Signatures of Chaos and the structure of eigenstates in holography

Julian Sonner

Moscow via Zoom

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Contents

Setting the Stage

AdS/CFT relates gravity (often in AdS) to unitary field theory (often CFT)

Goal: use AdS/CFT to investigate the physics of quantum BHs

Advantages: recent progress in low-D models (AdS_3/CFT_2, SYK,…)
Analytical and numerical control leads to conceptual insights: RMT/ETH type ideas linked to bulk physics

Disadvantages: we are not in flat space, and yet we are in low-D; however, analogs exist for many interesting questions.

Thermalization ➛ BH formation (& evaporation)

The models

AdS_3/CFT_2 at large central charge

Large-c sparse-spectrum (…) theories

Virasoro identity block (~ classical gravity) receives perturbative and non-perturbative corrections (~ controlled by G_N)

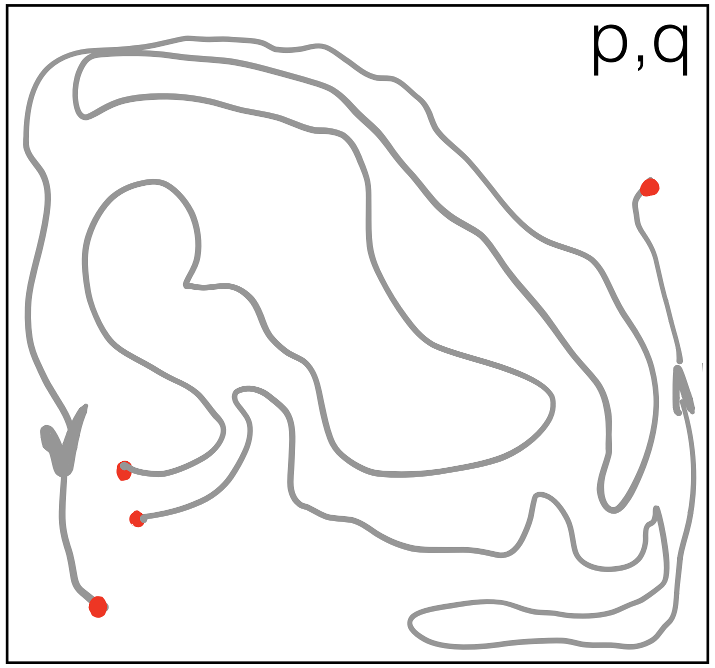
$nAdS_2/nCFT_1$ at 'large N_{SYK}' [log dim (H)]

Canonical example: SYK or similar disorder theories

Maximally chaotic quantum system governed by Diff(S^1)/SL(2) at low energies ➛ universal Schwarzian sector

Classical & quantum Lyapunov

Classical chaos: sensitivity to initial conditions



$$\frac{dq(t)}{dq(0)} = \{q(t), p(0)\} \sim e^{\lambda_L t}$$

$$\Downarrow$$

[Larkin, Ovchinnikov 1969]

$$\left\langle [q(t), p(0)]^2 \right\rangle_\beta \sim \hbar^2 e^{2\lambda_L t}$$

'out-of-time-order correlation function': $\langle W_t V W_t V \rangle_\beta \sim 1 - \hbar^2 e^{2\lambda_L t}$

(⇢ operator growth in chaotic systems)

[Maldacena, Shenker, Stanford]

Quantum thermalisation [Deutsch; Srednicki,…]

Classical thermalization: **ergodicity** & chaos

v.s.

Quantum thermalization: Eigenstate Thermalisation Hypothesis

$$\langle m|\mathcal{O}|n\rangle = \overline{\mathcal{O}}_{\mathrm{mc}}(\overline{E})\delta_{mn} + e^{-S(\overline{E})/2} f(\overline{E},\omega)R_{mn}$$

A generic excited state will then thermalise by **dephasing**

$$\langle\psi|\mathcal{O}|\psi\rangle = \sum_{i,j} c_i^* c_j e^{it(E_i-E_j)}\mathcal{O}_{ij} \longrightarrow \overline{O}\left(\bar{E}\right) + e^{-S}$$

expectation value
of non-extensive operator

dephasing:
spectral chaos

on average thermal
up to exponential in S

Comments

ETH is a modern take on spectral chaos à la Wigner/Dyson

- RMT is the canonical example of a system satisfying ETH: Wigner/Dyson spectral statistics
- Cf. integrable model: Poisson statistics, no thermalisation
- Very-late time physics (t ➛ ∞, more precisely $t \gtrsim e^N$)

Quantum Lyapunov (quantum Butterfly effect)

- Need semiclassical parameter ($\hbar$,1/N, …)
- Relevant timescale: ‘scrambling’ time

$$t_s \sim \log N$$

A priori these two notions of chaos are unrelated

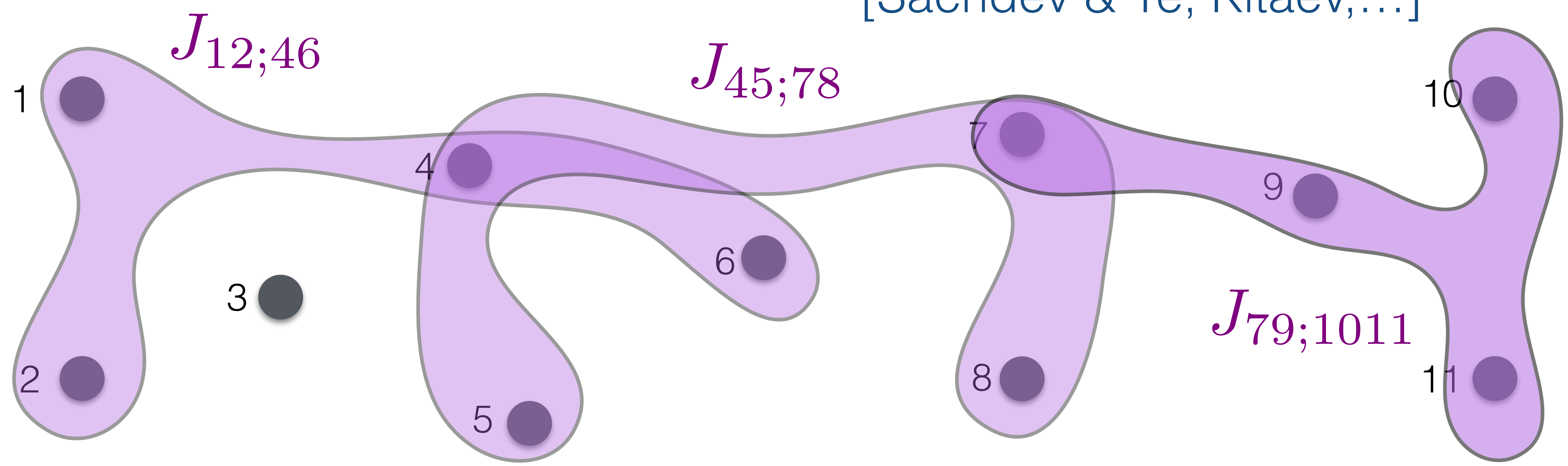
Contents

holographic lamp posts

SYK: a holographer's guinea pig

random disorder model with all-to-all couplings (complex fermions)

