Modular Hamiltonians in QFT, AQFT and holography

- Raúl Arias -

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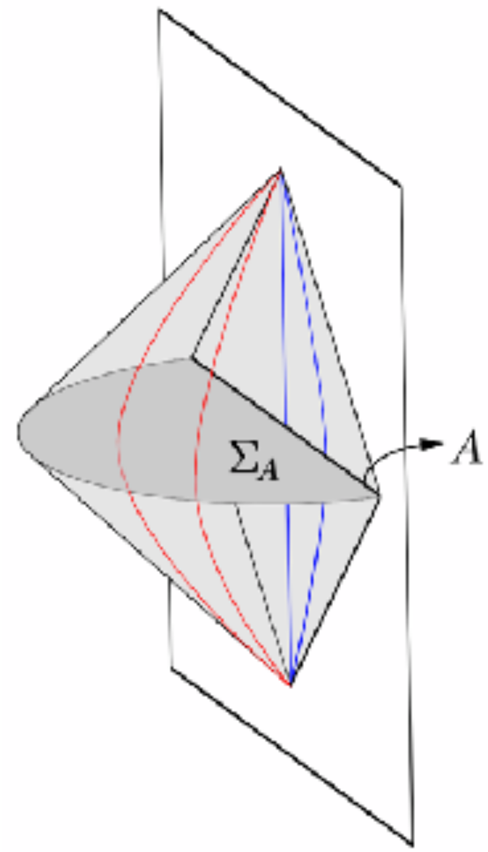

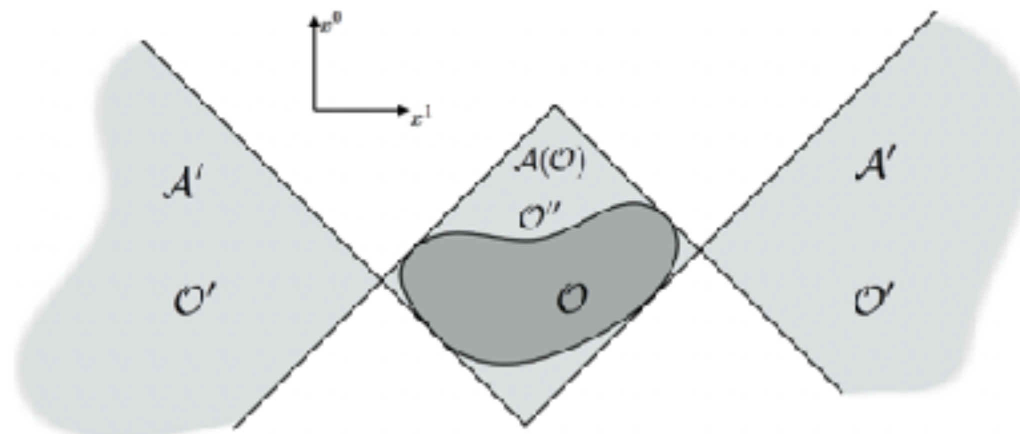

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Frontiers of holographic duality,
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SISSA

Outline

- Motivation
- Introduction to Tomita-Takesaki modular theory
- Some results in QFT for the vacuum
- Modular Hamiltonian for holographic excited states

  (A)QFT   Bulk

- Summary

Motivation

- Useful in QFT & Quantum Information

Fidelity
Relative Entropy
QRRE
Trace distance
.
.
.

Distinguishability between quantum states

$$S_{rel}(\rho||\sigma) = Tr(\rho \log \rho) - Tr(\rho \log \sigma)$$

In QFT is better to write it as

$$\Delta S = S(\rho) - S(\sigma)$$

$$S_{rel}(\rho||\sigma) = \Delta\langle K_\sigma \rangle - \Delta S$$

$$\Delta\langle K_\sigma \rangle = \langle K_\sigma \rangle_\rho - \langle K_\sigma \rangle_\sigma$$

- First Law of Entanglement

$$\delta S = \delta \langle K \rangle$$

Generalisation of the first law of thermodynamics

Useful in Condensed Matter and Stat. Phys.

