Complexity and Conformal Field Theory

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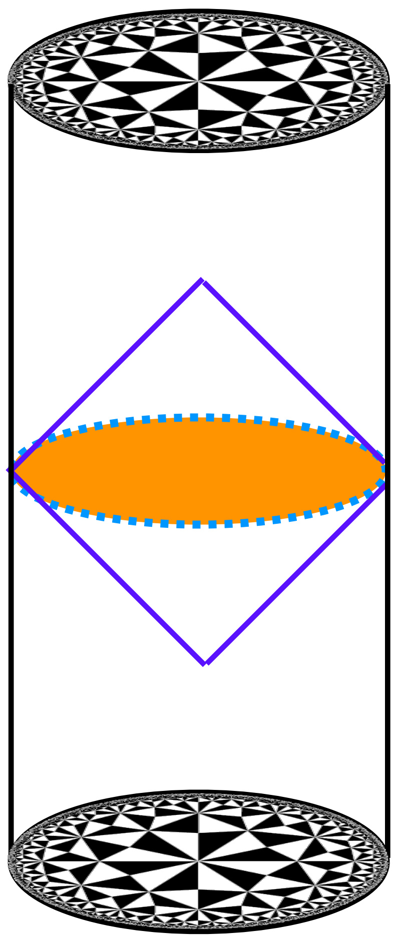
based on **2005.02415** with Mario Flory

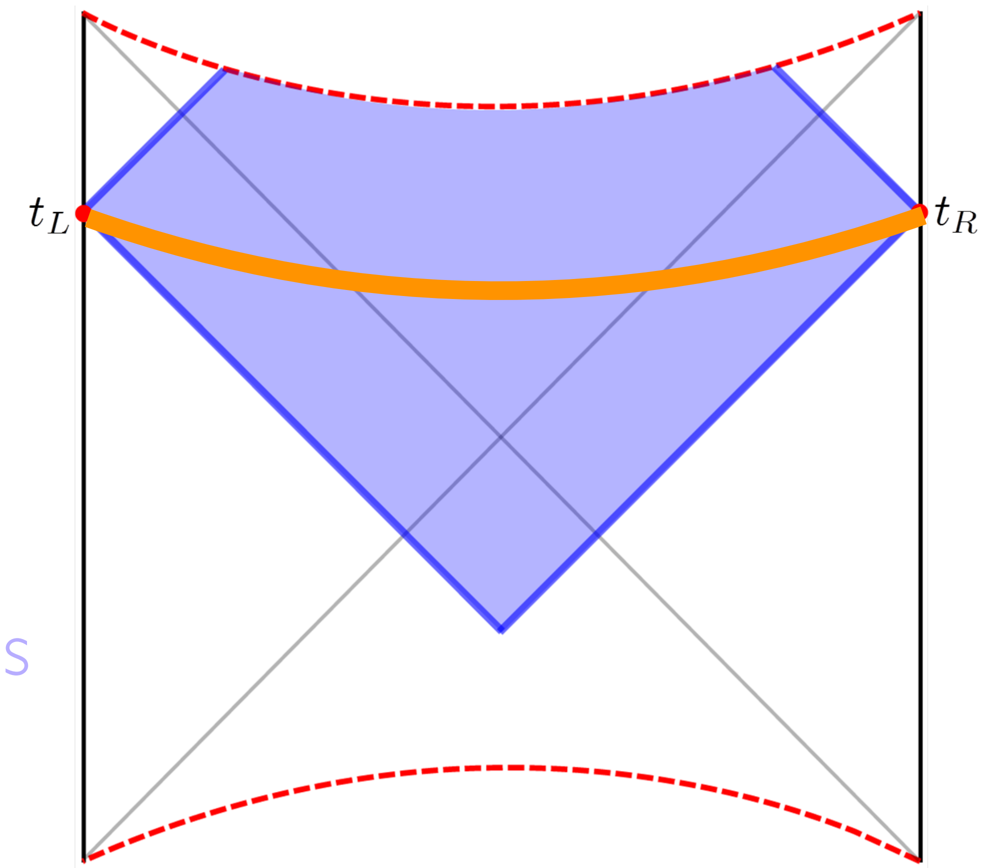
Holographic complexity proposals

1402.5674 by Susskind, …, 1509.07876 by Brown et al., …



$\mathcal{C}_V \sim$ Volume of codim-1 max volume bulk slice

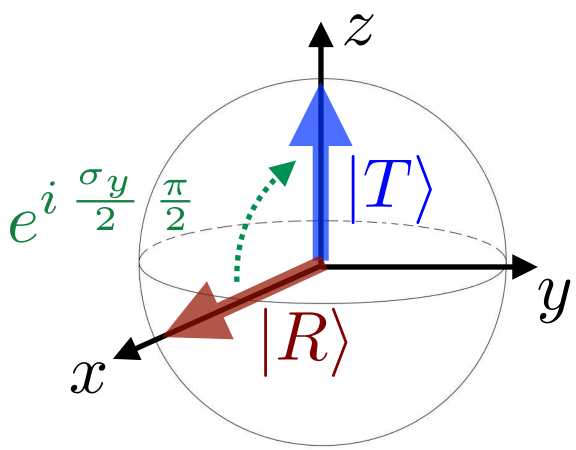
$\mathcal{C}_A \sim$ Action in codim-0 bulk region with null bdries