[Sachdev & Ye, Kitaev,…]



$$H = \frac{1}{(2N)^{3/2}} \sum_{i,j,k,l=1}^{N} J_{ij;kl} c_i^\dagger c_j^\dagger c_k c_l$$

Couplings $J_{ij;kl}$ are drawn from a Gaussian random distribution with $\overline{J_{ij;kl}} = 0$ and $\overline{|J_{ij;kl}|^2} = J^2$

solvable in large-N expansion and @ finite N (exact diagonalization)

bulk dual includes a gravity sector (“The Schwarzian”)

numerical results

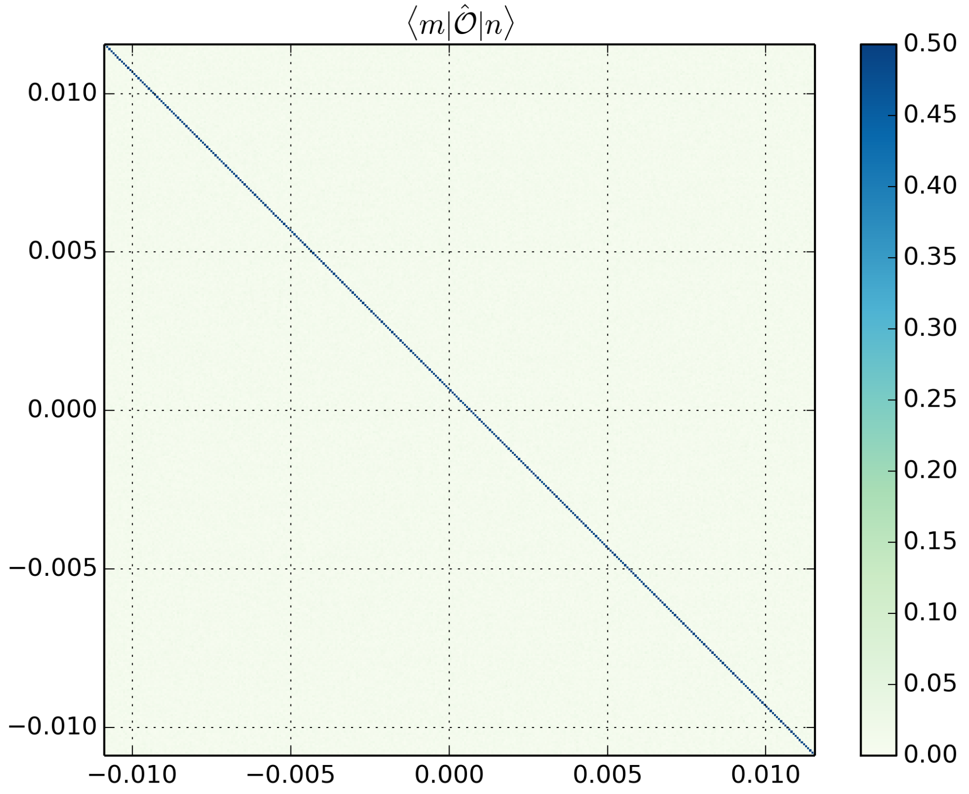
Finite N - results on eigenstates

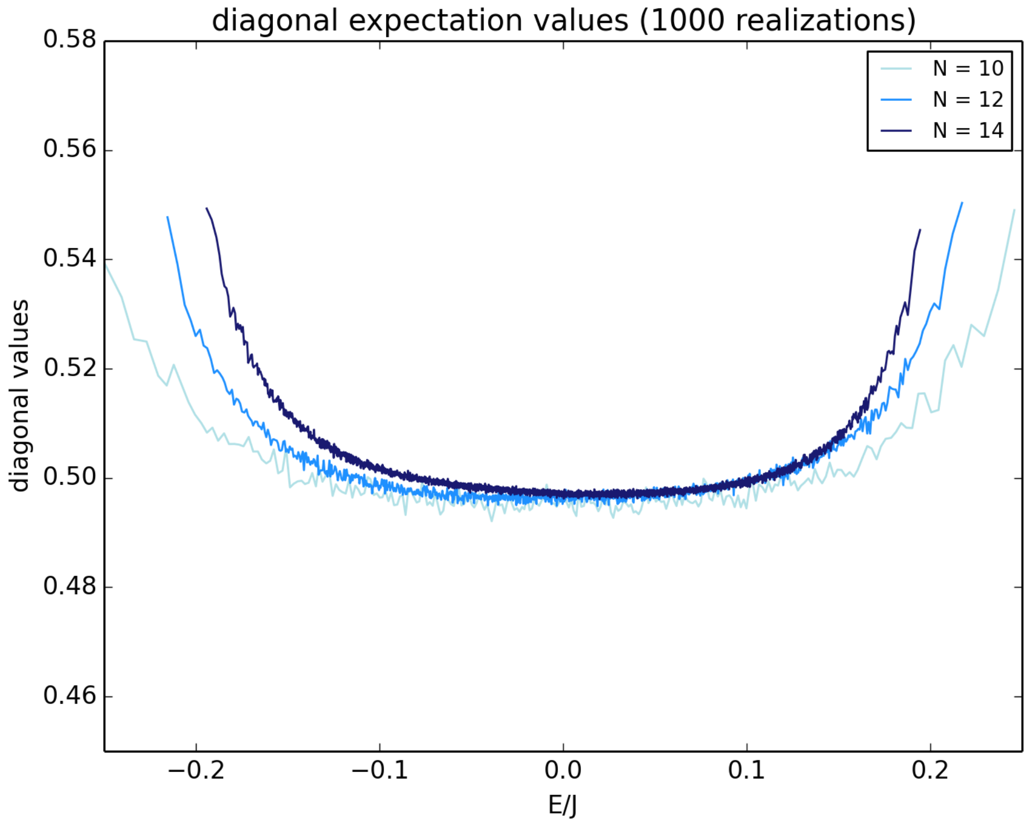
Solve SYK in exact diagonalization [Fu, Sachdev; JS & Vielma,…]