Called entanglement Hamiltonian. The entanglement spectrum characterise and identify topological states of matter

Li, Haldane $\cdots\cdots\blacktriangleright$ Non Abelian FQHE states

Fidkowski $\cdots\cdots\blacktriangleright$ Topological Insulators & SC

Yao, Qi $\cdots\cdots\blacktriangleright$ Kitaev Model

and can give information about the operator content of the theory

Läuchli 2013, Cardy, Tonni 2016, Di Giulio, R.A, Tonni 2019, Chang, You, Wen, Ryu 2019

Useful in Gravity

- Bekenstein bound: Universal bound on the entropy of a region — Casini '12
- Linearized Einstein equations — Lashkari, McDermott, Van Raamsdonk '13
- Relative Entropy = Bulk Relative Entropy — Jafferis, Lewkowycz, Maldacena, Suh '15

Tomita - Takesaki (or modular) theory

$\mathcal{A}$ → von Neumann Algebra on a Hilbert space $\mathcal{H}$

This relation defines the op. S (Tomita op.) → $SA|\Omega\rangle = A^\dagger|\Omega\rangle$

Cyclic and Separating

Unique polar decomposition

$$S = J\Delta^{1/2} = \Delta^{-1/2}J \qquad \Delta = S^\dagger S \qquad K \equiv -\log\Delta$$

Modular conjugation

Modular operator

Modular Hamiltonian

- TT theorem says that

$$JAJ = \mathcal{A}' \qquad \Delta^{it}\mathcal{A}\Delta^{-it} = \mathcal{A} \qquad \Delta^{it}\mathcal{A}'\Delta^{-it} \underset{\forall t \in \mathbb{R}}{=} \mathcal{A}'$$

which ensures that a uniparametric group of automorphisms on the algebra exist

$$\sigma_t(A) \equiv \Delta^{it}A\Delta^{-it}$$ → Modular flow

- Finite dimensional example

$$\mathcal{A} = M_n(\mathbb{C}) \otimes 1_n \qquad \mathcal{A}' = 1_n \otimes M_n(\mathbb{C}) \qquad \mathcal{H} = \mathbb{C}^n \otimes \mathbb{C}^n$$

Algebra of linear operators

A state on the algebra can be represented by a statistical operator

$$\rho \in M_n(\mathbb{C}) \qquad \rho = \sum_{i=1}^{n} \lambda_i |i\rangle\langle i| \qquad |\Omega\rangle = \sum_{j=1}^{n} \sqrt{\lambda_j} |j\rangle \otimes |j\rangle$$

Spectral decomposition

After some work

$$\Delta(|i\rangle \otimes |j\rangle) = \frac{\lambda_i}{\lambda_j} |i\rangle \otimes |j\rangle \qquad S(|i\rangle \otimes |j\rangle) = \sqrt{\frac{\lambda_i}{\lambda_j}} |i\rangle \otimes |j\rangle \qquad J(|i\rangle \otimes |j\rangle) = |j\rangle \otimes |i\rangle$$

$$= \rho \otimes \rho^{-1}$$

And so

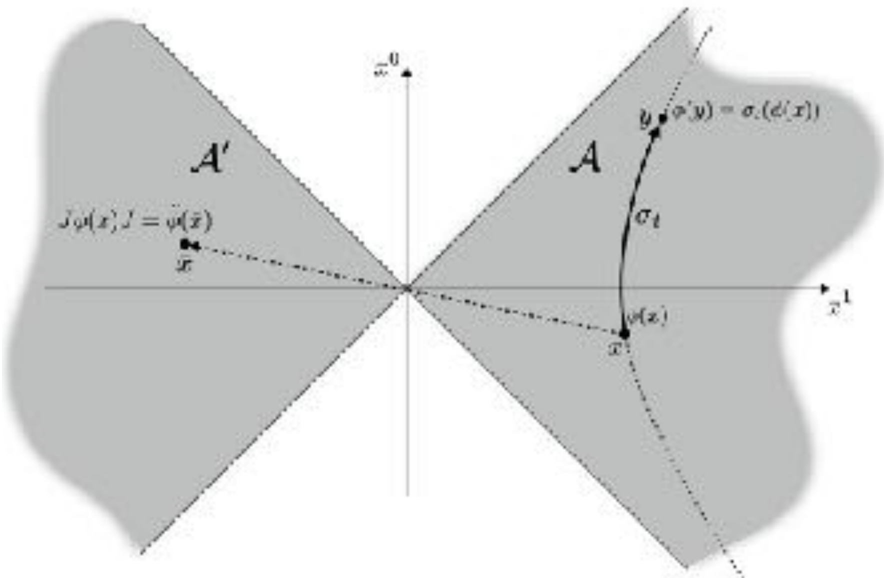
$$\Delta^{it} = \rho^{it} \otimes \rho^{-it} \qquad K = -\log \rho \otimes 1 + 1 \otimes \log \rho \qquad \sigma_t(A \otimes 1) = \rho^{it} A \rho^{-it} \otimes 1$$

