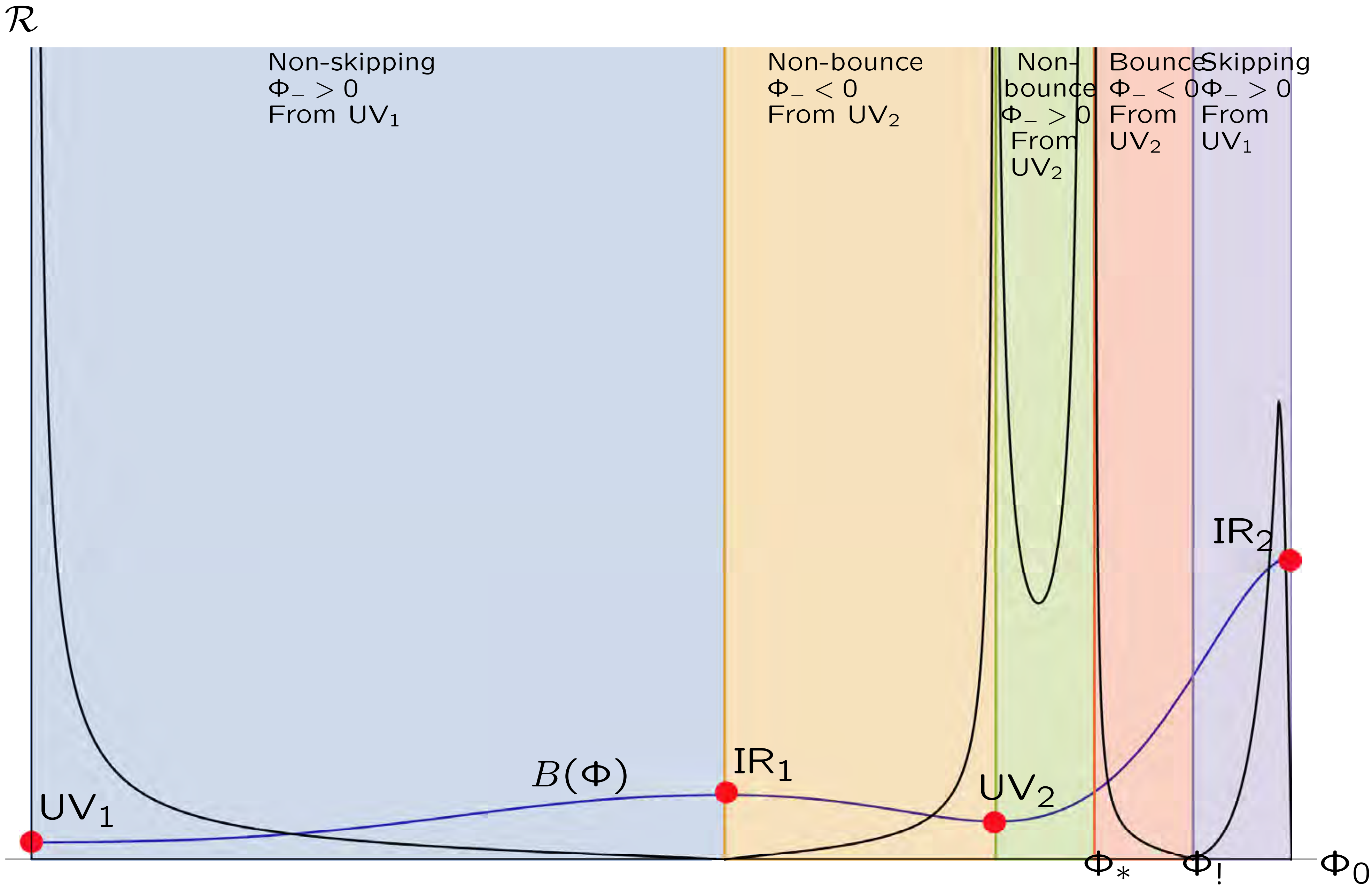
$$S^{\text{renorm}}_{EE}(\mathcal{R}|B_{ct} = (M\ell)^2\Omega_3\mathcal{R}^{-\frac{1}{2}}(B(\mathcal{R})-B_{ct})$$

Skipping flows at finite curvature

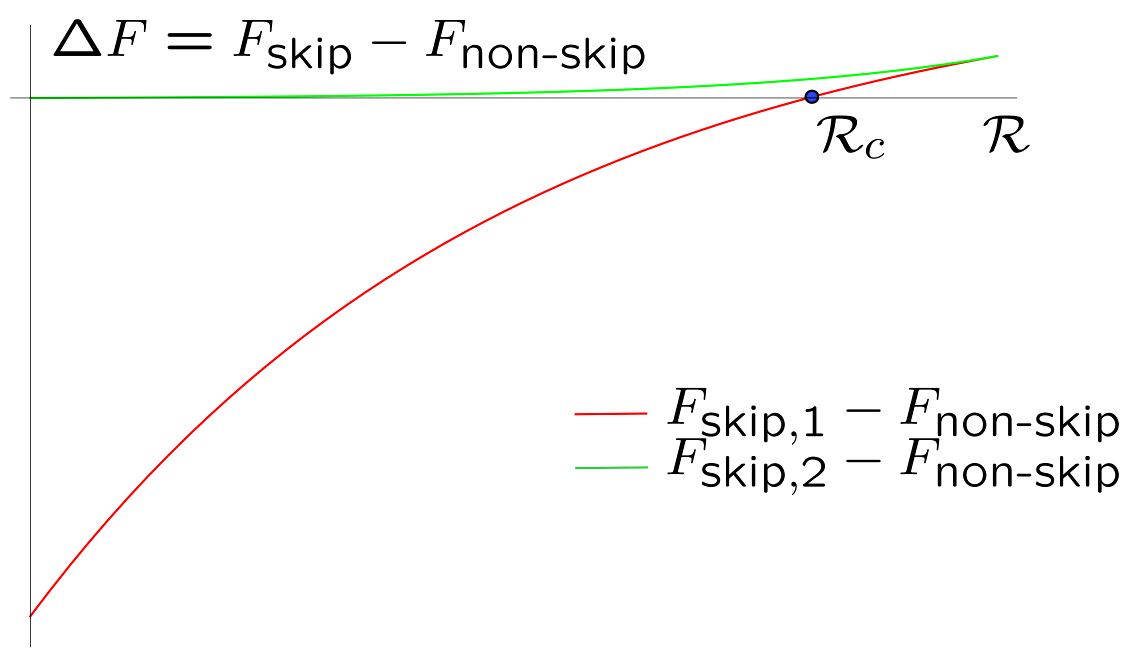




The solid lines represent the superpotential $W(\Phi)$ corresponding to the three different solutions starting from UV_1 which exist at small positive curvature. Two of them (red and green curves) are skipping flows and the third one (orange curve) is non-skipping. For comparison, we also show the flat RG flows (dashed curves)


