

Some work of Vyacheslav V. Shokurov

Caucher Birkar



Online talk on V.V. Shokurov's 70th birthday, Steklov Institute, Moscow, May 2020

Short biography

Vyacheslav Vladimirovich Shokurov was born on 18 May 1950 in Moscow, Russia.

He attended Moscow High School No 2.

He attended a seminar at this school run by Vasily Iskovskikh.



He did undergraduate at the Faculty of Mechanics and Mathematics of Moscow State University (1967–1972).

He did PhD under Yuri Manin from (1972–1975).



His first job was at the Institute of Technical and Scientific Information (1975–1982).

He then held a position at the Yaroslav State Pedagogical University (1982–1990).

He spent one year at the Institute for Advanced Studies in Princeton (1990–1991).

Then moved to Johns Hopkins University where he has been a professor ever since (1991).

He is also a non-tenured member of the Steklov Mathematical Institute in Moscow.

Short biography

Shokurov trained 9 PhD students: Terutake Abe, Florin Ambro, Caucher Birkar, Ivan Cheltsov, Yifei Chen, Sung Rak Choi, Joseph Cutrone, Nicholas Marshburn, Jihun Park.



He is part of a bigger academic family.

Valery Alexeev
Florin Ambro
Caucher Birkar
Fedor Bogomolov
Ivan Cheltsov
Yifei Chen
Sung Rak Choi
Kiryong Chung
Joseph Cutrone
Adrien Dubouloz
Kyunik Hong
Yong Hu
Zhengyu Hu
Jun-Muk Hwang
Yewon Jeong
Seung-Jo Jung
Ilja Karzhemanov
Ludmil Katzarkov
Hosung Kim
In-Kyun Kim
Soonyoung Kim

THE SHOKUROVS

Igor Krylov
Kyoung-Seog Lee
Yongnam Lee
Qifeng Li
Jesus Martinez Garcia
Jihun Park
Jinhyun Park
Jinhyun Park
Yuri Prokhorov
Dmitrijs Sakovics
Vyacheslav Shokurov
Andrew Wilson
Joonyeong Won
Jinsong Xu

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Shokurov worked on the Noether-Enriques-Petri theorem, modular symbols, cusp forms, Kuga varieties, etc.

But I focus on his work in higher dimensional algebraic geometry, that is, birational geometry. He has played a fundamental role in shaping this field.

He is also a very kind and generous person.

Today let's celebrate his birthday together with his amazing mathematical achievements.



Overview of birational geometry

We work over \mathbb{C} .

A variety means a quasi-projective algebraic variety.

Two varieties are birational if they have isomorphic open subsets.

Birational geometry aims to classify varieties up to birational isomorphism.

This means finding "nice" elements in each birational class, and classifying them, e.g. construct their moduli spaces.

It involves lots of birational and biregular geometry.

The canonical divisor K and its variants play a central role.

Overview of birational geometry

Each one-dimensional birational class contains a unique smooth projective curve.

The **genus** of a smooth projective curve X is

$$g = h^0(K_X) = h^1(\mathcal{O}_X) = \text{number of holes in } X$$

where X is considered as a compact Riemann surface.

$$\begin{cases} g = 0 & \Leftrightarrow \deg K_X < 0 & \Leftrightarrow X \simeq \mathbb{P}^1 \\ g = 1 & \Leftrightarrow \deg K_X = 0 & \Leftrightarrow X \text{ elliptic} \\ g \geq 2 & \Leftrightarrow \deg K_X > 0 & \Leftrightarrow X \text{ general type} \end{cases}$$

Study such X individually, eg investigate the linear system $|K_X|$.

Study such X collectively in families, eg investigate the **moduli space** M_g of curves of genus g .



genus=0
positive curvature



genus=1
zero curvature



genus=2
negative curvature

Overview of birational geometry

Let X be a smooth projective surface.

K_X may not be uniformly negative, trivial, or positive.

There is a sequence of birational transformations

$$X = X_1 \rightarrow X_2 \rightarrow \cdots \rightarrow X_t =: Y$$

such that either

- K_Y is negative along fibres of some fibration $Y \rightarrow Z$, or
- K_Y is trivial along fibres of some fibration $Y \rightarrow Z$, or
- K_Y is positive.

The above process is the classical **minimal model program** (MMP) for surfaces. It was known in the early 20th century.

It took many decades until birational geometry in dimension 3 took off.

Overview of birational geometry

Let X be a projective variety with "good" singularities.

