Stringy Euler numbers of toric Calabi-Yau hypersurfaces

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Log desingularizations and discrepancies

Let X be a normal quasi-projective $\mathbb{Q}$-Gorenstein algebraic variety. Take a resolution of singularities of X

$$\rho : Y \to X$$

whose exceptional locus is a union $\bigcup_{i=1}^{r} D_i$ of smooth irreducible divisors with only normal crossings. Set $I := \{1, \ldots, r\}$ and write

$$K_Y = \rho^* K_X + \sum_{i \in I} a_i D_i,$$

Definition

The rational numbers $a_i \in \mathbb{Q}$ $(i \in I)$ are called *discrepancies* of divisors D_i.

Singularities

Definition

Singularities of X are called at worst

- *terminal* if $a_i > 0$, $\forall i \in I$;
- *canonical* if $a_i \geqslant 0$, $\forall i \in I$;
- *log-terminal* if $a_i > -1$, $\forall i \in I$.

The stringy Euler number $\chi_{\mathrm{str}}(X)$

Definition (Version 1)

Let $\rho : Y \to X$ be a resolution and $K_Y = \rho^* K_X + \sum_{i \in I} a_i D_i$. Define for any subset $J \subseteq I$:

$$D_\emptyset := Y, \quad D_J := \bigcap_{j \in J} D_j \ \ (\emptyset \neq J \subseteq I).$$

The *stringy Euler number* of X is the rational number

$$\chi_{\mathrm{str}}(X) := \sum_{\emptyset \subseteq J \subseteq I} \chi(D_J) \prod_{j \in J} \left(\frac{1}{a_j + 1} - 1 \right)$$

$$= \sum_{\emptyset \subseteq J \subseteq I} (-1)^{|J|} \chi(D_J) \prod_{j \in J} \frac{a_j}{a_j + 1}.$$

(a product over $\emptyset$ is assumed to be 1)

Some properties of $\chi_{\mathrm{str}}(X)$

General remarks

- The rational number $\chi_{\mathrm{str}}(X)$ *does not depend* on the choice of a desingularization $\rho : Y \to X$. In particular, if X is smooth, then

$$\chi_{\mathrm{str}}(X) = \chi(X)$$

  (we can take $\rho = id$).
- If $\rho : Y \to X$ is a *crepant* desingularization ($a_i = 0 \; \forall i \in I$), then

$$\chi_{\mathrm{str}}(X) = \chi(Y).$$

  Examples: minimal desingularizations of ADE-singularities of surfaces.
- If X and X' are birational *K-equivalent*, then

$$\chi_{\mathrm{str}}(X) = \chi_{\mathrm{str}}(X').$$

The stringy Euler number $\chi_{\mathrm{str}}(X)$

Definition (Version 2)

The *stringy Euler number* of X is the rational number

$$\chi_{\mathrm{str}}(X) := \sum_{\emptyset \subseteq J \subseteq I} \chi(D_J^\circ) \prod_{j \in J} \left(\frac{1}{a_j + 1} \right),$$

where

$$D_J^\circ = D_J \setminus \bigcup_{j \notin J} D_j.$$

Stringy Euler numbers of toric varieties

Let $X = X_\Sigma$ be a $\mathbb{Q}$-Gorenstein d-dimensional **toric variety** of fan $\Sigma \subset N_\mathbb{R}$. Denote by $\omega : N_\mathbb{R} \to \mathbb{R}$ a Σ-piecewise linear function with $\omega(e_i) = 1$ for any primitive lattice generators e_i of 1-dimensional cone $\mathbb{R}_{\geqslant 0} e_i \in \Sigma(1)$. A toric desingularization $\rho : Y \to X$ is determined by a regular refinement $\widehat{\Sigma}$ of Σ such the smooth toric divisors D_j correspond to 1-dimensional cones $\tau_j := \mathbb{R}_{\geqslant 0} e_j \in \widehat{\Sigma}(1)$. One has $a_j = \omega(e_j) - 1$ and $\chi(D_J^\circ) = 0$ unless $|J| = d$ and D_J° is a torus fixed point corresponding to a d-dimensional cone $\sigma \in \widehat{\Sigma}(d)$. The second version implies:

$$\chi_{\rm str}(X) = \sum_{\sigma \in \widehat{\Sigma}(d)} \prod_{\tau_j \prec \sigma} \left(\frac{1}{a_j + 1} \right)$$