$\mathcal{R}$
Non-skipping $\Phi_- > 0$ From UV_1
Non-bounce $\Phi_- < 0$ From UV_2
Non-bounce $\Phi_- > 0$ From UV_2
Bounce $\Phi_- < 0$ From UV_2
Skipping $\Phi_- > 0$ From UV_1
IR_2
$B(\Phi)$
IR_1
UV_2
UV_1
Φ_*
$\Phi_!$
Φ_0

A quantum phase transition for UV_1

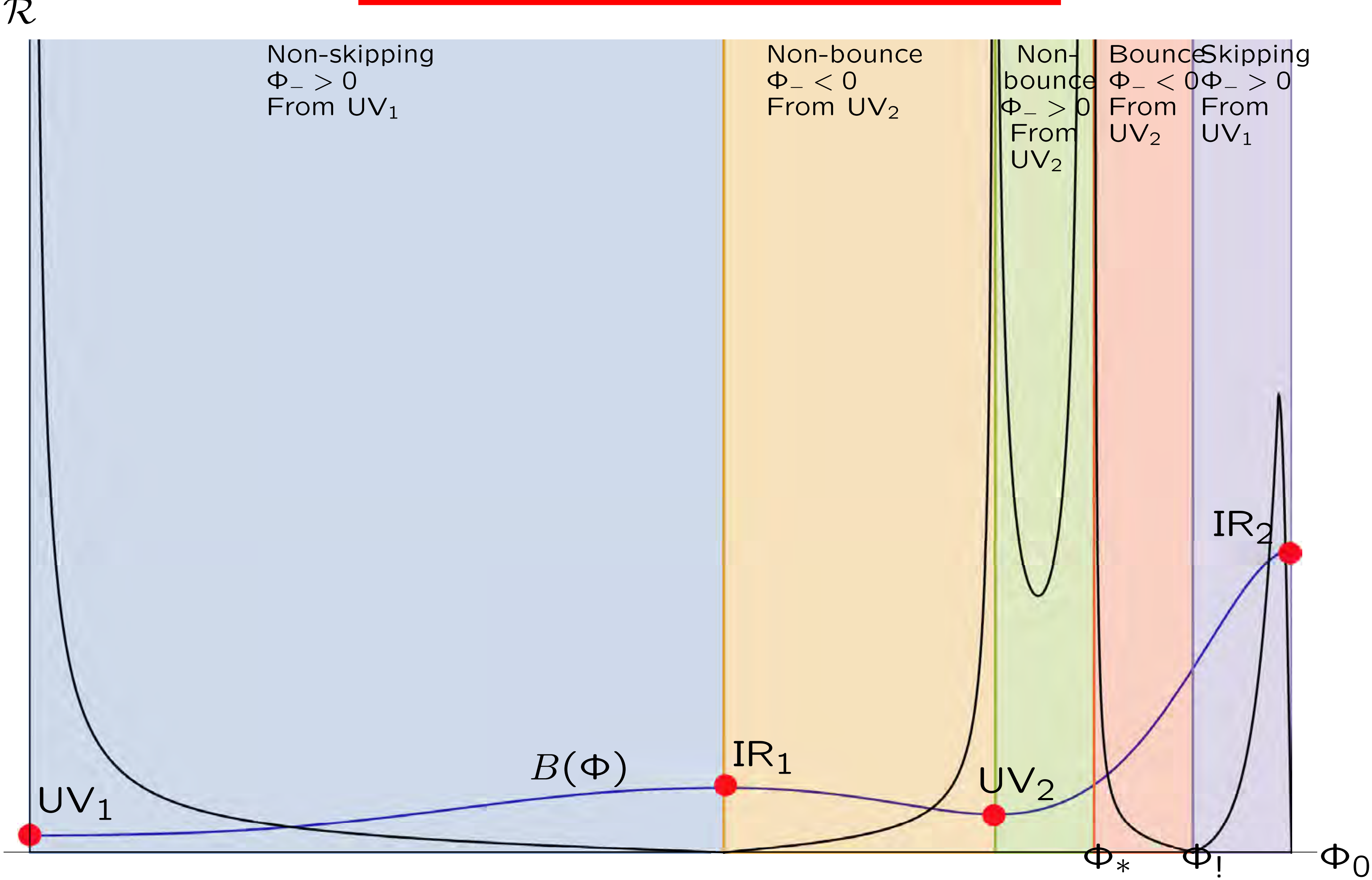


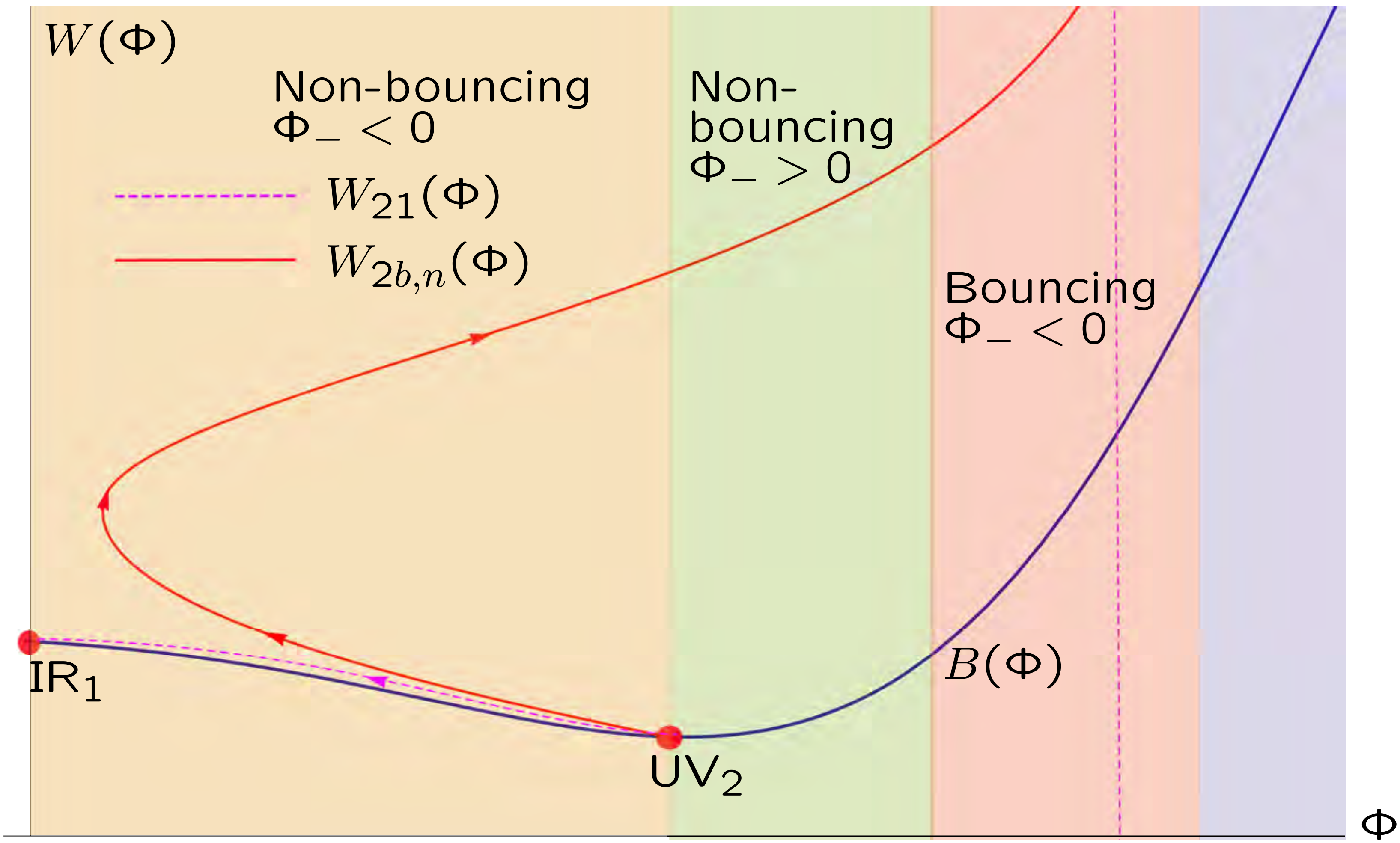
• Free energy difference between the skipping and the non-skipping solution.