We find (numerically) that indeed ETH applies in SYK [JS & Vielma]

$$\boxed{\langle m|\mathcal{O}|n\rangle = \overline{\mathcal{O}}_{\mathrm{mc}}(\overline{E})\delta_{mn} + e^{-S(\overline{E})/2} f(\overline{E},\omega) R_{mn}}$$

$$\mathcal{O} = \hat{n}_k \quad (\text{for some site} \quad k)$$



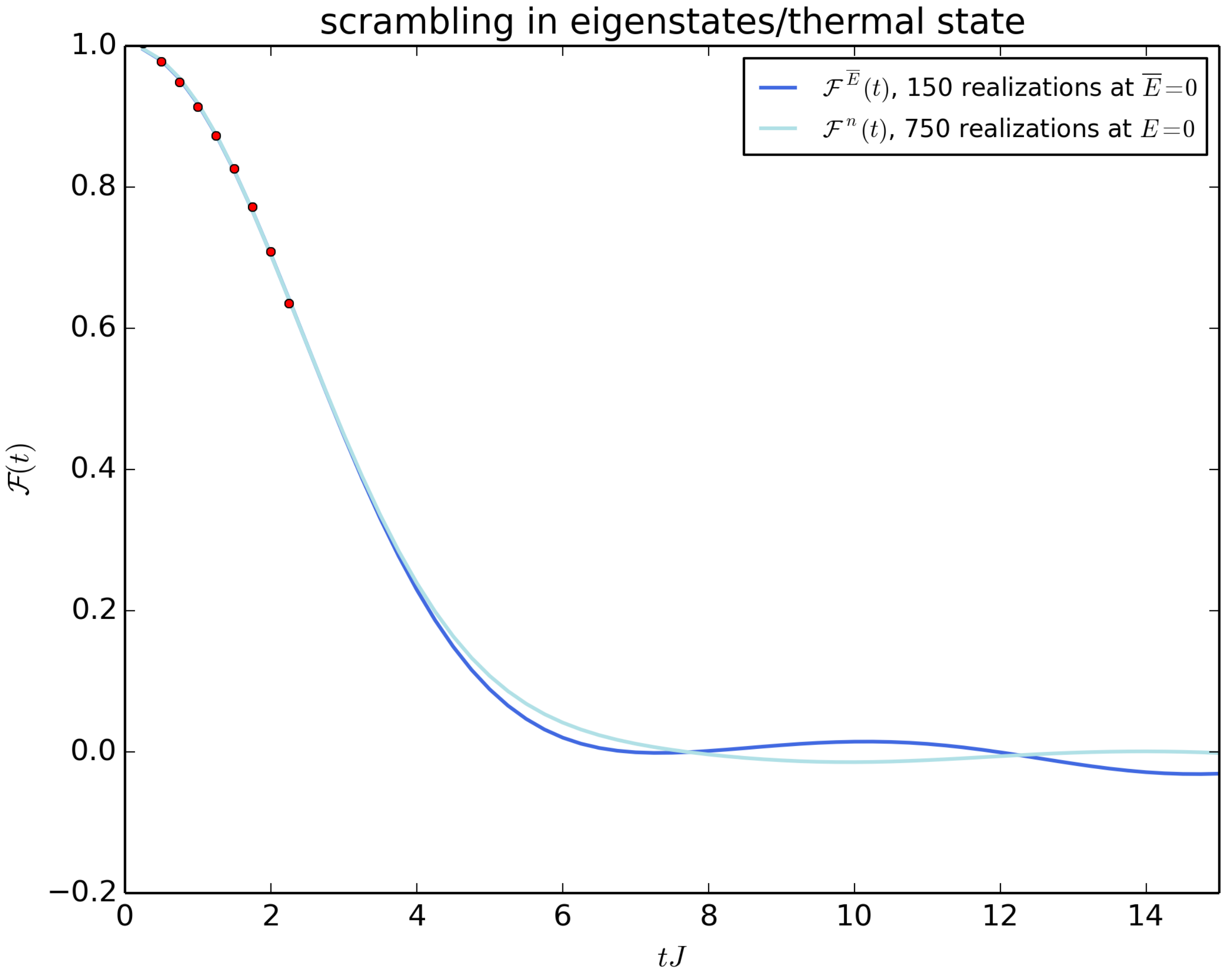


Off-diagonals: entropically suppressed with deviations from RMT (Thouless physics) [also: Garcia-Garcia & Verbaarschot; Altlands, Bagrets]

Scrambling in eigenstates

[JS & M. Vielma, 2017]

Compare OTOC in eigenstates to thermal result



OTOC indistinguishable from thermal up to scrambling time:

Conjecture: $\exists\ \lambda_{\mathrm{L}}^{\mathrm{ETH}} = \dfrac{2\pi}{\beta(E)}$

From states to operators

[Parker, Cao, Avdoshkin, Scaffidi, Altman, 2019]

Scrambling tied to growth of Heisenberg operator $e^{iHt}\mathcal{O}e^{-iHt}$

Size of operator ⇄ Complexity

Krylov complexity: measure size in a canonical basis:

$$|\mathcal{O}_n) \propto |\,[H,[H,[\dots\,,\mathcal{O}])$$

With respect to the inner product $(\mathcal{O}_1|\mathcal{O}_2) = \mathrm{Tr}\left[\mathcal{O}_1^\dagger \mathcal{O}_2\right]$

Orthonormalizing $|\mathcal{O}_n)$ $\implies$ Lanczos coefficients b_n

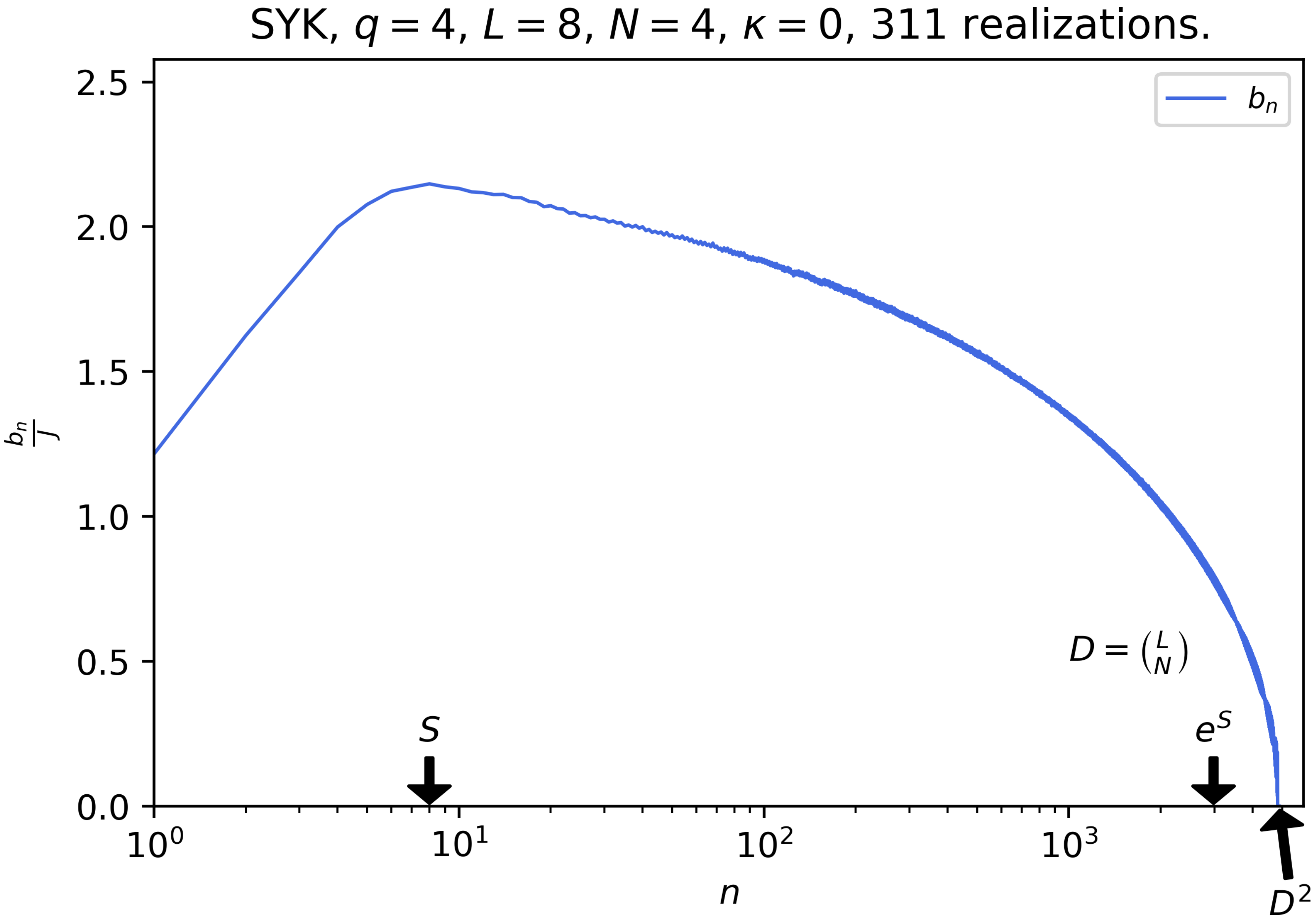
Conjecture: $b_n \sim \alpha n$ in chaotic systems with $\lambda_L \leq 2\alpha$