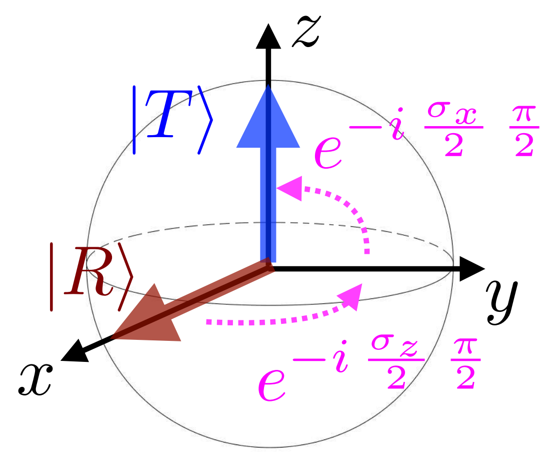
What do holography complexity proposals stand for in hQFT?

How to define complexity in QFT?

Complexity is concerns hardness of defining an operator U or a state $|T\rangle$ from a simpler state $|R\rangle$ using restricted resources, e.g. 1903.12621 by Brown & Susskind :



vs. e.g.



naively: $\frac{\pi}{2} < \frac{\pi}{2} + \frac{\pi}{2}$

say y-rot is $\mathcal{O}(10)$ harder to do: $10\times\frac{\pi}{2} > \frac{\pi}{2} + \frac{\pi}{2}$

The approach that naturally applies to QFTs comes from quant-ph/0502070 by Nielsen :

$$U = \mathcal{P}e^{-i\int_0^1 d\tau\, Q(\tau)}$$

with $Q(\tau) = \sum_I O_I\, \epsilon^I(\tau)$

$$\mathcal{C}_{L_1} \sim \min\left[\int_0^1 \mathrm{d}\tau \sum_I \Pi_I\, |\epsilon^I(\tau)|\right]$$

$$\mathcal{C}_{L_2} \sim \min\left[\int_0^1 \mathrm{d}\tau \sqrt{\sum_{I,J} \Pi_{IJ}\, \epsilon^I(\tau)\, \epsilon^J(\tau)}\right]$$

$\vdots$

One can* use complexity of U to define complexity of states: $|T\rangle = U|R\rangle$

Complexity in free QFTs - punchlines

First applications to free QFTs on a lattice / cMERA regularization:

1707.08570 by Jefferson & Myers, 1707.08582, 1807.07075, 1810.05151, …

- unitary gates (for bosons): $\mathcal{O}_I \sim \phi_j \phi_l, \quad \pi_j \pi_l \quad \text{and} \quad \phi_{(j}\pi_{l)}$
- for very fine-tuned cost function, $\mathcal{C}_{\text{certain } L_2}$, one can get exact results
- circuits then also use very non-local gates, e.g. $\mathcal{O}_I = \phi_{\text{here}}\, \phi_{\text{other galaxy}}$
- U(1) phases do not matter and are ignored

Features:

- the good: one can reproduce static properties of holographic complexity proposals (leading divergences in the vacuum, complexity of formation in TFD states, complexity of purification for subregions…)
- the bad (but expected): very different time dependence than holography
- problematic: calculations do not naturally generalize to holographic QFTs

Lessons from entanglement entropy in QFT

To my taste, $\mathcal{C}_V$ and $\mathcal{C}_A$ looked a lot like calculating HRT surfaces before first works on entanglement entropy in QFT (pioneers: 1980s, explosion > 2004)

History likes to repeat itself and perhaps there is a lesson for complexity buried in the development of entanglement entropy in QFTs

1986 article by Bombelli et al. and **hep-th/9303048** by Srednicki

Pioneering works looked a lot like studies of complexity in free QFTs, but an exponential progress occurred building on the universal CFT_{1+1} result

hep-th/9403108 by Holzhey et al. and **hep-th/0405152** by Cardy & Calabrese

$$S = \frac{c}{3} \log\left(l/l_{UV}\right)$$

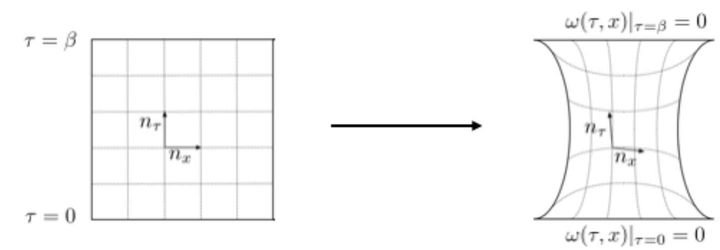
Idea behind **2005.02415** with Mario Flory and this talk: work towards such a universal result by revisiting complexity in CFT_{1+1}