"inner" modular Hamiltonian

Modular Hamiltonian for vacuum state in QFT

Any QFT in Rindler wedge and vacuum state

$$\mathcal{W}_R := \{x \in \mathbb{R}^d : x^1 > |x^0|\}$$



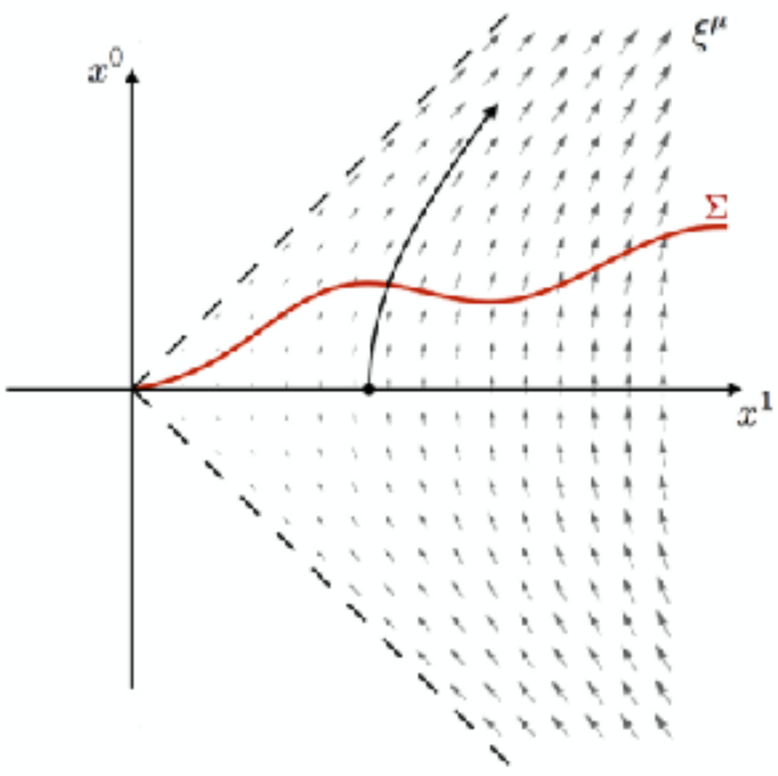
Bisognano - Wichmann '75

$$J = \Theta \qquad \Delta = e^{-K_f}$$

CRT operator   Boost generator

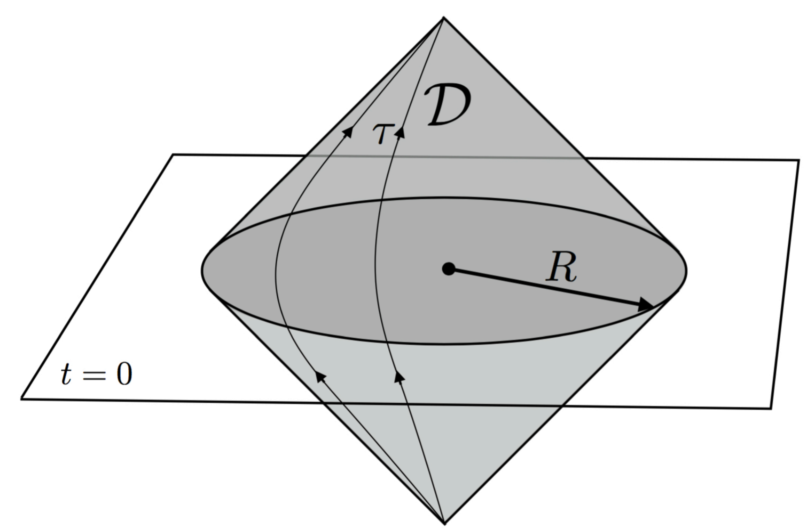
$$K = 2\pi \int_{x>0} d^{d-1}x \, x \, T_{00}(x)$$

- K is local because is related to a Killing symmetry. It generates a geometric flow along the boost orbits in the Rindler wedge