• The red curve corresponds to the on-shell action difference between the $W_{s,1}(\Phi)$ solution and the non-skipping solution.

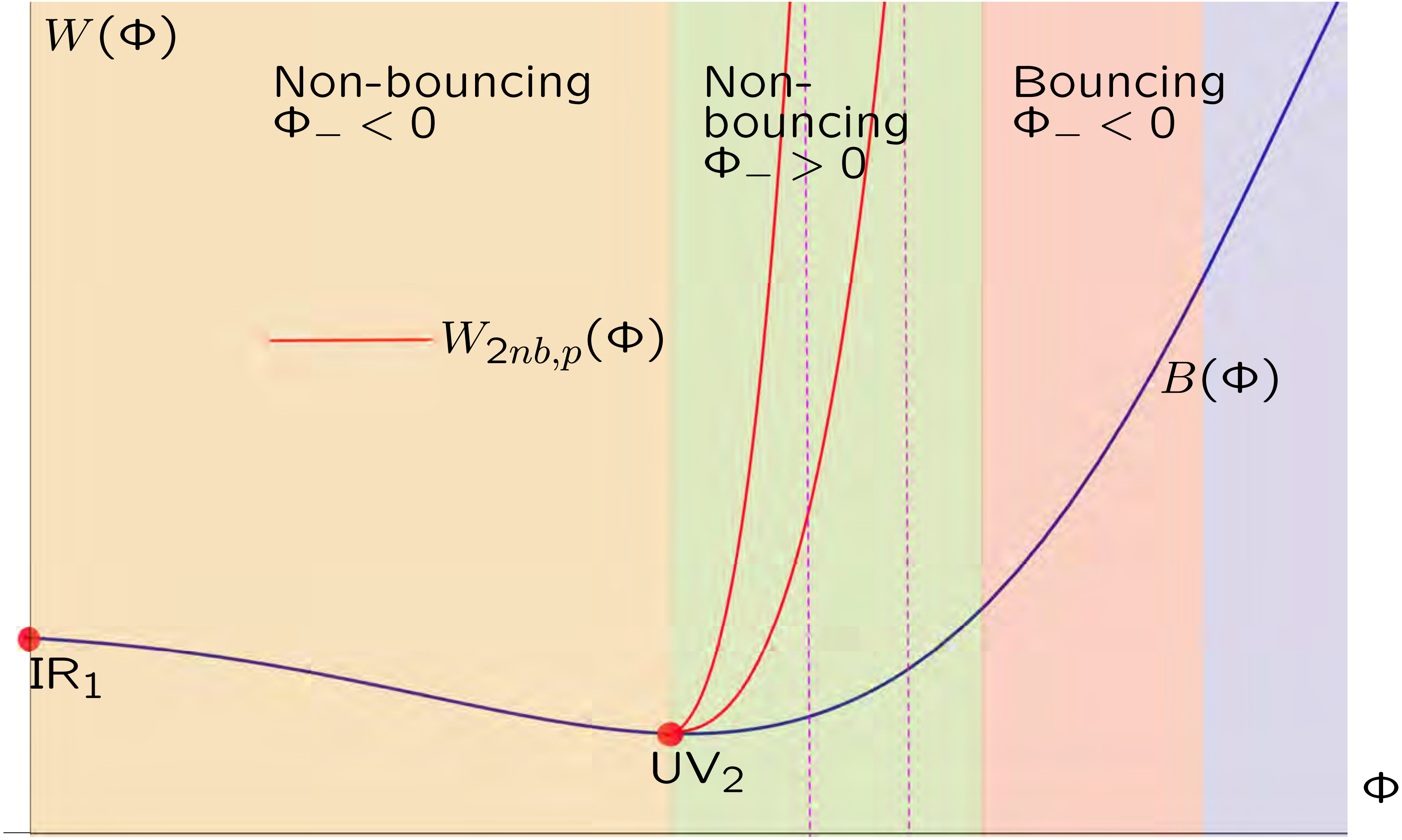
• The green curve corresponds to the on-shell action difference between the $W_{s,2}(\Phi)$ solution and the non-skipping solution $W_{ns}(\Phi)$.

The RG flows from UV_2





$W(\Phi)$
Non-bouncing
$\Phi_- < 0$
Non-
bouncing
$\Phi_- > 0$
$W_{21}(\Phi)$
$W_{2b,n}(\Phi)$
Bouncing
$\Phi_- < 0$
$B(\Phi)$
IR_1
UV_2
Φ



$W(\Phi)$
Non-bouncing
$\Phi_- < 0$
Non-
bouncing
$\Phi_- > 0$
Bouncing
$\Phi_- < 0$
$W_{2nb,p}(\Phi)$
$B(\Phi)$
IR_1
UV_2
Φ

Spontaneous breaking saddle points

- There are two flows with $\mathcal{R} \to \infty$

- One is the standard flow associated with UV$_2$. $\mathcal{R} \to \infty$ because $\phi_0 = 0$ although R_{UV} can be anything. The solution is exact AdS, with $\langle O \rangle = 0$.

- The $\mathcal{R} \to \infty$ solution associated with $\phi = \phi_*$ is a distinct branch of the theory.

- At $\phi = \phi_*$, ϕ_0 (the source) vanishes, therefore $\mathcal{R} \to \infty$ although R_{uv} =finite.

- The point $\phi = \phi_*$ (a single solution) is a one-parameter family of saddle points with $\phi_0 = 0$ but a non trivial (relevant) vev

$$\langle O \rangle = \xi_* \; R_{UV}{}^{\frac{\Delta_+}{2}}$$

- Therefore the CFT UV$_2$ has two saddle points at finite positive curvature R_{UV}. In one $\langle O \rangle = 0$ and in the other $\langle O \rangle \neq 0$.

Stabilisation by curvature

- The theories with $\phi_0 > 0$ and $\mathcal{R} < \mathcal{R}_*$ do not exist.