(equality for maximally chaotic systems)

Operator Chaos in SYK

[JS, , E Rabinovici, A. Sanchez-Garrido, R. Shir]

Lanczos sequence in ED for SYK:



In thermodynamic limit: $b_n \sim \mathcal{J} n$ [Parker et al.]

But this misses (by construction) the majority of post-scrambling dynamics

analytical results

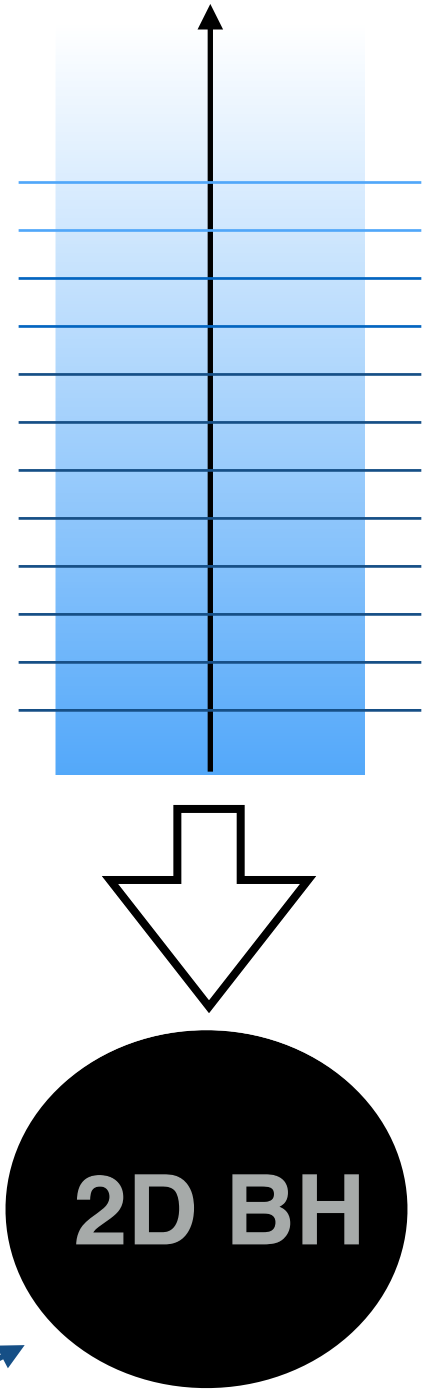
Spectrum & methods @ large N

[P. Nayak, JS & Vielma]

continuum: Schwarzian

$$S = -C \int_0^{\beta} \left\{ \tan\left(\frac{\pi\phi}{\beta}\right), \tau \right\} d\tau$$

$$\phi \in \mathrm{Diff}\left(S^1\right)$$



discrete tower of states

$$\mathcal{O}_n \sim c_i^\dagger \partial^{2n+1} c_i$$

$$h_n = 2\Delta + 1 + 2n + \epsilon_m$$

Schwarzian
~ monodromy method

Liouville between branes

Limit of conformal
six-pt functions
OPE coeffs.

ETH

ETH (?)

Large - N: the conformal sector

Theory of the $\mathcal{O}_n$ operators is a CFT [Gross, Rosenhaus]

Thus we can relate statements about ETH to 3-pt function

$$\langle \mathcal{O}_n \mathcal{O}_k \mathcal{O}_m \rangle \sim \langle E_n | \mathcal{O}_k | E_m \rangle := c_{nkm}$$

The OPE coefficients satisfy

$$\boxed{c_{nkm} = \overline{\mathcal{O}(\bar{E})}\delta_{mn} + \mathcal{O}(e^{-S})}$$

in the limit $n, m \gg k \sim \mathcal{O}(1)$ ("middle-weight operators")

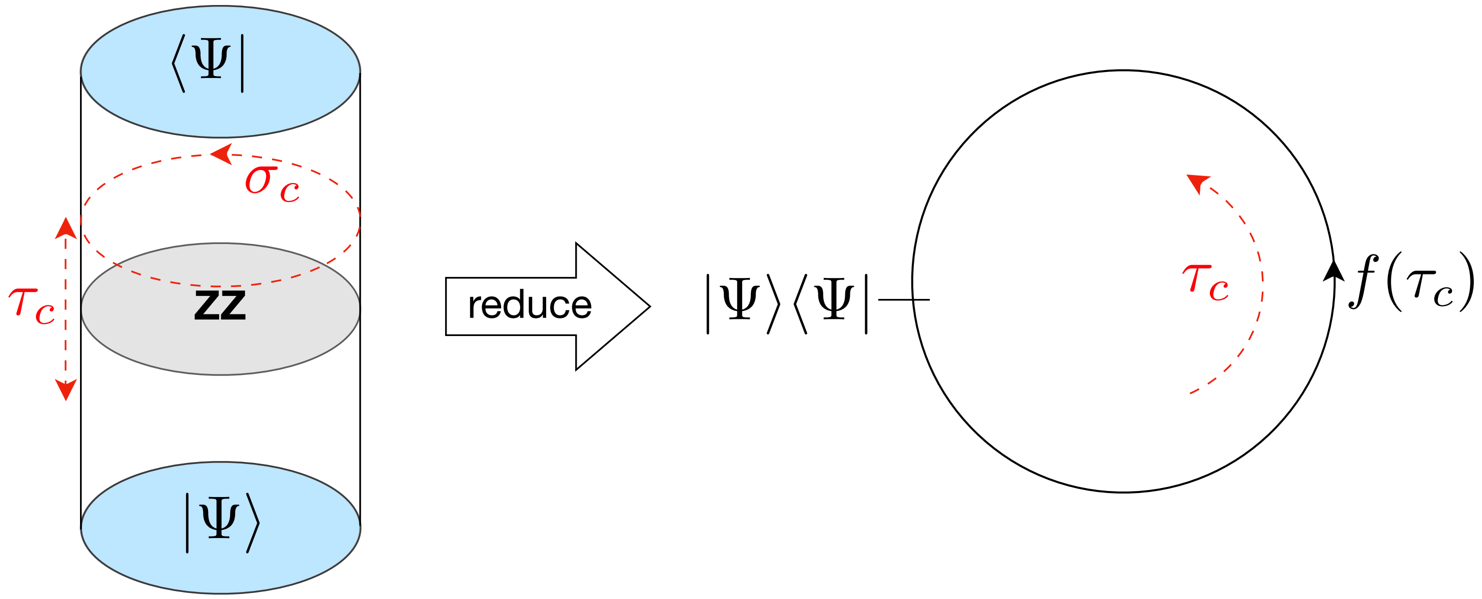
[P. Nayak, JS & Vielma]