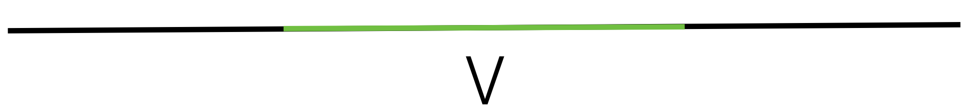
Conformal map

Casini-Huerta-Myers '12



$$K = 2\pi \int_{|x|<R} d^{d-1}x \, \frac{R^2 - r^2}{2R} T_{00}(x)$$

Free fermions in one interval



Dirac field

$$K = \int_V d^{d-1}x\, d^{d-1}y\, \psi^\dagger(x)\mathcal{K}(x,y)\psi(y)$$

Modular Hamiltonian

Kernel

Peschel '03

$$C(x-y) = \langle 0|\psi(x)\psi^\dagger(y)|0\rangle \qquad \mathcal{K} = -\log(C^{-1}-1)$$

How to compute K?

- Computing Eigenfunctions and Eigenvalues of C

Casini, Huerta '09

R.A, Casini, Huerta, Pontello '18

$$C(x-y) = \int_{-\infty}^{\infty} ds\; u_s(x)\frac{1+\tanh s}{2}u^*(y)$$

$$\mathcal{K}(x,y) = \int_{-\infty}^{\infty} ds\; u_s(x)\, 2\pi s\, u_s^*(y)$$

$$K = 2\pi \int_V dx\, \frac{x(L-x)}{L} T(x)$$

Local Temperature

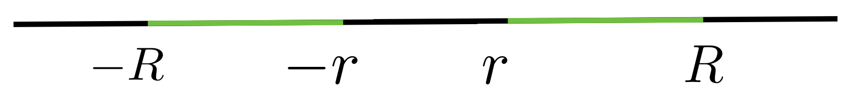
Energy density operator

Universality: Same for free Scalars and fermions independently of the mass

Blanco, R.A, Casini, Huerta '16 & R.A, Casini, Huerta, Pontello '17

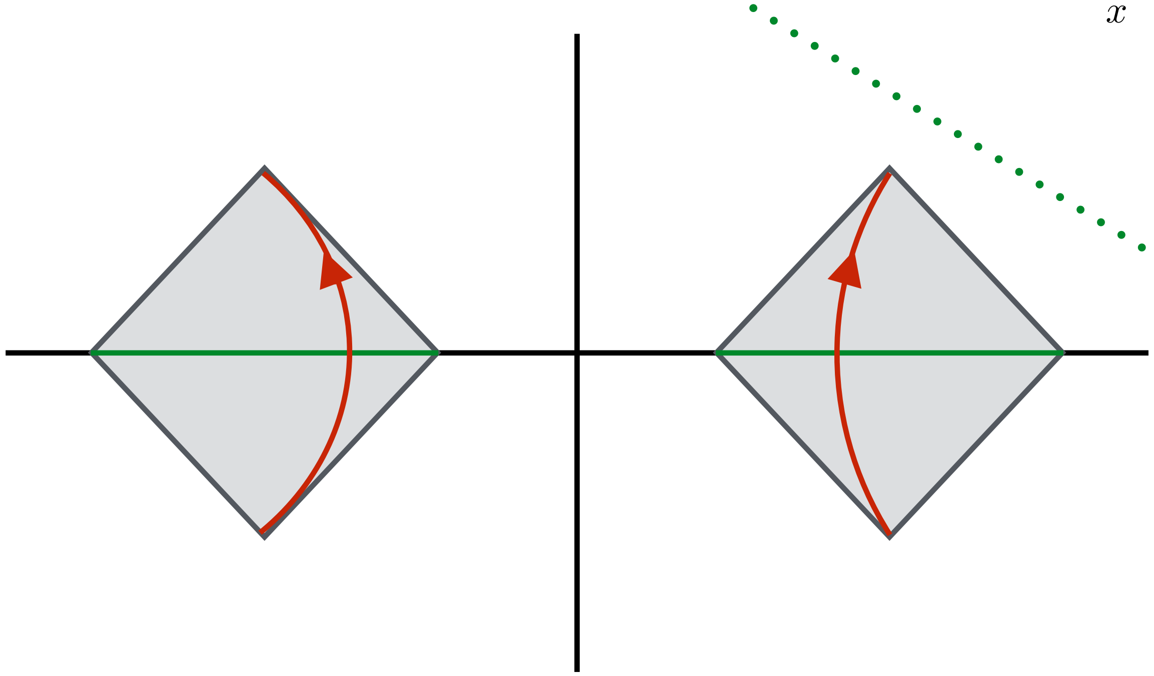
Free fermions and chiral scalar in two intervals