- But for $\mathcal{R} > \mathcal{R}_*$ there are two non-trivial saddle points

- This is an example of a theory that in flat space, it exists for $\phi_0 < 0$ but not for $\phi_0 > 0$.

- But the theory with $\phi_0 > 0$ exists when $\mathcal{R} > \mathcal{R}_*$.

• There is a simple example from weakly-coupled field theory that exhibits similar behavior:

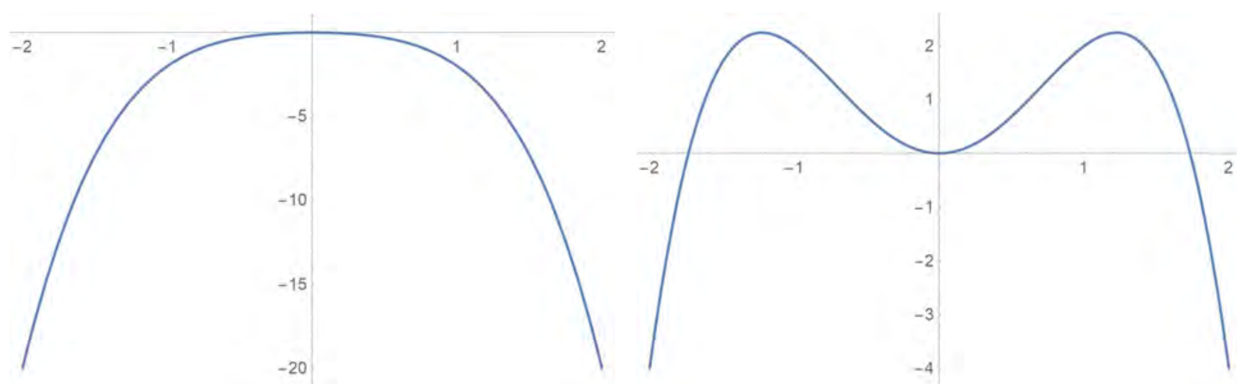
$$V_{flat}(\phi) = -\lambda\phi^4 - m^2\phi^2$$

• When $\lambda > 0$ the theory does not exist.