If a d-dimensional cone σ is generated by a $\mathbb{Z}$-basis $e_1, \ldots, e_d \in N$ and $\omega : N_{\mathbb{R}} \to \mathbb{R}$ is a linear function with positive values

$$\omega(e_i),\ 1 \leqslant i \leqslant d,$$

Then

$$\prod_{j=1}^{d} \left(\frac{1}{a_j + 1} \right) = \prod_{j=1}^{d} \left(\frac{1}{\omega(e_j)} \right)$$

equals to the lattice normalized volume of the d-dimensional rational simplex $\sigma \cap \{\omega(x) \leqslant 1\}$.

$$\chi_{\mathrm{str}}(X) = \sum_{\sigma \in \widehat{\Sigma}(d)} \mathrm{Vol}_d(\sigma \cap \{\omega(x) \leqslant 1\}) = \mathrm{Vol}_d(\{\omega(x) \leqslant 1\}) = \mathrm{Shed}(\Sigma).$$

Examples

Example (log-terminal singularity

Quotient $X := \mathbb{C}^2/\mu_n$ under the group action of $\mu_n := \langle \zeta \rangle$ on $\mathbb{C}^2$:

$$(x, y) \mapsto (\zeta x, \zeta^k y), \quad gcd(n, k) = 1.$$

One has $\chi(X) = 1$, but $\chi_{\mathrm{str}}(X) = n = |\mu_n|$.

Example (terminal singularity

Quotient $X := \mathbb{C}^4/\mu_2$ under the group action of $\mu_2 := \langle \zeta \rangle$ on $\mathbb{C}^4$:

$$(x_1, x_2, x_3, x_4) \mapsto (-x_1, -x_2, -x_3, -x_4).$$

One has $\chi(X) = 1$, but $\chi_{\mathrm{str}}(X) = 2 = |\mu_n|$.

Applications to MMP

Theorem

If two minimal models X and X' are birational, then

$$\chi_{\mathrm{str}}(X) = \chi_{\mathrm{str}}(X').$$

This implies that the stringy Euler number χ_{str} is well-defined function on the birational classes $\langle X \rangle_{\mathrm{bir}}$ of algebraic varieties X of non-negative Kodaira dimension $\kappa(X)$.

Remark

The stringy Euler number of the birational class of an algebraic surface S with $\kappa(S) \geqslant 0$ equals the usual Euler number of its minimal birational model.

Remark

If the stringy Euler number of a given birational class $\langle * \rangle_{\mathrm{bir}}$ of algebraic varieties has non-integral value, then this birational class does not contain a smooth minimal model.

Applications to toric MMP (according to M. Reid)

Theorem (M. Reid, 1983)

Let X and X' be two projective $\mathbb{Q}$-Gorenstein toric varieties such that X' is obtained from X by either a toric divisorial Mori contraction $f : X \to X'$, or by a toric flip $f : X \dashrightarrow X'$, then one has

$$\chi_{\mathrm{str}}(X) > \chi_{\mathrm{str}}(X').$$

Since the stringy Euler number of a projective $\mathbb{Q}$-Gorenstein toric variety is a positive integer, the above monotone property implies a termination of toric flips.

Non-degenerate toric hypersurfaces

Let $M \cong \mathbb{Z}^d$ be a lattice of rank d. We consider M as the lattice of characters of a d-dimensional algebraic torus $\mathbb{T}_d \cong (\mathbb{C}^*)^d$.

Definition

A Laurent polynomial $f(\mathbf{t}) \in \mathbb{C}[M] \cong \mathbb{C}[t_1^{\pm 1}, \ldots, t_d^{\pm 1}]$ is called *non-degenerate* if there exists a smooth projective torus embedding $\mathbb{T}_d \hookrightarrow X$ such that the Zariski closure $\overline{Z}_f$ of the affine hypersurface $Z_f := \{f(\mathbf{t}) = 0\}$ is smooth and $\overline{Z}_f$ together with toric divisors $D_1, \ldots, D_r$ form a set of smooth normal crossing divisors in X. If $\Delta := Newt(f) \subset M_{\mathbb{R}} := M \otimes \mathbb{R}$, then the non-degeneracy of f is a Zariski open condition on the coefficients a_m of

$$f(\mathbf{t}) = \sum_{m \in A} a_m \mathbf{t}^m, \quad \Delta = \mathrm{conv}(A).$$

There are several different equivalent definitions of non-degenerate hypersufaces (or non-degenerate Laurent polynomials)

Canonical Calabi-Yau varieties

Definition

A d-dimensional smooth projective normal variety X with at worst Gorenstein canonical singularities is called *canonical Calabi-Yau* variety if

- the canonical divisor K_X is trivial;
- $h^i(X, \mathcal{O}_X) = 0$ $(0 < i < d)$.