Large - N: the Schwarzian sector

At low-energy break conformal symmetry. Leading soft-mode physics

$$\int \mathscr{D}\mu(f)e^{C\int \{f(\tau),\tau\}d\tau}$$

We can construct this path integral by reducing from 2D



2D boundary
Liouville

1D Schwarzian
path integral

$$\boxed{\begin{aligned}&\mathrm{tr}\left[(\,\cdot\,)\,|\Psi\rangle\langle\Psi|\right]\\ &\sim \mathrm{tr}\left[(\,\cdot\,)\,\rho_{\beta_\Psi}\right]+e^{-C}\end{aligned}}$$

[LMTV; Nayak, JS, Vielma]

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[P. Nayak, JS & Vielma]

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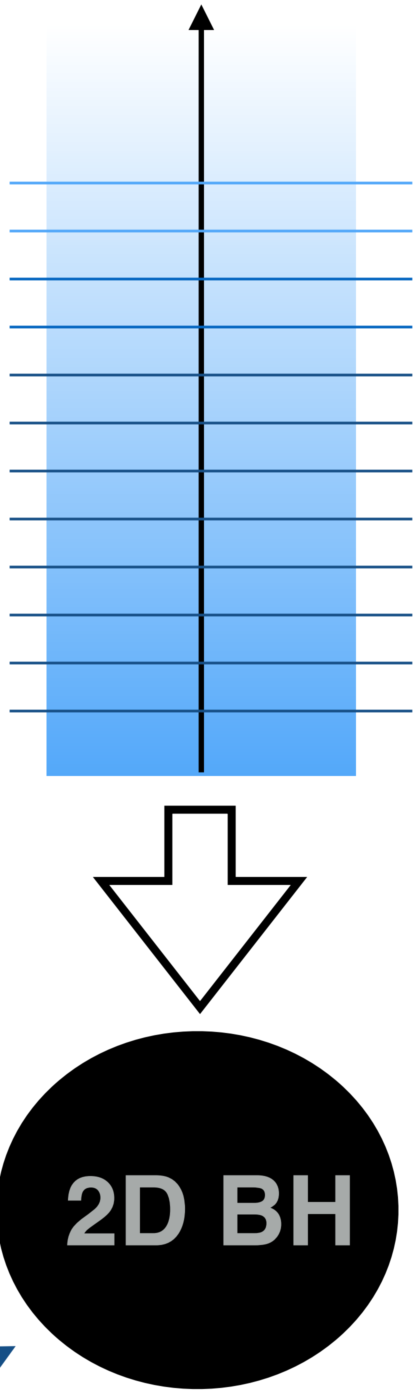

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$$\mathcal{O}_n \sim c_i^\dagger \partial^{2n+1} c_i$$

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Schwarzian
~ monodromy method

Liouville between branes

Limit of conformal
six-pt functions
OPE coeffs.

ETH

ETH (?)

Contents

1. Introduction

context and motivation;

2. Holographic ETH I: eigenstates in SYK

results from ED & at large-N; scrambling in eigenstates

3. Holographic ETH II: eigenstates in CFT$_2$ & Schwarzian

thermalisation & scrambling in eigenstates; extended ETH

Working with eigenstates in CFT$_2$ [T Anous, JS]

Let us now consider our second holographic lamp post:
sparse-spectrum CFT$_2$ [HKS,…]

Insert a primary operator at the origin

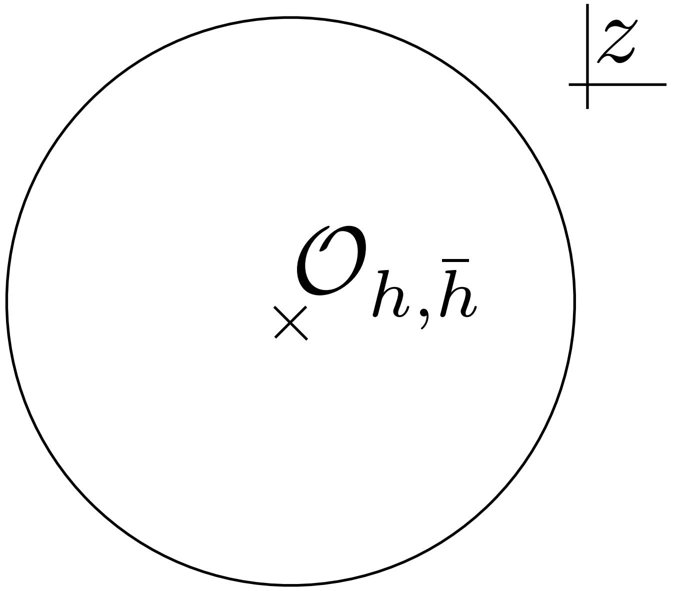
⇩

`prepare eigenstate`
`on cylinder`

⇩

This allows us to compute

$$\boxed{\langle h,\bar{h}|\mathcal{Q}|h'\overline{h'}\rangle = \langle n|\mathcal{Q}|n'\rangle}$$



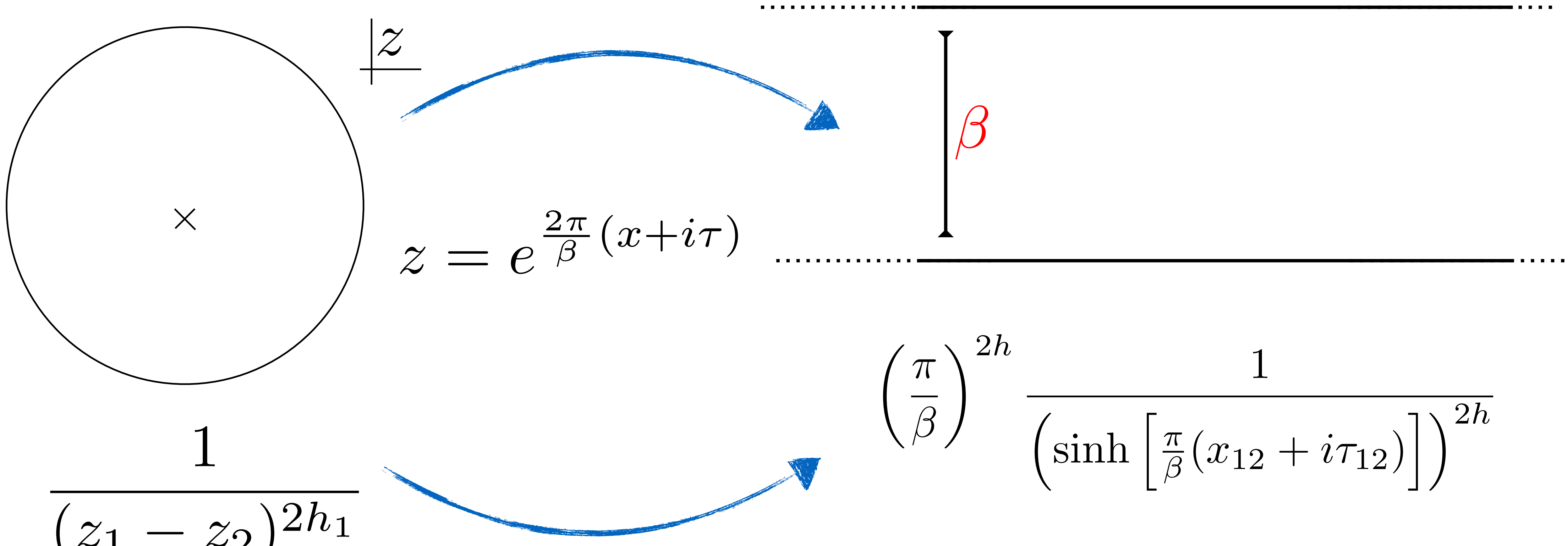
$$|h,\bar{h}\rangle := \mathcal{O}_{h,\bar{h}}(0)|0\rangle$$