Map to an equivalent problem in the complex plane, where multiplicative boundary conditions are imposed on the intervals



$$K = \boxed{2\pi \int_A dx \, \frac{(x^2 - r^2)(R^2 - x^2)}{2(R-r)(x^2 + rR)} T(x)} + i\pi \int_A dx \, \psi^\dagger(x) \, \frac{rR(x^2 - r^2)(R^2 - x^2)}{(R-r)\, x\,(x^2 + rR)^2} \psi(\bar{x})$$

$$\bar{x} = -\frac{rR}{x}$$



Quasi-local

Casini, Huerta '09

R.A, Casini, Huerta, Pontello '18

$$H = \frac{1}{2} \int dx : J^2(x) :$$

$$K = \int_{A\times A} dx\, dy\, J(x)\mathcal{K}(x,y)J(y)\,,$$

$$\mathcal{K} = -\frac{i}{2}\frac{V}{|V|}\log\left(\frac{|V|+1/2}{|V|-1/2}\right) C^{-1}\,, \qquad iC(x-y) := [J(x), J(y)]$$

$$V = -iC^{-1}F - \frac{1}{2}\,. \qquad F(x,y) := \langle J(x)J(y)\rangle$$

$$\mathcal{K}(x,y) = \sum_{k=1}^{2} \int_{-\infty}^{\infty} ds\, u_{k,s}(x)\pi|s|u_{k,s}(y)^*$$

$$K(x,y) = \boxed{K_{loc}(x,y)} + K_{nonloc}(x,y)$$

Completely non local

R.A, Casini, Huerta, Pontello '18

Modular Hamiltonian for excited states in QFT

Less is know in excited states:

- Expectation values in excited states of a CFT — Lashkari ‘15
- Deformed ball shaped regions in CFT — Sárosi, Ugajin ‘17
- Modular flows in AQFT setup — Lashkari, Liu, Rajagopal ’18

$$\Delta_\Psi = \Psi \Delta_0^{1/2} (\Psi_J^\dagger \Psi_J)^{-1} \Delta_0^{1/2} \Psi^\dagger \qquad \Psi_J = J_0 \Psi J_0 \qquad |\Psi\rangle = \Psi|0\rangle$$

- Rindler and shape deformed Rindler spaces — Balakrishnan, Parrikar ’20

$$K_\lambda = c_\lambda + K + \sum_{n=1}^{\infty} \frac{1}{n!} \delta^n K,$$

$$\delta^n K = n! \frac{(-i)^{n-1}}{(2\pi)^{n-1}} \int d\mu_n \int_{-\infty}^{\infty} ds_1 \cdots \int_{-\infty}^{\infty} ds_n \, f_{(n)}(s_1 + i\tau_1 \cdots, s_n + i\tau_n) O(s_1, Y_1) \cdots O(s_n, Y_n)$$

Holographic excited states

$$|\Psi_\lambda\rangle = U_\lambda|\Psi_0\rangle = \mathcal{P}\, e^{-\int_{\tau<0} d\tau\ (H+\mathcal{O}.\,\lambda(\tau))}|\Psi_0\rangle \qquad \Leftrightarrow \qquad \langle\phi_\Sigma|\Psi_\lambda\rangle \equiv \int \mathcal{D}_{\phi_\Sigma;\lambda}\Phi e^{-S_E[\Phi]}$$

- Coherent states of the perturbative bulk fields

Botta-Cantcheff, Martinez, Silva '15
Christodoulou, Skenderis '16
Botta-Cantcheff, Martinez, Silva '18

$$|\Psi_\lambda\rangle \propto e^{\int dk\ \lambda_k a_k^\dagger}|\Psi_0\rangle$$

- Coherence is broken if we add corrections in 1/N

- Always exist an associated classical geometry that captures the EE via RT

Faulkner et.al. '17

- Any near-AdS spacetime can be described in this way

Marolf et.al. '17

- Works as generating states

$$|\Psi_\lambda\rangle \equiv \prod_w e^{-\frac{|\lambda_w|^2}{2}} \sum_n \frac{(\lambda_w)^n}{\sqrt{n!}}(a_w^\dagger)^n|\Psi_0\rangle.$$