Two natural questions

The following two natural questions were considered in [B. 2017]

Questions

1. How to characterize d-dimensional lattice polytopes $\Delta \subset M_{\mathbb{R}}$ such that non-degenerate hypersurfaces Z_f with the Newton polytope Δ are birational to canonical Calabi-Yau varieties X?
2. How to compute the stringy Euler number $\chi_{\rm str}(X)$ of canonical Calabi-Yau varieties X by a combinatoria formula based on the Newton polytope Δ?

Canonical Fano polytopes

Definition

A d-dimensional lattice polytope $\Delta \subset M_{\mathbb{R}}$ is called *canonical Fano polytope*, if it contain exactly one lattice point p in its interior Δ°. For simplicity we assume that $p = 0 \in M$.

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Theorem (Khovanskiî, 1978)

The geometric genus p_g of a non-degenerate toric hypersurface Z_f defined by Laurent polynomial f with Newton polytope Δ equals $\Delta^{\circ} \cap M$. In particular, $p_g = 1$ if and only if Δ is canonical Fano polytope.

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Theorem

There exists a natural bijection between d-dimensional canonical Fano polytopes Δ up to $GL(d, \mathbb{Z})$-isomorphism and d-dimensional $\mathbb{Q}$-Gorenstein toric Fano varieties X_{Δ} with at worst canonical singularities up to isomorphism.

Canonical Fano polytopes

For any fixed dimension d there exist only finitely many d-dimensional canonical Fano polytopes up to a $GL(d, \mathbb{Z})$-isomorphism.

- There exists exactly one canonical Fano polytope of dimension 1: $\Delta = [-1, 1]$.
- There exist exactly 16 canonical Fano polytopes of dimension 2.
- There exist exactly $674,688$ three-dimensional canonical Fano polytopes (Kasprzyk, 2010)
- The complete list of all 4-dimensional canonical Fano polytopes is still unknown.

Combinatorial duality

Denote $N := \mathrm{Hom}(M, \mathbb{Z})$, $M_{\mathbb{R}} := M \otimes \mathbb{R}$, $N_{\mathbb{R}} := N \otimes \mathbb{R}$, and

$$\langle *, * \rangle \;:\; M_{\mathbb{R}} \times N_{\mathbb{R}} \to \mathbb{R}$$

the natural pairing.

Definition

A d-dimensional canonical Fano polytope $\Delta \subset M_{\mathbb{R}}$ is called *reflexive* if the *polar dual* polytope

$$\Delta^* := \{ y \in N_{\mathbb{R}} \;:\; \langle x, y \rangle \geqslant -1, \;\; \forall x \in \Delta \}$$

is also a canonical Fano polytope.

If Δ is reflexive, then Δ^* is also reflexive and $(\Delta^*)^* = \Delta$. This duality perfectly agrees with Mirror Symmetry. There exists a natural 1-to-1 correspondence between k-dimensional faces $\theta \prec \Delta$ and $(d - k - 1)$-dimensional faces $\theta^* \prec \Delta^*$

The Hodge numbers of two d-dimensional smooth Calabi-Yau varieties V and V^* that are mirror symmetric to each other must satisfy the equalities

$$h^{p,q}(V) = h^{d-p,q}(V^*)$$

for all p, q $(0 \leqslant p, q \leqslant d)$. In particular, the Euler number $\chi = \sum_{p,q}(-1)^{p+q}h^{p,q}$ must satisfy the equality

$$\chi(V) = (-1)^d \chi(V^*).$$

Combinatorial formula

Theorem (B., Dais 1994)

Let Δ be a d-dimensional reflexive polytope. Then the stringy Euler number of a general CY hypersurface $X \subset \mathbb{P}_\Delta$ equals

$$\chi_{\rm str}(X) = \sum_{k=1}^{d-2} (-1)^{k-1} \sum_{\theta \prec \Delta \,:\, \dim(\theta)=k} Vol_k(\theta) \cdot Vol_{d-k-1}(\theta^*).$$

If $X^* \subset \mathbb{P}^*_\Delta$ be the CY hypersurface corresponding to the dual polytope Δ^*, then

$$\chi(X) = (-1)^{d-1}\chi(X^*).$$

Quasi-smooth Calabi-Yau hypersurfaces

Definition

A Calabi-Yau hypersurface $X_w \subset \mathbb{P}(w_0, w_1, \ldots, w_d)$ defined by a weighted homogeneous polynomial W of degree $w = \sum_{i=0}^{d} w_i$ is called quasi-smooth if the partial derivatives $\partial W / \partial z_i$ $(0 \leqslant i \leqslant d)$ form a regular sequence in $\mathbb{C}[z_0, z_1, \ldots, z_d]$.