A little Cardy-ology

Conformal correlators obey tensor transformation laws

$$\langle \mathcal{O}(z_1)\mathcal{O}(z_2)\cdots\rangle = (w')^{h_1}\,(w')^{h_2}\cdots\langle \mathcal{O}(w(z_1))\mathcal{O}(w(z_2))\cdots\rangle$$

Can use this to map plane to periodically identified strip



$$z = e^{\frac{2\pi}{\beta}(x+i\tau)}$$

$$\frac{1}{(z_1-z_2)^{2h_1}}$$

$$\left(\frac{\pi}{\beta}\right)^{2h}\frac{1}{\left(\sinh\left[\frac{\pi}{\beta}(x_{12}+i\tau_{12})\right]\right)^{2h}}$$

= thermal 2pt function!

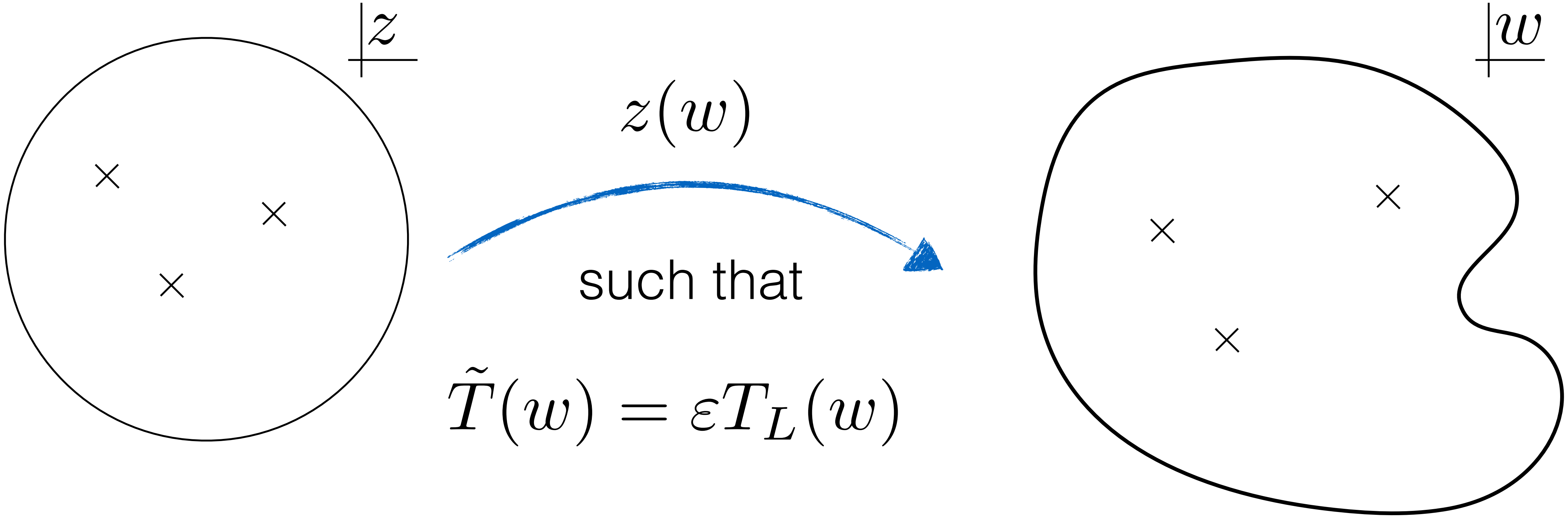
Eigenstate geometry

Schematically, for $\langle h|\mathcal{Q}(z_1)\cdots\mathcal{Q}_{n_L}|h\rangle$, we have

$$\boxed{T(z) = T_H(z) + \varepsilon T_L(z)}$$

Heavy states
that make the geometry

Light operators
(the non-extensive ones)



$z(w)$

such that

$$\tilde{T}(w) = \varepsilon T_L(w)$$

Idea: Casimir energy of w-geometry takes care of heavy states

Thermal eigenstates

[T Anous, JS]

heavy states approximate thermal ensemble up to exponential corrections

$$\langle E|\mathcal{Q}_1(t_1)\cdots\mathcal{Q}_{n_L}(t_{n_L})|E\rangle = \mathrm{Tr}\left[e^{-\beta H}\mathcal{Q}_1(t_1)\cdots\mathcal{Q}_{n_L}(t_{n_L})\right] + \mathcal{O}\left(e^{-c}\right)$$

True for id domination

$$|\alpha| = \sqrt{\frac{24h}{c} - 1} =: \frac{2\pi}{\beta}$$

Non-id contributions get corrected

By applying this to a basis of light operators we have the statement

$$\mathrm{Tr}_{\mathcal{H}_{\mathcal{Q}^c}}|E\rangle\langle E| = \mathrm{Tr}_{\mathcal{H}_{\mathcal{Q}^c}}e^{-\beta H} + \mathcal{O}(e^{-c})$$

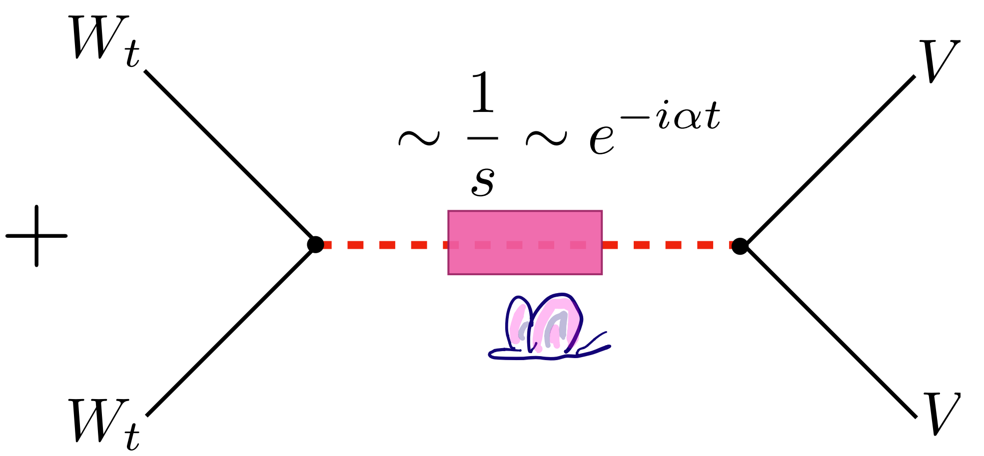
2D holographic CFT satisfy an extended version of ETH