Modular Hamiltonian

Rindler Wedge $\cdots\cdots\blacktriangleright$ Thermofield dynamics (TFD)

$$|\Psi_\lambda\rangle = (U_\lambda \otimes \mathbb{I})|1\rangle\rangle = U_\lambda\,|1\rangle\rangle, \qquad U_\lambda \equiv \mathcal{P}\, e^{-\int_{I_-} d\tau\ (K_0 + \mathcal{O}.\,\lambda(\tau))}$$

Cyclic and Separating

Identity state

- Reduced density matrix

$$\rho_\lambda \equiv \mathrm{Tr}_{\widetilde{\mathcal{A}}}\ |\Psi_\lambda\rangle\langle\Psi_\lambda| = U_\lambda\ U_\lambda^\dagger = U_\lambda(-\pi,\pi) = \mathcal{P}\, e^{-\int_{S^1} d\tau\ (K_0 + \mathcal{O}.\,\lambda(\tau))}$$

In a very special case

$$K_\lambda = K_0 + \int_{\Sigma_R} \lambda(x)\mathcal{O}(x)\ \sqrt{g_{\Sigma_R}}dx^{d-1} \qquad K_0 = \int_{\Sigma_R} T^{\mu\nu} n_\mu \tau_\nu\ \sqrt{g_{\Sigma_R}}\,dx^{d-1}$$

✓ Modular Evolution

✓ Map to the spherical entangling surface and modular flow

Tomita Takesaki in field theory

- Vacuum state

$$\boxed{\left(\tilde{\Phi}(\tilde{x},t) - \Phi^{\dagger}(x,t-i\beta/2)\right)\,|\Psi_0\rangle = 0}$$

Thermal state condition

$$\tilde{\ } \equiv J \qquad \Delta^{1/2} = U_0(i\beta/2)\otimes\tilde{U}_0(-i\beta/2) = e^{-\beta(K_0-\tilde{K}_0)/2}$$

$$S\,\Phi|\Psi_0\rangle = \Phi^{\dagger}|\Psi_0\rangle$$

- Excited state

$$\left(\tilde{\Phi}(\tilde{x},t) - U_\lambda(-i\beta/2)\Phi^{\dagger}(x,t)U_\lambda(i\beta/2)\right)\,|\Psi_\lambda\rangle = 0$$

- Special case $\lambda(\tau,x)=\lambda(x)$

$$K_\lambda \longrightarrow \Delta_\lambda^{1/2} \longrightarrow S_\lambda\,\Phi|\Psi_\lambda\rangle = \Phi^{\dagger}|\Psi_\lambda\rangle,$$

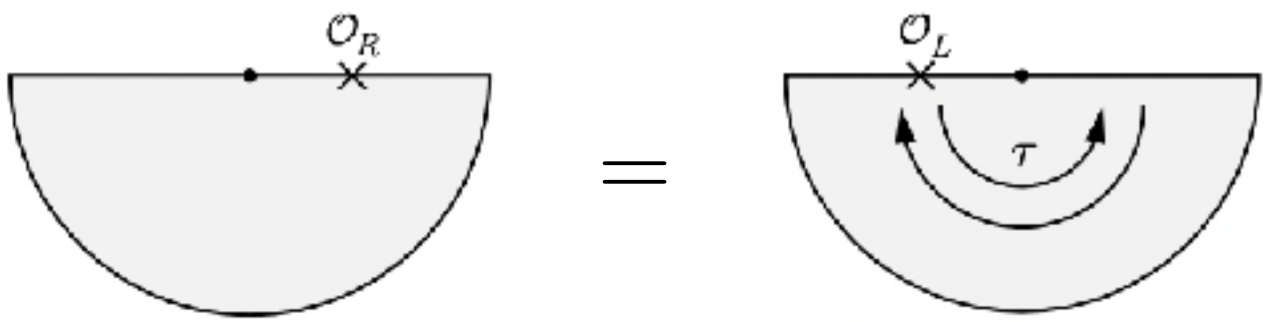
But we can't ensure that the result is the one related with the modular operator of the TT theory in the general case