We say X is $\left\{ \begin{array}{ll} \textbf{Fano} & \text{if } K_X \text{ is anti-ample, eg } \mathbb{P}^n \\ \textbf{Calabi-Yau} & \text{if } K_X \text{ is trivial, eg abelian varieties} \\ \textbf{canonically polarised} & \text{if } K_X \text{ is ample} \end{array} \right.$

Example: $X \subset \mathbb{P}^n$ a hypersurface of degree r :

$$\left\{ \begin{array}{ll} r < n + 1 & \Rightarrow X \text{ is Fano} \\ r = n + 1 & \Rightarrow X \text{ is Calabi-Yau} \\ r > n + 1 & \Rightarrow X \text{ is canonically polarised} \end{array} \right.$$

Example: X a smooth projective curve:

$$\left\{ \begin{array}{lll} \textit{genus} = 0 & \iff X \text{ Fano} & \iff X \simeq \mathbb{P}^1 \\ \textit{genus} = 1 & \iff X \text{ Calabi-Yau} & \iff X \text{ elliptic curve} \\ \textit{genus} \geq 2 & \iff X \text{ canonically polarised} & \iff X \text{ general type.} \end{array} \right.$$

The big step of birational classification is summarised in the next conjecture.

Conjecture (**Minimal model and abundance**)

Each smooth variety X is birational to a projective variety Y with good singularities such that either

Y admits a Fano fibration, or

Y admits a Calabi-Yau fibration, or

Y is canonically polarised.

Known cases:

- dimension 2: (Castelnuovo, Enriques)(Zariski, Kodaira, Shafarevich, etc) 1900,
- dimension 3 (Kawamata, Kollár, Miyaoka, Mori, Reid, Shokurov)(Fano, Hironaka, Itaka, Iskovskikh, Manin, etc) 1970's-1990's,
- any dimension for X of general type (BCHM=B-Cascini-Hacon-M^cKernan, after Shokurov, etc) 2006.

Overview of birational geometry

How to get Y from X ?

If X is of general type, i.e. K_X is big, i.e. Kodaira dimension equals dimension, then

$$Y = \operatorname{Proj} \bigoplus_{m \geq 0} \mathcal{O}_X(mK_X)$$

assuming we already know the algebra is finitely generated over \mathbb{C} .

In practice, we run the MMP giving a sequence of birational transformations

$$X = X_1 \xrightarrow{\text{div contraction}} X_2 \xrightarrow{\text{flip}} X_3 \dashrightarrow \cdots \dashrightarrow X_t = Y$$

This is a generalisation of the classical MMP for surfaces.

Running the MMP requires many local and global ingredients.

Overview of birational geometry

This includes base point free theorem, cone and contraction theorem, flips, termination, understanding singularities, etc.

Flips and singularities are unavoidable new features in dimension ≥ 3 .

The MMP works in the more general framework of pairs (X, B) consisting of a variety X and a boundary divisor $B = \sum b_i B_i$ with coefficients $b_i \in [0, 1]$.

In some special situation one can run MMP on arbitrary divisors, e.g. Fano type varieties, toric varieties.

The non-vanishing theorem

An important ingredient of the MMP is existence of contractions, analogues of contraction of -1 -curves on surfaces.

Let X be a projective variety with klt singularities.

The aim is to make K_X nef, if possible.

Let A be an ample divisor. We use A to measure how far K_X is from being nef.

Let $t = \inf\{s \in \mathbb{R}^{\geq 0} \mid K_X + sA \text{ is nef}\}$.

If $t = 0$, then K_X is nef. So assume $t > 0$.

Theorem (Shokurov non-vanishing, 1985):

We have

$$h^0(X, m(K_X + tA)) \neq 0$$

for some $m \in \mathbb{N}$.

The non-vanishing theorem

Theorem (Kawamata-Shokurov base point freeness, 1985):
The linear system $|m(K_X + tA)|$ is base point free for some $m \in \mathbb{N}$.

The theorem gives a non-trivial contraction $X \rightarrow Z$ by contracting all curves $C \subset X$ with $(K_X + tA) \cdot C = 0$. So $X \rightarrow Z$ is a Fano contraction, that is, $-K_X$ is ample over Z .

Conversely, every Fano contraction $X \rightarrow Z$ arises as above.

If $X \rightarrow Z$ is birational, then it is composed of flipping and divisorial contractions. If not, then it is a Fano fibration, e.g. Mori fibre space.

In conjunction with Mori's theory of **extremal rays**, the above process gives an algorithm for running the MMP.

Others also contributed to this story, e.g. Benveniste, Kollár, Reid.

Example: Let $z \in Z$ be a smooth point and $f: X \rightarrow Z$ be the blow up at z . Let $A = -K_X + f^*H$ for some sufficiently ample H . Then $t = 1$ and $K_X + A = f^*H$ simply defines the contraction $X \rightarrow Z$.

Another important ingredient of the MMP is **existence of flips**.