Quantum butterfly

[T Anous, JS]

Corollary: 4-pt OTOC of $\mathcal{Q}$ ⇨ given by single “papillon” exchange:

$$\langle \mathcal{O}\, W_t\, V\, W_t\, V\, \mathcal{O} \rangle \sim 1 +$$



$$1 - \frac{\#}{N} e^{\lambda_h t}$$

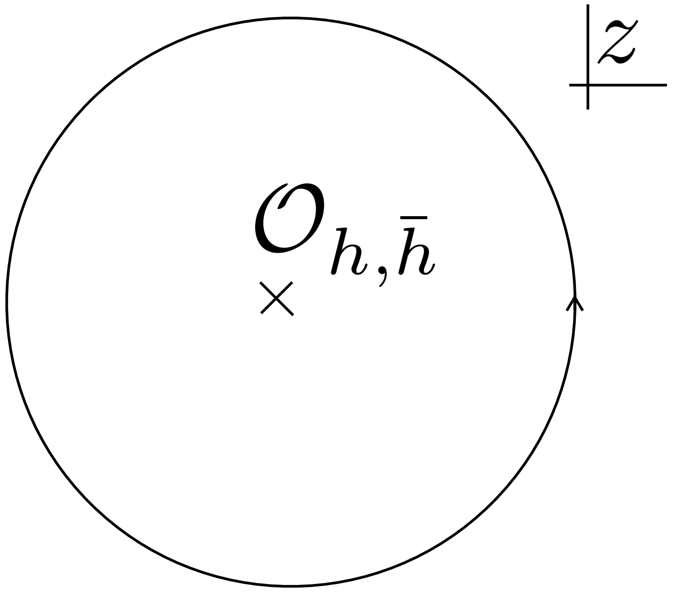
With quantum Lyapunov exponent $$\lambda_h(E) = \frac{2\pi}{\beta} = |\alpha|$$

There is a phase transition to a non-ergodic phase with **oscillatory OTOC**

Phases of scrambling [T Anous, JS]

CFT$_2$ & SYK: Universal description in terms of Diff(S^1) [Nayak, JS, Vielma]

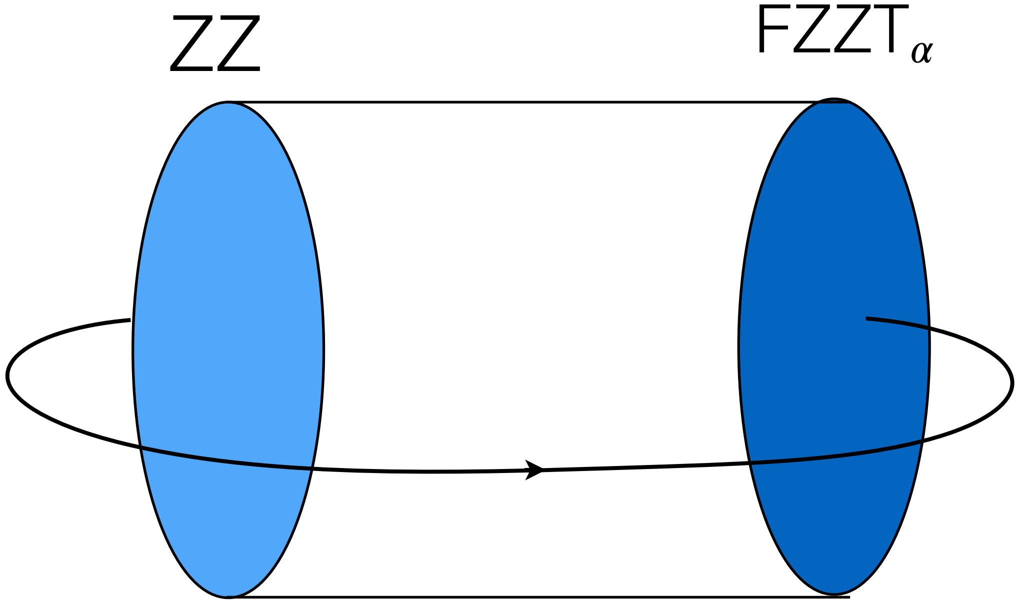
Sparse CFT$_2$



$$\mathrm{Tr} M_{S^1} = -2\cosh \pi\alpha$$

$$\langle E_\alpha | \cdots | E_\alpha \rangle$$

Schwarzian



$$\mathrm{Tr} M_{S^1} = -2\cosh \pi\alpha$$

$$\langle\langle E_\alpha | \cdots | E_\alpha \rangle\rangle$$

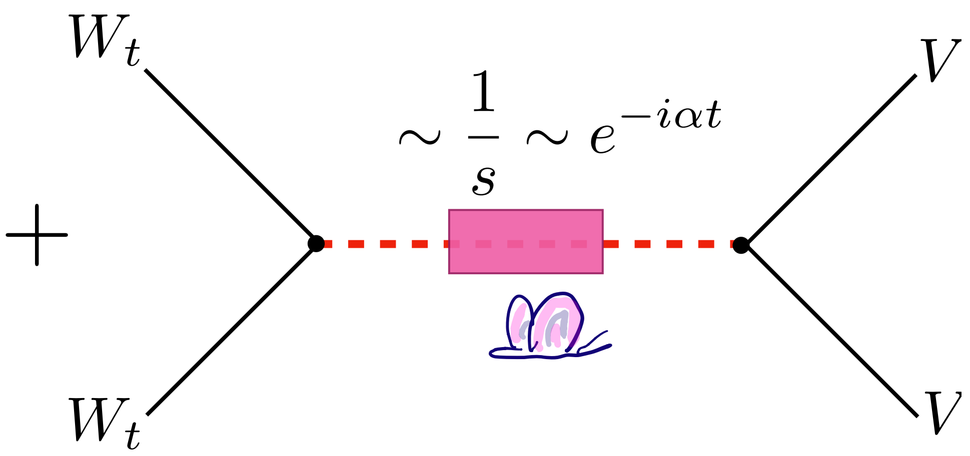
Monodromy classified according to coadjoint orbits of Diff(S^1)

Quantum butterfly transition

[T Anous, JS]

[P Nayak, JS, M Vielma]

$$\left.\begin{array}{r}\langle E_\alpha|W_tVW_tV|E_\alpha\rangle \\ \langle\langle E_\alpha|W_tVW_tV|E_\alpha\rangle\rangle\end{array}\right\} \sim \ 1 \ +$$



Hyperbolic orbit (**α** ∊ iℝ): chaotic “papillon” exchange

with maximal Lyapunov $\lambda_E = \dfrac{2\pi}{\beta(E)}$

Parabolic orbit: universal critical transition?