Tomita Takesaki on the bulk

But, we know the precise holographic dual of our excited states and

$$\left(\tilde{\Phi}(\tilde{x},t) - \Phi^{\dagger}(x,t-\ i\beta/2)\right)\ |\Psi_0\rangle = 0$$

$$\boxed{\left(\Phi_R(t) - \Phi_L(t-i\beta/2)\right)|\Psi_0\rangle \ = \left(\Phi_R(t) - U_0(-i\pi,0)\Phi_L(t)U_0(0,i\pi)\right)|\Psi_0\rangle \ = 0}$$



$$|\Psi_0\rangle \ \equiv U_0(0,-i\pi)\otimes \mathbb{I}|1\rangle\rangle = \mathbb{I}\otimes U_0(0,-i\pi)|1\rangle\rangle$$

$$\Big\downarrow \Delta$$

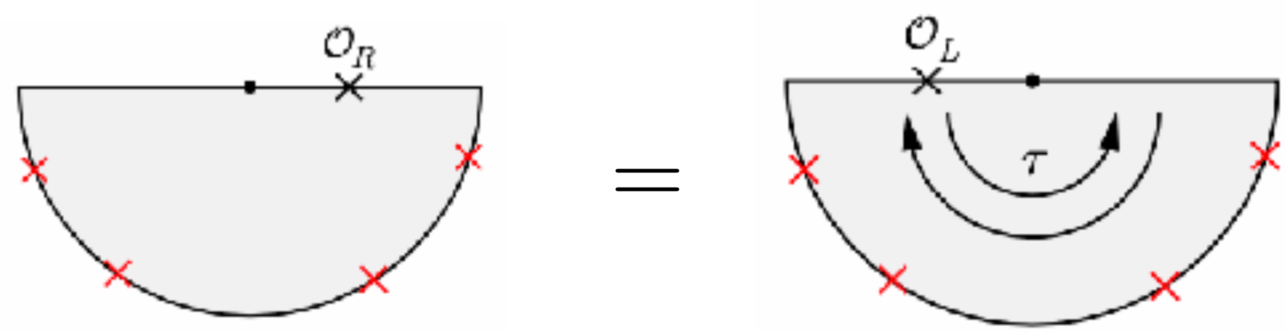
$$S\Phi_R(t)|\Psi_0\rangle = \Phi^{\dagger}_R(t)|\Psi_0\rangle = \Phi_R(t)|\Psi_0\rangle$$

Excited states

$$\left(\Phi_R(t) - U_\lambda(-i\pi,0)\Phi_L(t)U_\lambda(0,i\pi)\right)|\Psi_\lambda\rangle = 0$$

$$\Big\downarrow$$

$$S_\lambda\Phi_R(t)|\Psi_\lambda\rangle = \Phi^{\dagger}_R(t)|\Psi_\lambda\rangle = \Phi_R(t)|\Psi_\lambda\rangle$$



Botta-Cantcheff, Martinez, Silva '19

- Closed expression for the modular operator

R.A, Botta-Cantcheff, Martinez, Zarate '20

$$\Delta_\lambda = \rho_\lambda^{-1} \otimes \rho_\lambda \,, \qquad \langle \phi_+ | \rho_\lambda | \phi_- \rangle = \int \mathcal{D}\Phi_{\lambda;\phi_\pm} \, e^{-S_E[\Phi]} \,.$$

We can also obtain the result for any other subsystem connected to this via an isometry of AdS

Computing the bulk modular Hamiltonian at large N

$$ds^2 = -u^2 dt^2 + \frac{du^2}{1+u^2} + \left(1+u^2\right)\,\left(d\chi^2 + \sinh^2\chi \, d\Omega^2_{d-2}\right)$$

Orthonormal basis

$$\Phi(u,y,t) = \sum_n a_n^\dagger \phi_n(u,y,t) + h.c. \qquad |\Psi_\lambda\rangle = D(\lambda)|\Psi_0\rangle = D(\lambda)e^{-\pi H}|1\rangle\rangle \qquad D(\lambda_n)a_n D^\dagger(\lambda_n) = a_n + \lambda_n$$

Canonically quantised field

Displacement operator

- Following simple algebraic computations…

$$K_\lambda = \sum_n w_n \, : (a_n + \lambda_n)(a_n^\dagger + \lambda_n^*) : \qquad \lambda_n \equiv \lim_{|u|\to\infty} \int_{\Sigma_R} dy \int_0^\pi d\tau \, \lambda(y,\tau)\, \phi_n(u,y,-i\tau)$$

At large N the holographic excitations consist of a family of canonical transformations