Remark

The weighted projective space $\mathbb{P}(w_0, w_1, \ldots, w_d)$ is a Gorenstein toric Fano variety if and only if each w_i divides w. In the latter case, one can choose W in Fermat-form:

$$W = \sum_{i=0}^{d} z_i^{w/w_i}.$$

Quasi-smooth Calabi-Yau hypersurfaces

Classifications:

- There exist exactly 95 families of quasi-smooth Calabi-Yau 2-folds (K3-surfaces) in $\mathbb{P}(w_0, w_1, w_2, w_3)$; (M. Reid, 1979)
- There exist exactly 7555 families of quasi-smooth Calabi-Yau 3-folds in $\mathbb{P}(w_0, w_1, w_2, w_3, w_4)$; (Kreuzer, Skarke 1998)
- There exist exactly $1,100,055$ families of quasi-smooth Calabi-Yau 4-folds in $\mathbb{P}(w_0, w_1, w_2, w_3, w_4, w_5)$ (Lynker, Schimmrigk, Wisskirchen 1998), (Brown, Kasprzyk 2015)

Vafa's formula

Vafa's formula

$$\chi_{\mathrm{orb}}(X_w) = \frac{1}{w} \sum_{l,r=0}^{w-1} \prod_{0 \leqslant i \leqslant d \,:\, lq_i, rq_i \in \mathbb{Z}} \left(1 - \frac{1}{q_i}\right).$$

In this formula, one denotes $q_i := \frac{w_i}{w}$ $(0 \leqslant i \leqslant d)$, and one assumes

$$\prod_{0 \leqslant i \leqslant d \,:\, lq_i, rq_i \in \mathbb{Z}} \left(1 - \frac{1}{q_i}\right) = 1$$

if $lq_i, rq_i \notin \mathbb{Z}$ for all $i \in \{0, \ldots, d\}$.

Orbifold Euler number

Theorem, Ono-Roan 1993

Let $S^{2d+1} \subseteq \mathbb{C}^{d+1} \setminus \{0\}$ be the unit sphere. Consider the compact smooth $(2d-1)$-dimensional real manifold $S_w := S^{2d+1} \cap \{W = 0\}$ together with the S^1-fibration $S_w \to X_w$ which is the restriction of the Seifert S^1-fibration $S^{2d+1} \to \mathbb{P}(w_0, w_1, \ldots, w_d)$ to a quasi-smooth Calabi-Yau hypersurface X_w of degree $w = \sum_{i=0}^{d} w_i$. Then the S^1-equivariant K-groups $K^i_{S^1}(S_w)$ $(i = 0, 1)$ have finite rank and

$$\operatorname{rank} K^0_{S^1}(S_w) - \operatorname{rank} K^1_{S^1}(S_w) = \frac{1}{w} \sum_{l,r=0}^{w-1} : \prod_{0 \leqslant i \leqslant d \,:\, lq_i, rq_i \in \mathbb{Z}} \left(1 - \frac{1}{q_i}\right).$$

In particular, the last number is an integer.

The Laurent polynomial of Hori-Vafa

Consider d-dimensional algebraic torus:

$$\mathbb{T}_w^d := \{(x_0, x_1, \ldots, x_d) \in (\mathbb{C}^*)^{d+1} \mid \prod_{i=0}^{d} x_i^{w_i} = 1\} \subseteq (\mathbb{C}^*)^{d+1}$$

whose lattice of characters M_w is determined by the short exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Z}^{d+1} \to M_w \to 0,$$

where the map $\mathbb{Z} \to \mathbb{Z}^{d+1}$ sends 1 to $(w_0, w_1, \ldots, w_d) \in \mathbb{Z}^{d+1}$. If x_i $(0 \leqslant i \leqslant d)$ are standard basis of characters of $(\mathbb{C}^*)^{d+1}$, the the sum $\sum_{i=0}^{d} x_i$ is a regular function on $\mathbb{T}_w^d$, a Laurent polynomial $f_w(\mathbf{t})$ that we call Hori-Vafa polynomial of weighted projective space $\mathbb{P}(w_0, w_1, \ldots, w_d)$.