Complexity in CFT_{1+1} — existing approaches

Discussed free boson/fermion results cover also massless theories, but they do not utilize specifically CFT_{1+1} structure and do not generalize to $hCFTs_{1+1}$

1703. 00456 by the Kyoto group, …

One can view the path-integral optimization



as a version of circuit complexity, but puzzles remain **1904.02713** with Camargo, Jefferson & Knaute

Remarkably, **1811.03097** by Berlin, Lewkowycz & Sarosi was able to reproduce $\mathcal{C}_V$ in some cases (also in AdS_3) using bdry sources; however, CFT understanding is limited

To move forward, in **2005.02415** with Mario Flory we decided to revisit the setups of **1807.04422** by Caputa & Magan and **2004.03619** by Erdmenger, Gerbershagen & Weigel (see Johanna's talk on April 28^{th})

Setup

1807.04422 by Caputa & Magan; 2004.03619 by Erdmenger, Gerbershagen & Weigel; 2005.02415 with Mario Flory

We consider CFT_{1+1} on a Lorentzian cylinder with spatial coord $0 \leq \sigma < 2\pi$

We look at unitary circuits realizing diffeomorphisms of S^1 in a gradual form:

$\sigma \to f(\sigma)$ via $f(\tau, \sigma)$ with $f(\tau = 0, \sigma) = \sigma$ and $f(\tau = 1, \sigma) = f(\sigma)$

Such circuits are realized by unitaries of the form $U = \mathcal{P} e^{-i \int_0^1 \mathrm{d}\tau \, Q(\tau)}$

with $Q(\tau) = \int_0^{2\pi} \frac{\mathrm{d}\sigma}{2\pi} T(\sigma) \, \dot{f}(\tau, F(\tau, \sigma))$ and $f(\tau, F(\tau, \sigma)) = \sigma$

$T(\sigma)$: right- or left-moving component of $T_{\mu\nu}$

$\dot{f}(\tau, F(\tau, \sigma))$: $\epsilon(\tau, \sigma) = - \frac{\dot{F}(\tau, \sigma)}{F'(\tau, \sigma)}$

Reminder: $Q(\tau) = \sum_I O_I \, \epsilon^I(\tau)$

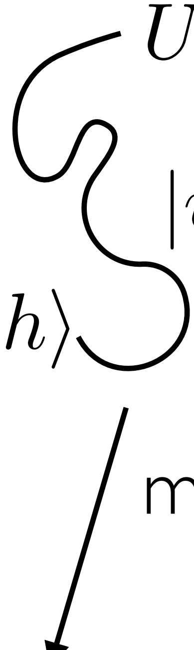
Fubini-Study metric and CFT_{1+1}

1807.04422 by Caputa & Magan; **2005.02415** with Mario Flory

From the point of view of matching with holography, it would be ideal to assign cost to circuits in a way that requires minimum of choices*

This can be realized by weighting layers of circuits via distance travelled in Hilbert space upon acting on some state **1707.08582** with Chapman, Marocchio & Pastawski :

$U|h\rangle$



$$|\psi(\tau)\rangle = \mathcal{P} e^{-i\int_0^\tau \mathrm{d}\hat{\tau}\, Q(\hat{\tau})}|h\rangle$$

$|h\rangle$

minimal traversed length: $\quad |\langle\psi(\tau)|\phi(\tau+\mathrm{d}\tau)\rangle| = 1 - G^{FS}_{\tau\tau}\mathrm{d}\tau^2 + \ldots$

$$\mathcal{C}_{FS} = \min\left[\int_0^1 \mathrm{d}\tau \sqrt{G^{FS}_{\tau\tau}}\right] = \min\left[\int_0^1 \mathrm{d}\tau \sqrt{\langle\psi(\tau)|Q(\tau)^2|\psi(\tau)\rangle - \langle\psi(\tau)|Q(\tau)|\psi(\tau)\rangle^2}\right]$$