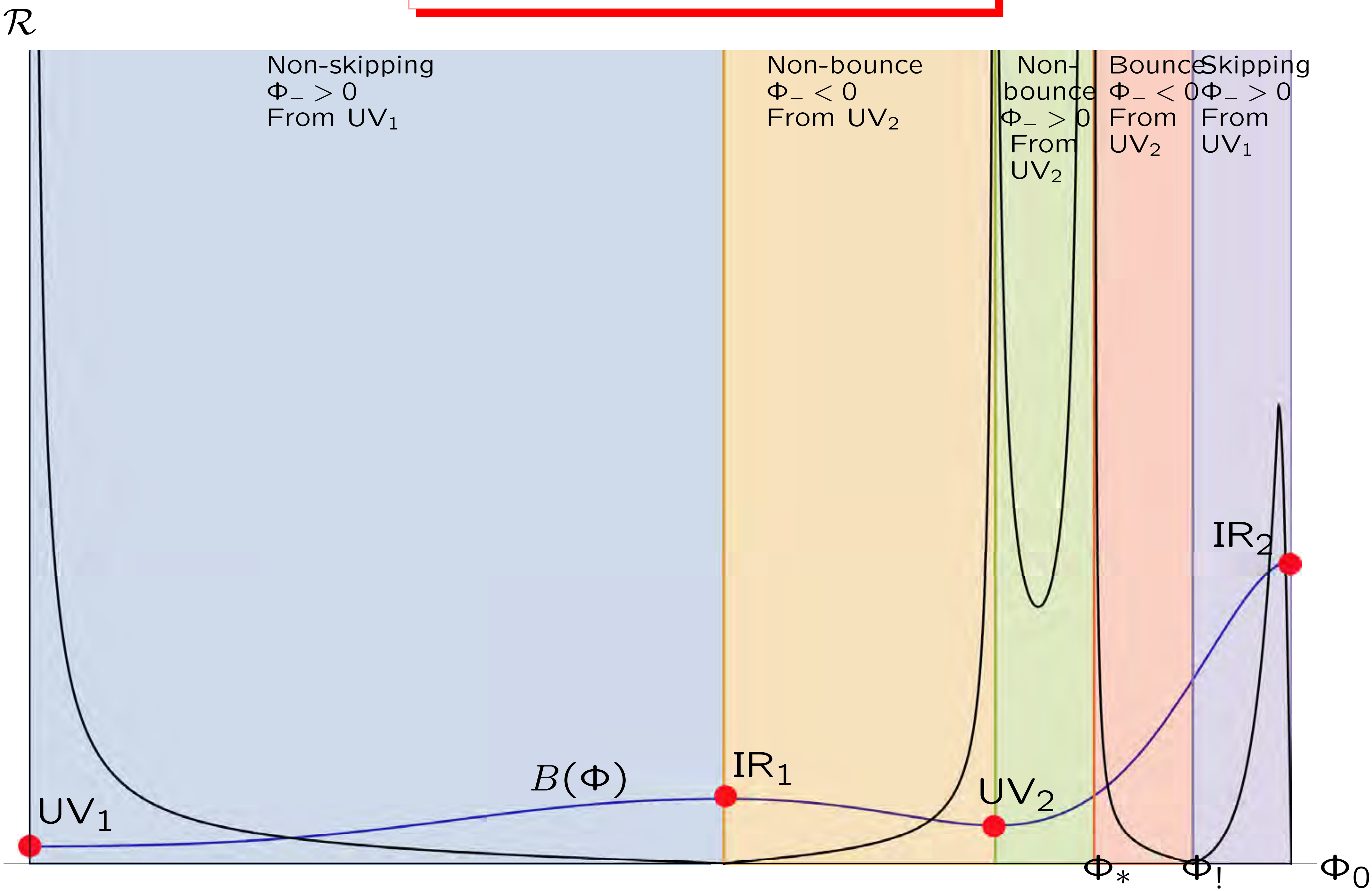
• At sufficiently high curvature

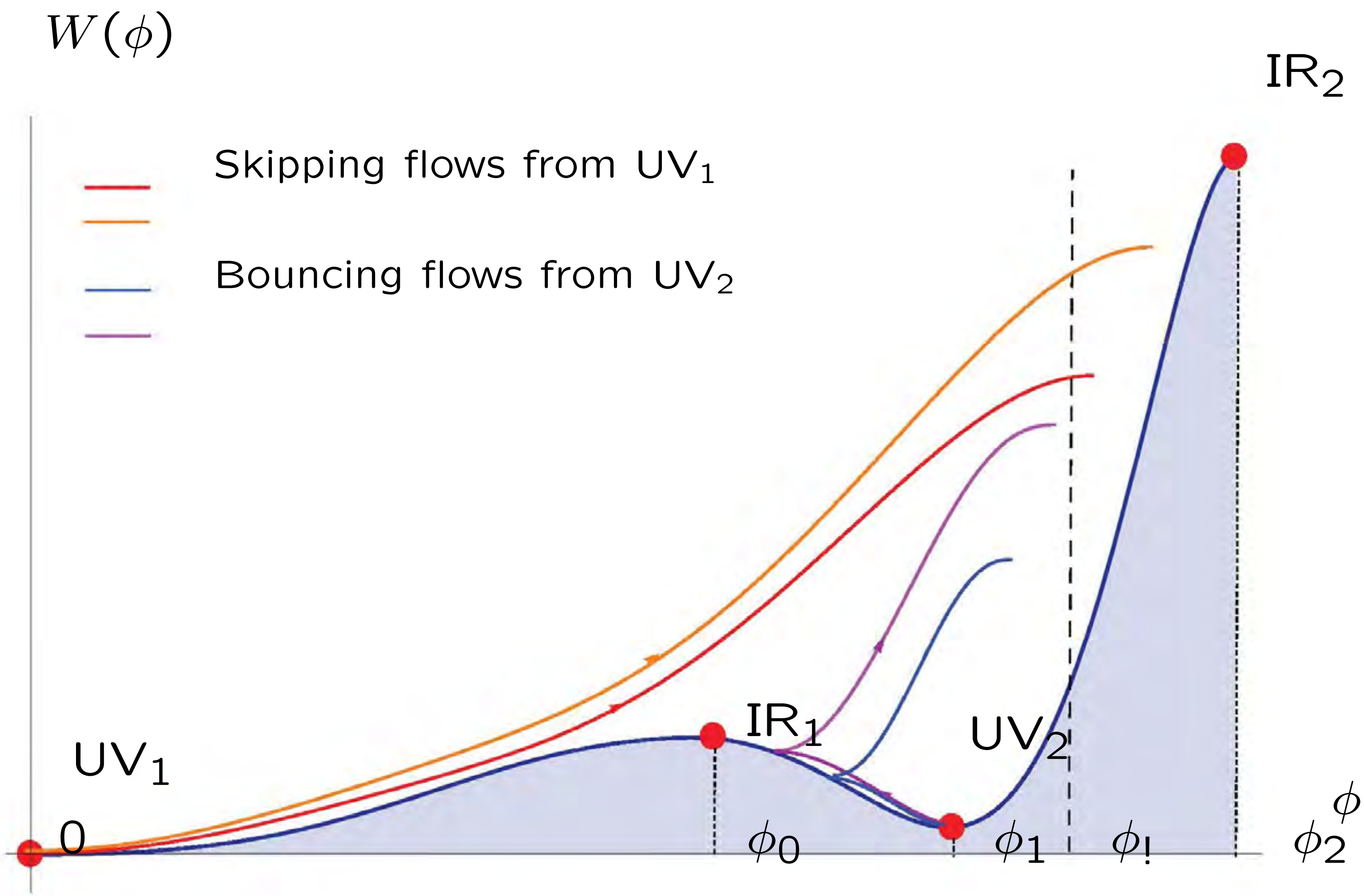
$$V_R(\phi) = -\lambda\phi^4 - m^2\phi^2 + \frac{1}{6R^2}\phi^2$$

the theory develops new extrema:

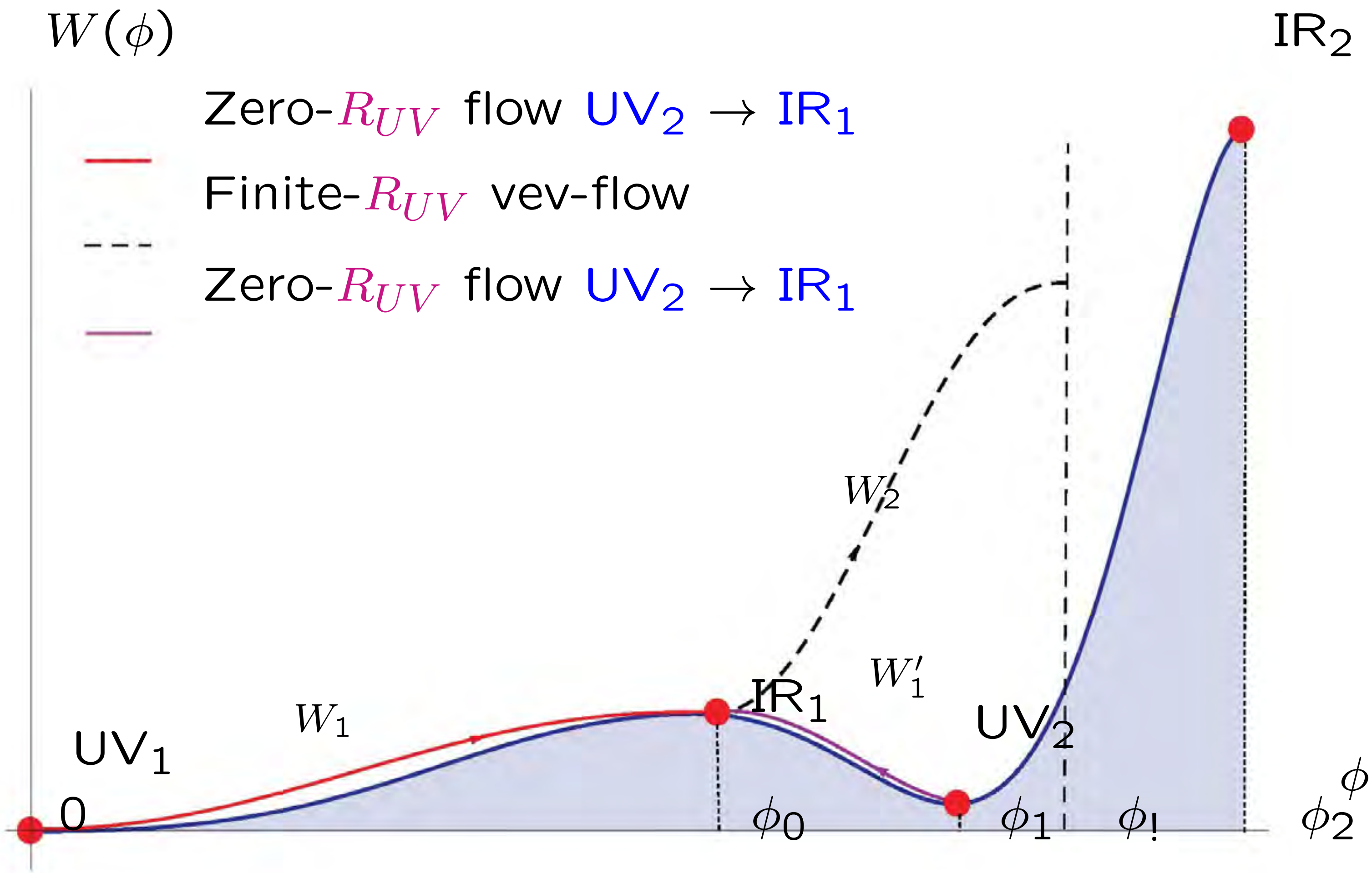


The $\Phi_!$ saddle-point





$W(\phi)$
IR$_2$
Skipping flows from UV$_1$
Bouncing flows from UV$_2$
IR$_1$
UV$_2$
UV$_1$
0
ϕ_0
ϕ_1
$\phi_!$
ϕ_2
ϕ



$W(\phi)$
IR$_2$
Zero-R_{UV} flow UV$_2$ $\rightarrow$ IR$_1$
Finite-R_{UV} vev-flow
Zero-R_{UV} flow UV$_2$ $\rightarrow$ IR$_1$
W_2
W_1'
IR$_1$
W_1
UV$_2$
UV$_1$
ϕ
0
ϕ_0
ϕ_1
$\phi_!$
ϕ_2

• $\Phi_!$ cannot be reached from either UV$_1$ or UV$_2$ but only from IR$_1$.