Elliptic orbit (**α** ∊ ℝ): non-ergodic oscillatory OTOC

Complexity growth [A Saha, JS, M Vielma, M Visser]

Similar ideas allow one to compute the growth rate of the Lanczos sequence.

Start with moments of Liouvillian superoperator:

$$\mu_{2n} := (\mathcal{O}|\mathcal{L}^{2n}|\mathcal{O})$$

2n commutators with H

Then a recursive algorithm exists, based on the relation

$$b_1^2 \cdots b_n^2 = \det(\mu_{i+j})_{0\leq i,j\leq n}$$

For large-c, sparse spectrum 2D CFT:

$$b_n \sim \pi T \sqrt{n(n-1)}$$

saturation: $2\lambda_L = \alpha$

Summary

SYK & CFT$_2$: solvable models of strongly-correlated many-body systems. Interesting in their own right & have a bulk dual

Correlations in individual eigenstates are exponentially close to thermal ones, with non-perturbative corrections → extended ETH

(At subleading order find GGE for primary state) [Dymarski et al. Maloney et al.]

Eigenstate scramble as fast as the thermal ensemble
→ e.g.typical states contain (much of) BH interior

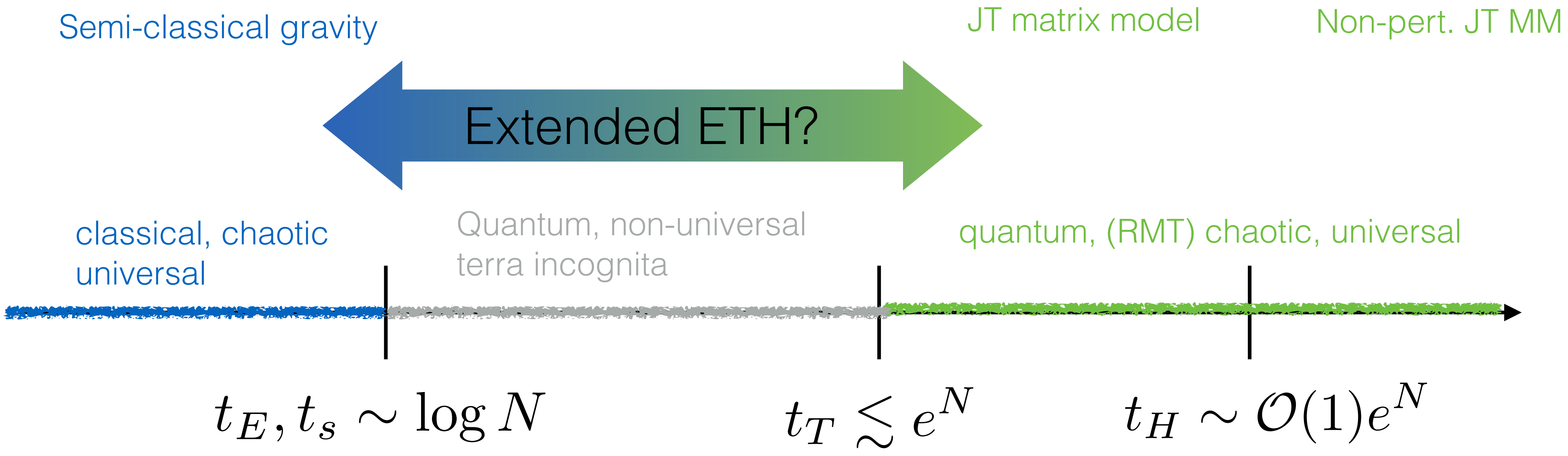
Oscillatory OTOC → non-ergodic phases [de Boer et al.]

Krylov complexity growth another measure of maximal chaos

Timescales?

Chaotic time scales

level spacing $\Delta \sim (\mathrm{dim}\mathcal{H})^{-1} \sim e^{-N}$



Semi-classical gravity

JT matrix model

Non-pert. JT MM

Extended ETH?

classical, chaotic
universal

Quantum, non-universal
terra incognita

quantum, (RMT) chaotic, universal

$t_E, t_s \sim \log N$

$t_T \lesssim e^N$

$t_H \sim \mathcal{O}(1)e^N$

t_s	Scrambling or ‘Lyapunov’ time: OTOC decay
t_E	Ehrenfest time: classical to quantum crossover
t_T	Thouless time: system becomes ergodic (RMT)
t_H	Heisenberg time (plateau): system becomes non-perturbative in e^N

thank you for your attention

Large - N

[Sachdev; Parcollet & Georges; Kitaev; Jevicki, Suzuki, Yoon; Polchinski & Rosenhaus,…]

at large-N, it is often useful to use bi-local effective action
(q-body version of the model)

$$\frac{S}{N} = -\frac{1}{2}\mathrm{Tr}\log\left(\partial_\tau - \Sigma_{\tau,\tau'}\right) + \frac{1}{2}\int d\tau d\tau' \left[\Sigma_{\tau,\tau'}G_{\tau,\tau'} - \frac{J^2}{q^2}G^q_{\tau,\tau'}\right]$$

we have introduced two Hubbard-Stratonovich fields, **Σ**, and G

deep IR is conformal, broken by ***β***J correction.

$$S = -C\int \{f(\tau), t\}\, d\tau \quad \rightleftarrows \quad \text{Liouville QM} \quad f'(\tau) = e^{\phi(\tau)}$$

Schwarzian action; reformulate as Liouville quantum mechanics

Large - N: the conformal sector

Theory of the $\mathcal{O}_n$ operators is a CFT (with GFT limit) [Gross, Rosenhaus]

Thus we can relate statements about ETH to 3-pt function

$$\langle \mathcal{O}_n \mathcal{O}_k \mathcal{O}_m \rangle \sim \langle E_n | \mathcal{O}_k | E_m \rangle := c_{nkm}$$

The OPE coefficients should then satisfy

$$\boxed{c_{nkm} = \overline{\mathcal{O}(\bar{E})} \delta_{mn} + \mathcal{O}(e^{-S})}$$

in the limit $n, m \gg k \sim \mathcal{O}(1)$ (“middle-weight operators”)

ETH in conformal sector

[Nayak, JS, Vielma]

OPE coefficients can be deduced from fermion 6-pt functions

In the middleweight limit planar diagrams dominate. We find

$$\langle \mathcal{O}_m \mathcal{O}_k \mathcal{O}_n \rangle = \mathfrak{g}(E,d) \times \left[2^{-2(2E-1)} \sqrt{2\pi E} \frac{\Gamma(4E-1)}{\Gamma(2E-d)\Gamma(2E+d)} \right]$$

E is average energy and d is the difference. In the relevant limit (E ➛ ∞):

$$\boxed{c_{nkm} \longrightarrow f_k(E)\delta_{m,n} + e^{-E\ln 2} R(E,d)}$$

smooth function of E

entropic suppression

conformal sector of SYK satisfies ETH