The Laurent polynomial of Hori-Vafa

Example

Consider a sequence of weights $w_0, w_1, \dots, w_d$ such that $w_0 = 1$. Then one gets a splitting of the above short exact sequence and obtain an isomorphism $M_w \cong \mathbb{Z}^d$ such that the lattice vectors $v_1, \dots, v_d \in \mathbb{Z}^d$ can be chosen as the standard $\mathbb{Z}$-basis and $v_0 = (-w_1, \dots, -w_d)$. Then the Laurent polynomial $f_w \in \mathbb{C}[t_1^{\pm 1}, \dots, t_d^{\pm 1}]$ has the form

$$f_w(\mathbf{t}) = \sum_{i=0}^{d} \mathbf{t}^{v_i} = \frac{1}{t_1^{w_1} \cdots t_d^{w_d}} + t_1 + \cdots + t_d.$$

The Laurent polynomial of Hori-Vafa

Example

If all weights w_i are equal 1, we obtain the well-known polynomial

$$f(\mathbf{t}) = \frac{1}{t_1 \cdots t_d} + t_1 + \cdots + t_d$$

for usual d-dimensional projective space.

This Laurent polynomial describes LG-mirror of $\mathbb{P}^d$.

The Newton polytope of $f_w(\mathbf{t})$

The Newton polytope of the Hori-Vafa polynomial $f_w(\mathbf{t})$ is the lattice simplex Δ_w with lattice vertices $v_0, v_1, \ldots, v_d \in M_w$ generating the lattice M_w and satisfying the relation

$$\sum_{i=0}^{d} w_i v_i = 0.$$

The origin $0 \in M$ is an interior lattice point of Δ_w. Moreover, it is easy to show that the Laurent polynomial $f_w(\mathbf{t})$ is non-degenerate.

Mirrors of quasi-smooth Calabi-Yau hypersurfaces

Theorem (B., Schaller, 2020)

Let $(w_0, w_1, \ldots, w_d)$ be a sequence of positive integers such that there exists a quasi-smooth Calabi-Yau hypersurface $X_w \subseteq \mathbb{P}(w_0, w_1, \ldots, w_d)$. Then the affine hypersurface $Z_w \subseteq \mathbb{T}^d_w$ defined by the Laurent polynomial $f_w(\mathbf{t}) = \sum_{i=0}^{d} \mathbf{t}_i^{v}$ is birational to a $(d-1)$-dimensional Calabi-Yau variety X_w^* and one has

$$\chi_{\rm str}(X_w^*) = (-1)^{d-1} \frac{1}{w} \sum_{l,r=0}^{w-1} \prod_{0 \leqslant i \leqslant d \,:\, lq_i, rq_i \in \mathbb{Z}} \left(1 - \frac{1}{q_i}\right) = (-1)^{d-1} \chi_{\rm orb}(X_w),$$

where $q_i = \frac{w_i}{w}$ $(i \in I)$.

Mirror conjecture for quasi-smooth CY hypersurfaces

The above theorem supports the following statement:

Conjecture (B., Schaller, 2020)

Assume that a weighted projective space $\mathbb{P}(w_0, w_1, \dots, w_d)$ contains a quasi-smooth Calabi-Yau hypersurface X_w of degree $w = \sum_{i=0}^{d} w_i$. Then the affine hypersurface $Z_w \subseteq \mathbb{T}_w^d$ defined by the Laurent polynomial f_w is birational to a mirror of X_w.

Two examples of Skarke

Example

Let $X_{43} \subset \mathbb{P}(1,1,6,14,21)$. Then X_{43} not quasi-smooth, but it is birational to a smooth CY 3-fold Y with $h^{1,1}(Y) = 21$, $h^{2,1}(Y) = 273$, and $\chi(Y) = -504$.
On the other hand, the affine hypersurface $Z_{43} \subseteq (\mathbb{C}^*)^4$

$$\frac{1}{t_1 t_2^6 t_3^{14} t_4^{21}} + t_1 + t_2 + t_3 + t_4 = 0.$$

is birational to a 3-dimensional Calabi-Yau variety X_{43}^* with the stringy Euler number

$$\chi_{\mathrm{str}}(X_{43}^*) = 506 \neq 504 = -\chi_{\mathrm{str}}(X_{43}) = -\chi(Y).$$

Therefore, X_{43}^* is not a mirror of X_{43} or Y.

Two examples of Skarke

Example

The Newton polytope of a general hypersurface $X_{13} \subseteq \mathbb{P}(1,1,2,4,5)$ is reflexive, but this CY-hypersurface is not quasi-smooth. One can show that the affine hypersurface $Z_{43} \subseteq (\mathbb{C}^*)^4$

$$\frac{1}{t_1 t_2^2 t_3^4 t_4^5} + t_1 + t_2 + t_3 + t_4 = 0.$$

is birational to a 3-dimensional Calabi-Yau variety X_{13}^* with the stringy Euler number

$$\chi_{\rm str}(X_{13}^*) = \frac{1032}{5} \notin \mathbb{Z}.$$

Therefore, X_{13}^* can not be a mirror of X_{13}.