• The Flow from IR$_1$ to $\Phi_!$ has zero source and a vev

$$\langle O \rangle = \xi_! \, R_{UV}{}^{\frac{\Delta_+}{2}}$$

• At the IR$_1$ we have an AdS boundary.

• As $\mathcal{R} \quad \equiv \quad R_{\rm UV}\phi_0{}^{-\frac{2}{\Delta_-}}$, $\mathcal{R} \to 0$ when $\phi_0 \to 0$.

• This is again a one-parameter family of saddle points with different curvature where the theory is driven by the vev of an irrelevant operator.

• As before the CFT at IR$_1$ has two saddle points at finite curvature: one with $\langle O \rangle = 0$, and one with $\langle O \rangle \neq 0$.

• The one with $\langle O \rangle = 0$ has lower free energy.

Dependence of $\mathcal{F}_i$ on $B(\mathcal{R}), C(\mathcal{R})$

In terms of the two functions $B(\mathcal{R})$ and $C(\mathcal{R})$ the candidate $\mathcal{F}$ functions can be written as

$$\frac{\mathcal{F}_1(\mathcal{R})}{(M\ell)^2\Omega_3} = -\frac{4}{3}\mathcal{R}^{\frac{1}{2}}(2B'(\mathcal{R}) + C''(\mathcal{R}) + \mathcal{R}\ B''(\mathcal{R}))$$

$$\frac{\mathcal{F}_2(\mathcal{R})}{(M\ell)^2\Omega_3} = -2\mathcal{R}^{-\frac{3}{2}}(-(C(\mathcal{R}) - C(0)) + \mathcal{R}C'(\mathcal{R}) + \mathcal{R}^2B'(\mathcal{R}))$$

$$\frac{\mathcal{F}_3(\mathcal{R})}{(M\ell)^2\Omega_3} = -\frac{4}{3}\mathcal{R}^{-\frac{1}{2}}(B(\mathcal{R}) + C'(\mathcal{R}) - B(0) - C'(0)) + \mathcal{R}B'(\mathcal{R}))$$

$$\frac{\mathcal{F}_4(\mathcal{R})}{(M\ell)^2\Omega_3} = -\mathcal{R}^{-\frac{3}{2}}(C(\mathcal{R}) - C(0)) + \mathcal{R}(B(\mathcal{R}) - B(0))$$

RETURN

$\mathcal{F}$-functions and $\mathcal{F}$-theorems

• I will call "global" C-theorem, the existence of a function, C on the space of CFTs that satisfies

$$C(CFT_{UV}) \quad > \quad C(CFT_{IR})$$

• I will call "local" C-theorem, the existence of a function $C(\log\mu)$ on the space of QFTs (a function of the RG flow parameter), that satisfies locally

$$\frac{dC}{du} < 0 \quad , \quad C(\mu=\infty) = C(CFT_{UV}) \quad , \quad C(\mu=0) = C(CFT_{IR})$$

• In odd dimensions, there are no conformal anomalies and therefore, no obvious candidates for a C-function.

• A global F-function for 3d CFTs was proposed to be the renormalized "free energy" (or partition function) of a CFT on the 3-sphere.

Jafferis, Jafferis+Klebanov+Pufu+Safdi

• There is no general proof, but it has been checked in perturbative and supersymmetric examples.

• But the associated (renormalized) partition function fails to be a monotonic F-function along the flow.

Klebanov+Pufu+Safdi, Taylor+Woodhead

• An interpolating F-function satisfying the F-theorem was proposed to be the (appropriately renormalized) S^2 entanglement entropy in flat space.

Myers+Sinha, Myers+Casini+Huerta, Liu+Mezzei

• There is a general proof that in 3d it is always monotonic (but the proof cannot be extended to 5d).

,Casini+Huerta

• As we have seen, the partition function of the sphere contains a part that is related to entanglement entropy.

• We therefore concluded that de Sitter entanglement entropy and the S^3 partition function are tightly connected.

• Now that we have complete control of the holographic sphere partition function, we will use it to define variants of the F-function.

New $\mathcal{F}$-functions

• To obtain a "local" $\mathcal{F}$-function we must have a function $\mathcal{F}(\mathcal{R})$, with $\mathcal{R}$ some parameter along the flow, which exhibits the following properties:

♠ At the fixed points of the flow, the function $\mathcal{F}(\mathcal{R})$ takes the values $\mathcal{F}_{\text{UV}}$ and $\mathcal{F}_{\text{IR}}$ respectively that are given by the "global" F-function.

♠ The function $\mathcal{F}(\mathcal{R})$ evolves monotonically along the flow,

$$\frac{d}{d\mathcal{R}}\mathcal{F}(\mathcal{R}) \leq 0\,,$$

♠ There is an extra option for stationarity at the beginning and end of the flow. This is optional.

• We will use $\mathcal{R}$ as an interpolating variable between

$$IR: \ \mathcal{R} \to 0 \qquad \text{and} \qquad UV: \mathcal{R} \to \infty$$

and demand

1. $\mathcal{F}$ must be UV and IR finite.

2. It must also satisfy:

$$\lim_{\mathcal{R}\to\infty} \mathcal{F}(\mathcal{R}) \equiv \mathcal{F}_{UV} = 8\pi^2(M\ell_{UV})^2$$

$$\lim_{\mathcal{R}\to 0} \mathcal{F}(\mathcal{R}) \equiv \mathcal{F}_{IR} = 8\pi^2(M\ell_{IR})^2$$

$$\frac{d\mathcal{F}}{d\mathcal{R}} \geq 0$$

• The sphere free energy is a function of $\mathcal{R}$ and a UV cutoff Λ.

• It is UV divergent as $\Lambda \to \infty$. The detailed structure of the general UV divergences are known.

• It is IR divergent as $\mathcal{R} \to 0$. The detailed structure of the general IR divergences is known.

• The subtraction of UV divergences is standard and the renormalized partition function of a generic QFT on S^3 depends on two arbitrary scheme dependent constants.

• There are four distinct ways of subtracting the IR divergences. When this is done, the resulting $\mathcal{F}$ functions are scheme independent ($\mathcal{F}_{1,2,3,4}$).