Modular flow

- In this context is easy to compute

$$e^{-isK_\lambda}\Phi(u,y,0)e^{isK_\lambda} = \Phi(u,y,s) + f_\lambda(u,y,s) - f_\lambda(u,y,0)$$

$$= \Phi(\gamma^\mu(s)) + f_\lambda(\gamma^\mu(s)) - f_\lambda(\gamma^\mu(0))$$

$\gamma^\mu(s) = (u, y, t + 2\pi s)$

cf. with Lashkari, Liu, Rajagopal '18

$$\Phi(u,y,s) \equiv \sum_n a_n^\dagger \, \phi_n(u,y,s) \, + \, h.c. \qquad f_\lambda(u,y,s) \equiv \sum_n \lambda_n^* \phi_n(u,y,s) \, + \, c.c.$$

Classical solution of the e.o.m

- The BDHM prescription allow us to say something on the dual CFT

Banks, Douglas, Horowitz, Martinec '98

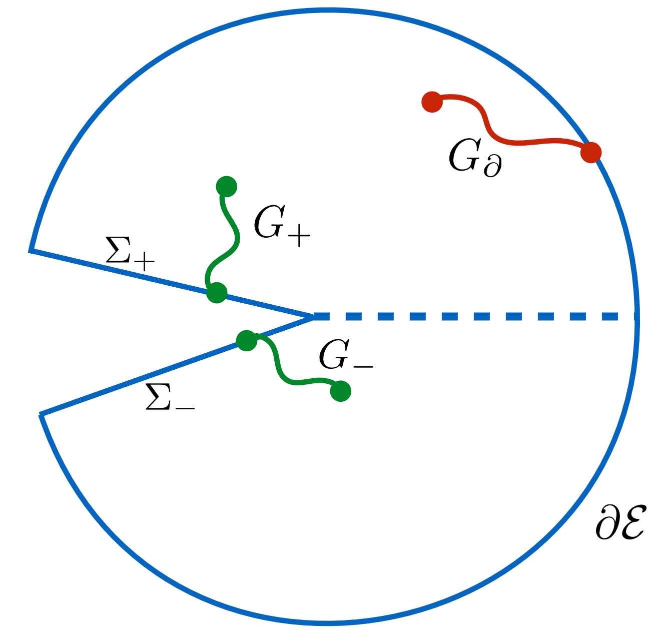
$$\mathcal{O}(y,0) = \lim_{u\to\infty} |u|^\Delta \, \Phi(u,y,0)$$

Assuming a duality between the bulk/QFT modular flows

$$\mathcal{O}(y,s) = \lim_{u\to\infty} |u|^\Delta \, e^{-isK_\lambda}\Phi(u,y,0)e^{isK_\lambda}$$

$$\mathcal{O}(y,s) = \lim_{u\to\infty} |u|^\Delta \, \Phi(u,y,s) + \lim_{u\to\infty} |u|^\Delta \, [f_\lambda(u,y,s) - f_\lambda(u,y,0)]$$

The gravity dual of modular Hamiltonians for arbitrary entangling surfaces



- As an operator equation

$$K_{CFT} = \frac{\hat{A}}{4G} + K_\lambda^{grav} + K_\lambda^{matter}$$

Jafferis, Lewkowycz, Maldacena, Suh '15

- Even if we don't have a Killing vector to foliate the Euclidean part

$$\langle +|K_\lambda^{grav}|-\rangle = \frac{1}{8\pi G}\int_{\Sigma_+} K^+\sqrt{h^+} + \frac{1}{8\pi G}\int_{\Sigma_-} K^-\sqrt{h^-}$$

Induced metric

$$\langle +|K_\lambda^{matter}|-\rangle = \int_{\Sigma_+} \phi^+\Pi^+\sqrt{h^+} + \int_{\Sigma_-} \phi^-\Pi^-\sqrt{h^-} + \int_{\partial\mathcal{E}} \lambda\partial_{\hat{n}}\lambda\sqrt{h}$$

Extrinsic curvature

field configuration

- Solution for the field

Bulk to bulk

$$\Phi(x) = \int_{\Sigma^\pm} G_\pm(x-y)\,\phi^\pm(y)\,dy + \int_{\partial\mathcal{E}} G_\partial(x-z)\,\lambda(z)\,dz$$

Bulk to bdy.

- Natural candidates for any region

Summary

- Basic notions of Tomita - Takesaki modular theory

- Examples in the QFT context for the vacuum state

- Consider a particular family of excited states

- Identify and compute the modular Hamiltonian, modular and conjugation operators in QFT but also in the bulk

- Prescription to compute matrix elements of the modular Hamiltonian in the gravity theory for any region

Thanks!