This is needed when the contraction $f: X \rightarrow Z$ does not contract divisors giving too singular Z . To continue the MMP we need a flip, a diagram $X \rightarrow Z \leftarrow X^+$ where X^+ has good singularities.

This exists if the algebra

$$\mathcal{R} = \bigoplus_{m \geq 0} f_* \mathcal{O}_X(mK_X)$$

is finitely generated locally over \mathcal{O}_Z .

Theorem (Mori, 1988): Flips exist in dimension 3.

Theorem (Shokurov, 1992): Log flips exist in dimension 3.

This was established in one of the best papers of Shokurov: "3-fold log flips".

Shokurov also proved termination of flips in dimension 3. The log version was proved by Kawamata.

Up to the early 90's the log MMP in dimension 3 was completed through:

- the cone and contractions and flips and terminations mentioned above,
- the log abundance theorem of Miyaoka, Kawamata, Keel-Matsuki-M^cKernan,
- contributions of Reid, Kollár, and others.

And then there was a period of drought.

Until ... another groundbreaking paper of Shokurov appeared: "Prelimiting Flips".

Theorem (Shokurov, 2001): Log flips exist in dimension 4.

Theorem (Shokurov, 2006): Minimal models exist in dimension 4.

As a first year PhD student I tried to read Shokurov's prelimiting flips paper.

Eventually I met him in person in 2002 who kindly agreed to be my second advisor.



Theorem (Hacon-M^cKernan, 2005) Log flips exist assuming MMP in lower dimension.

Theorem (B-Cascini-Hacon-M^cKernan, 2006) Log flips exist.

Shokurov's ideas played a key role in these theorems.

One of the earliest results of Shokurov in birational geometry concerns **Fano** varieties.

Theorem (Shokurov, 1979):

Let X be a smooth Fano 3-fold. Then there is a smooth surface $S \sim -K_X$. More generally, if $-K_X = rL$ for some Cartier divisor L , then there is a smooth surface $S \sim L$.

This and other results of Shokurov on Fano 3-folds complemented work of Iskovskikh, Mori, and Mukai on classification of Fano 3-folds.

The theorem is also the origin of his theory of **complements**.

In general an n -complement on a Fano variety X is given by some $D \sim -nK_X$ so that $(X, B := \frac{1}{n}D)$ has log canonical singularities.

Relative complements are similarly defined for Fano contractions $X \rightarrow Z$.

Shokurov used complements in his proof of existence of 3-fold flips.

He also used them in the classification of 3-fold singularities.

The theory of complements was further developed by Shokurov and Prokhorov.

Shokurov conjectured that bounded complements exist.

Theorem (B, 2016): On klt Fano varieties X , an n -complement exists for n depending only on $\dim X$.

The theorem plays an important role in

- boundedness of Fano varieties (B),
- Jordan property of Cremona groups (Prokhorov-Shramov),
- study of singularities (B, Han-Liu-Shokurov, Moraga),
- study of log Calabi-Yau fibrations (B, Di Cerbo-Svaldi),
- construction of moduli spaces for K-stable Fano varieties (Xu, etc).

Besides proving many results, Shokurov also proposed various conjectures.

Conjecture (ACC for log canonical thresholds):

For a variety X with log canonical singularities and \mathbb{Q} -Cartier integral divisor $M \geq 0$, define

$$\text{lct}(M, X) = \sup\{t \in \mathbb{R} \mid (X, tM) \text{ is log canonical}\}.$$

Then the set of all such t in fixed dimension satisfies the ascending chain condition.

This was proved by Hacon-M^cKernan-Xu (2012).

Conjecture (ACC for minimal log discrepancies):

Given a variety X , let $\text{mld}(X)$ be the minimal log discrepancy of X . Then the set of such $\text{mld}(X)$ in fixed dimension satisfies the ascending chain condition.

This is still open in dimension ≥ 3 .

Conjecture (Boundedness of complements):

Assume that $X \rightarrow Z$ is a Fano fibration where $\text{mld}(X) \geq \epsilon > 0$. Then there is a klt n -complement for $-K_X$ over Z for n depending only on $\epsilon, \dim X$.

This is still open in dimension ≥ 3 . The boundedness of complements discussed earlier was for $\epsilon = 0$ and lc complements.

Conjecture (Boundedness of singularities in Fano type fibrations):

Assume (X, B) is a pair with $\text{mld}(X, B) \geq \epsilon > 0$ such that $(X, B) \rightarrow Z$ is log Calabi-Yau and of Fano type. There is an induced log structure (Z, B_Z) . Then $\text{mld}(Z, B_Z) \geq \delta$ for some $\delta > 0$ depending only on $\epsilon, \dim X$.

This is still open in dimension ≥ 3 in full generality.

Thank you for listening!

Happy Birthday Slava!