• We can construct also two distinct F-functions starting directly from the de Sitter entanglement entropy ($\mathcal{F}_{5,6}$).

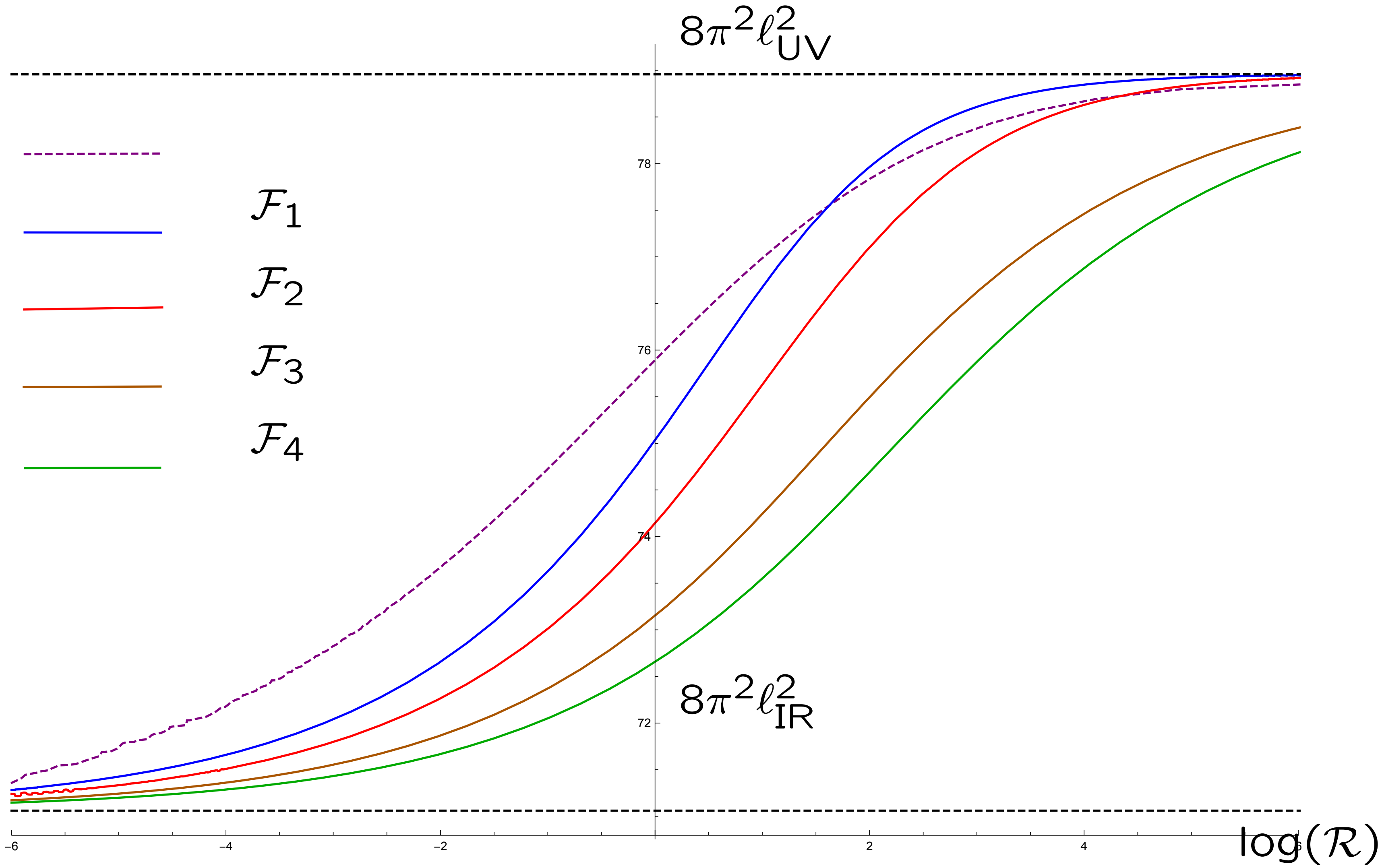
• It turns out that

$$\mathcal{F}_5 = \mathcal{F}_1 \quad , \quad \mathcal{F}_6 = \mathcal{F}_3$$

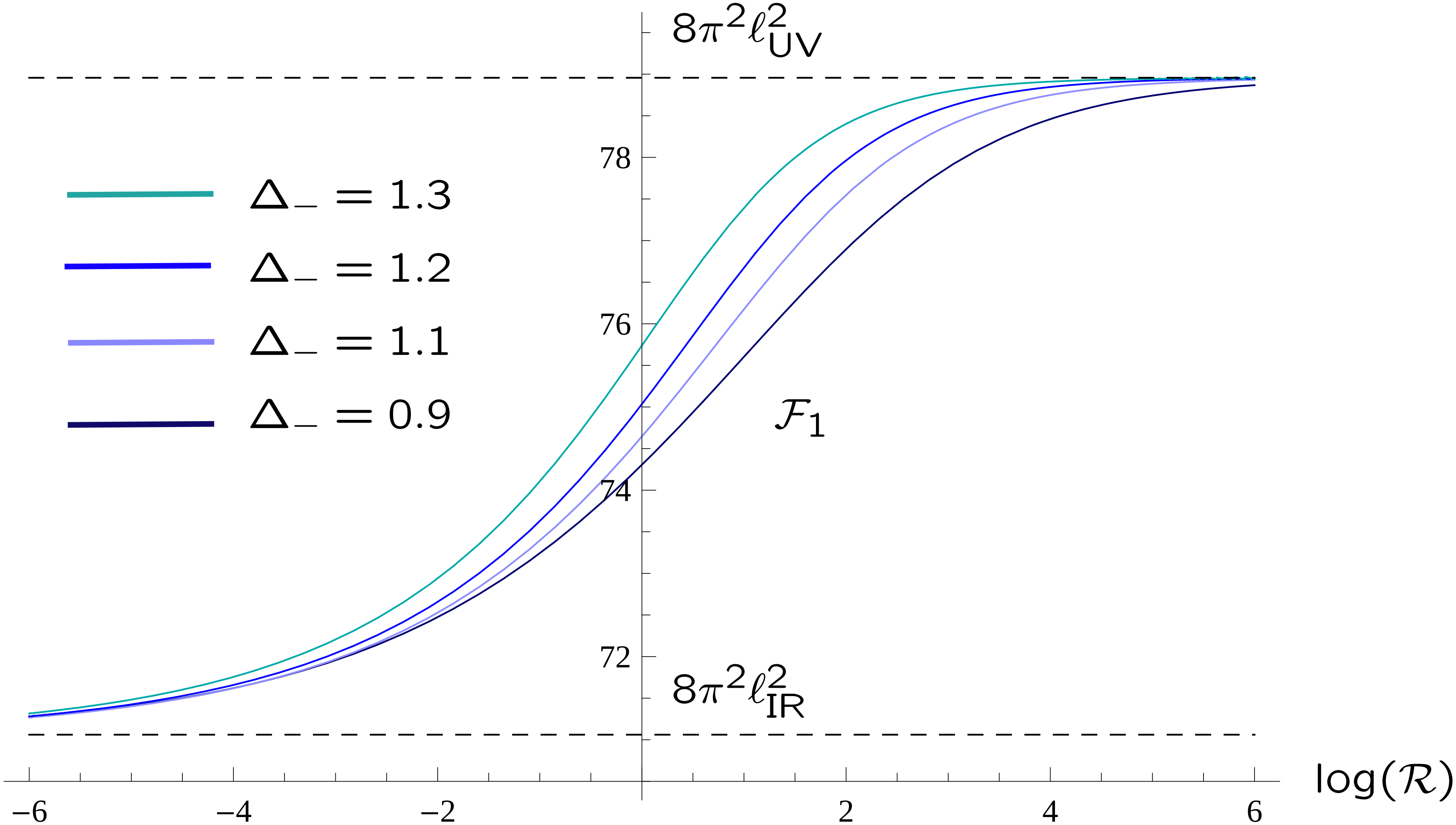
• We must also supplement these functions with the prescription that when $\Delta < \frac{d}{2}$, then instead of the partition function we must use the effective action (ie. its Legendre transform).