Fubini-Study and other cost functions

2005.02415 with Mario Flory

Following 1807.04422 by Caputa & Magan we take $|h\rangle$ to be energy eigenstate on a circle:

$$\mathcal{C}_{FS} = \min\left[\int_0^1 \frac{d\tau}{2\pi}\sqrt{\iint_0^{2\pi} d\sigma d\kappa \frac{\dot f(\tau,\sigma)}{f'(\tau,\sigma)}\frac{\dot f(\tau,\kappa)}{f'(\tau,\kappa)}\Pi(\sigma-\kappa)}\right]$$

with $$\Pi(\sigma-\kappa) = \langle h|T(\sigma)T(\kappa)|h\rangle - \langle h|T(\sigma)|h\rangle\,\langle h|T(\kappa)|h\rangle = \frac{c}{32\,\sin^4\left[(\sigma-\kappa)/2\right]} - \frac{h}{2\,\sin^2\left[(\sigma-\kappa)/2\right]}$$

We can also define

$$\mathcal{C}_{FS-Nielsen} = \min\left[\int_0^1 \mathrm{d}\tau\sqrt{\langle h|Q(\tau)^2|h\rangle - \langle h|Q(\tau)|h\rangle^2}\right]$$

$$= \min\left[\int_0^1 \frac{\mathrm{d}\tau}{2\pi}\sqrt{\int\int_0^{2\pi} \mathrm{d}\sigma\mathrm{d}\kappa \frac{\dot F(\tau,\sigma)}{F'(\tau,\sigma)}\frac{\dot F(\tau,\kappa)}{F'(\tau,\kappa)}\Pi(\sigma-\kappa)}\right]$$

Virtues 2005.02415 with Mario Flory

The key asset of the cost functions we considered is they are 2nd order in ∂_τ

This is crucial (!!!) for a well posed variational (initial value) problem between 2 generic transformations $f(\tau=0,\sigma)$ and $f(\tau=1,\sigma)$

Earlier works focused predominantly on $\min \int_0^1 \mathrm{d}\tau\, \langle \psi(\tau)|Q(\tau)|\psi(\tau)\rangle$

$$\sim \min \left[\int_0^1 \mathrm{d}\tau \int_0^{2\pi} \mathrm{d}\sigma \frac{\dot{f}}{f'} \left(\mathrm{const} + \{f,\sigma\}\right) \right]$$

first order in ∂_τ (c.f. proper L_1 with $\left|\frac{\dot{f}}{f'}\right|$) U(1) phase

Also, choosing $\int_0^1 \frac{\mathrm{d}\tau}{2\pi} \sqrt{\int_0^{2\pi} \mathrm{d}\sigma \mathrm{d}\kappa \frac{\dot{F}(\tau,\sigma)}{F'(\tau,\sigma)} \frac{\dot{F}(\tau,\kappa)}{F'(\tau,\kappa)} \Big(a\,\delta(\sigma-\kappa) + b\,\delta''(\sigma-\kappa) \Big)}$

gives Camassa-Holm $a=b$, Korteweg-de Vries $b=0$ & Hunter-Saxton $a=0$ eoms

Optimal circuits and their properties

KdV, CH, HS are all well known PDEs. Therefore, we focused on Fubini-Study:

$$\mathcal{C}_{FS} = \min\left[\int_0^1 \frac{d\tau}{2\pi}\sqrt{\iint_0^{2\pi} d\sigma d\kappa \frac{\dot{f}(\tau,\sigma)}{f'(\tau,\sigma)}\frac{\dot{f}(\tau,\kappa)}{f'(\tau,\kappa)}\Pi(\sigma-\kappa)}\right]$$

It represents geodesic motion in the metric $g_{\sigma\kappa} = \dfrac{\frac{c}{32\sin^4[(\sigma-\kappa)/2]} - \frac{h}{2\sin^2[(\sigma-\kappa)/2]}}{f'(\tau,\sigma)f'(\tau,\kappa)}$

Finding optimal circuits requires solving integro-partial differential eom, e.g. for

$$f(\tau=0,\sigma) = \sigma$$
$$f(\tau=1,\sigma) = \sigma + A\sin(\sigma) \quad \text{with } A \ll 1$$

we got $f(\tau,\sigma) = \sigma + A\,\tau\,\sin(\sigma) + A^2\,\dfrac{c\tau^2 - c\tau + 20h\tau^2 - 20h\tau}{4(c+8h)}\,\sin(2\sigma) + \ldots$

1811.03097 by Berlin, Lewkowycz & Sarosi

This was seen to agree with $\mathcal{C}_V$ from 1806.08376 by Flory and Miekley

???

Outlook

$CFTs_{1+1}$ provide an invaluable lab for understanding complexity beyond free QFTs in a way that might ultimately allow to make contact with holography

In 2005.02415 with Mario Flory we introduced and explored complexity for CFT_{1+1} that

- does not assign cost predominantly to U(1) factors
- is based on a well-posed variational problem with an initial and final cond

Our work opens many interesting directions, which include

- realizing circuits as path-integrals in curved geometry specified by $f(\tau, \sigma)$
- connecting them to the path-integral optimization and holography
- detailed comparison with holographic complexity in AdS_{1+2}