• All $\mathcal{F}_{1,2,3,4}$ passed many checks both in holography and standard perturbation theory

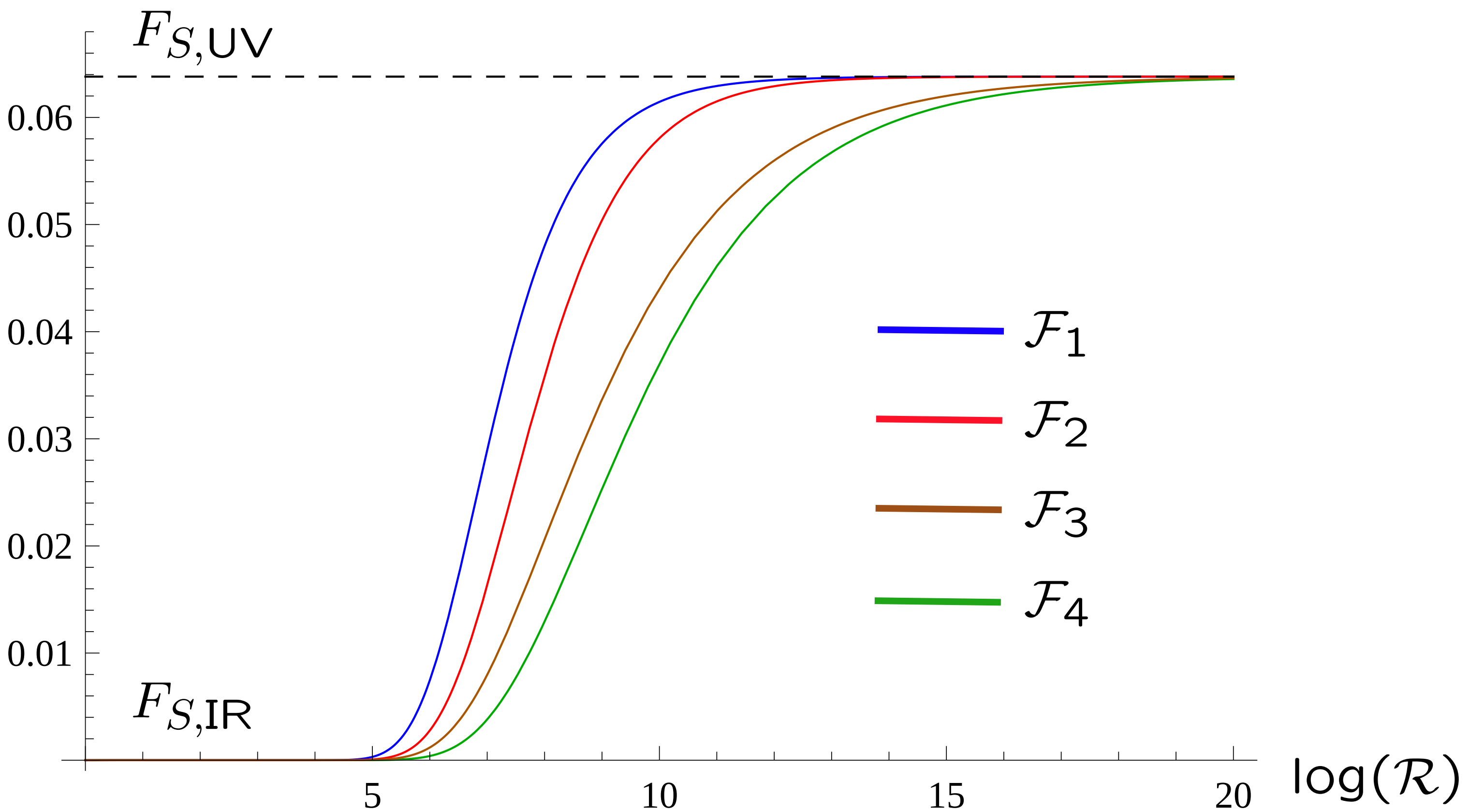
♠ All $\mathcal{F}_{1,2,3,4}$ are monotonic in many numerical holographic examples we analyzed when $\Delta > \frac{3}{2}$.



$\mathcal{F}_{1,2,3,4}$ vs. $\log(\mathcal{R})$ for a holographic model with Mex Hat potential and $\Delta_- = 1.2$.



$\mathcal{F}_1$ vs. $\log(\mathcal{R})$ for a holographic model with $\Delta_- = 0.9$ (dark blue), 1.1, (light blue), 1.2 (blue) and 1.3 (cyan).



Legendre-transformed $\mathcal{F}_{1,2,3,4}$ for a theory of a free massive boson on S^3.

♠ There is no general proof of monotonicity so far.

Detailed plan of the presentation

- Title page 0 minutes
- Introduction 2 minutes
- The goal 3 minutes
- Holographic RG: the setup 7 minutes
- The first order formalism 10 minutes
- General Properties of the superpotential 11 minutes
- The extrema of V 14 minutes
- The standard holographic RG Flows 15 minutes
- Bounces 18 minutes

- Detour: entanglement entropy 43 minutes
- Holographic QFTs 46 minutes
- The Holographic dictionary 48 minutes
- C-functions C-theorems 50 minutes
- The C-function in 4 dimensions 51 minutes
- F-functions 55 minutes
- Renormalization 59 minutes
- The vanilla flows at finite curvature, II 60 minutes
- The IR limits, II 62 minutes
- The first order flows 64 minutes
- The interpretation of parameters 66 minutes
- Detour:Curvature-dependent β-functions and geometric flows 70 minutes
- UV and IR divergences of F and S_{EE} 71 minutes

- Stabilisation by curvature 141 minutes
- The $\Phi_!$ saddle-point 144 minutes
- Dependence of $\mathcal{F}_i$ on $B(\mathcal{R}), C(\mathcal{R})$ 147 minutes
- F-functions and F-theorems 150 minutes
- New F-functions in 3d 156 minutes