Derived prismatic cohomology and perfectoidization

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Arithmetic geometry seminar, Moscow 2020

These are the slides for an expositary talk at the Arithmetic geometry seminar in Moscow. The talk mainly follows B. Bhatt's lecture notes available at http://www-personal.umich.edu/ bhattb/teaching/prismatic-columbia/ , specifically, lectures 7 and 8. Most of the statements either belong to B. Bhatt and B. Scholze and can be found in their paper https://arxiv.org/abs/1905.08229, or are simple corollaries of standard facts. Since the talk was given via Zoom, some slides were intentionally left blank to provide some space for answering questions. The last two statements were proved in the follow-up talk, its slides will be available at http://www.mathnet.ru/php/seminars.phtml?option_lang=rus &presentid=27265 .

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- In this talk, we introduce prismatic and Hodge-Tate cohomology for arbitrary p-complete A/I-algebras for a bounded prism (A, I).
- We study introduce a filtration on Hodge-Tate cohomology, which allows us to gain some control over this cohomology theory.
- It would be great to assign a universal perfectoid ring S_{perfd} over arbitrary p-complete A/I-algebra S. In a sense, we fulfil this dream, but in general, S_{perfd} will not be a classical algebra.
- Still, for a large class of A/I-algebras, namely for semiperfectoid A/I-algebras, S_{perfd} is a classical (concentrated in deg 0) perfectoid ring. Moreover, we show that the map $S \to S_{perfd}$ in this case is surjective, by reproving and utilizing the key lemma from the Andre's paper on the direct summand (any perfectoid ring admits all p-power roots fpqc locally in the category of perfectoid rings).

Non-abelian derived functors

Definition/Proposition

Let $F : \mathrm{Poly}_A \longrightarrow \mathcal{C}$ be a functor from the category of polynomial commutative A-algebras to some cocomplete (∞-)category $\mathcal{C}$. Suppose also that F commutes with coproducts and filtered colimits. Then there exists a unique *extension* $LF : CAlg_A \longrightarrow \mathcal{C}$, such that

1) LF restricts as F to the subcategory $\mathrm{Poly}_A \subset \mathrm{CAlg}_A$;
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1) LF restricts as F to the subcategory $\mathrm{Poly}_A \subset \mathrm{CAlg}_A$;
2) LF commutes with filtered colimits and geometric realizations.

These properties suggest a method of computing $LF(B)$ explicitly for an arbitrary A-algebra B.

Bar-resolution

There is a surjective counit A-algebra map from $A[B]$ to B, which can be extended to a functorial simplicial (bar) resolution

$$A[A[A[B]]] \substack{\rightarrow\\ \rightarrow\\ \rightarrow} A[A[B]] \rightrightarrows A[B]$$

of B.

By the properties of LF above,

$$LF(B) = |LF(A[A[A[B]]]) \substack{\rightarrow\\ \rightarrow\\ \rightarrow} LF(A[A[B]]) \rightrightarrows LF(A[B])|.$$

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When we want to prove that two derived functors agree, we have to prove that there exists a *functorial* isomorphism between their restrictions to the subcategory of polynomial algebras.

First example: cotangent complex

Definition

Let A be a commutative ring. The cotangent complex functor $L_{-/A} : \mathrm{CAlg}_A \longrightarrow D(A)$ is the left derived functor of the functor $\mathrm{Poly}_A \longrightarrow D(A)$ given by $B \mapsto \Omega_{B/A}[0]$.

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Note that when the target category is $D(A)$ (or any other target category we will be interested in) the simplicial complex computing $LF(B)$ can be transformed into a left half-plane bicomplex and then totalized.

Properties of cotangent complex

Proposition

Let B, C be A-algebras.
0) $L_{B/A}$ comes with a B-action and gives an object in $D(B)^{\leq 0}$.
1) $H^0(L_{B/A}) \simeq \Omega^1_{B/A}$.
2) If $A \to B$ is smooth, then $L_{B/A} \simeq \Omega_{B/A}[0]$. In particular, if $A \to B$ is etale, then $L_{B/A} \simeq 0$.
3) The triangle $L_{B/A} \otimes C \to L_{C/A} \to L_{C/B}$ is exact for any map $B \to C$ of A-algebras.
4) $H^{-1}(L_{B/A}) \simeq I/I^2$ for any surjective map $A \to B = A/I$. If I is generated by a (finite) regular sequence, then $L_{B/A} \simeq I/I^2[1]$.
5) $L_{C/A} \otimes^L_A B \simeq L_{C \otimes^L_A B}$.

Filtered derived category

Fix a base ring k of characteristic p. Denote by $D(k)^{\mathbb{N}}$ the filtered derived category. An object F_{-} of $D(k)^{\mathbb{N}}$ is given by a diagram $\{F_0 \to F_1 \to \cdots\}$ in $\mathrm{D(k)}$. For convenience, we set $F_{i<0} := 0$.

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We will be interested in three types of functors from $D(k)^{\mathbb{N}}$ to $D(k)$. First, we have a sequence of functors and natural transformations $ev_0 \to ev_1 \to \cdots : D(k)^{\mathbb{N}} \longrightarrow D(k)$ sending a sequence to its nth term.

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Second example: (derived) de Rham cohomology

Definition

Let k be an $\mathbb{F}_p$-algebra. Derived de Rham cohomology $dR_{-/k} : \mathrm{CAlg}_k \longrightarrow D(k)^{\mathbb{N}}$ is the left derived functor of the functor $\mathrm{Poly}_k \longrightarrow D(k)^{\mathbb{N}}$ given by $R \mapsto \{F_i(\Omega^*_R/k)\}$ where F_* is the canonical filtration.

By abuse of notation, we will often denote the object $UdR_{R/k} \in D(k)$ by $dR_{R/k}$ as well.

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Denote by $R^{(1)}$, the (derived) Frobenius twist of R: $R^{(1)} := R \otimes^L_{k,\phi} k$. By functoriality of Frobenius on polynomial algebras, $dR_{R/k}$ comes with $(R^{(1)})$-linear structure.

Derived Cartier isomorphism

Proposition

For any $R \in CAlg_k$, there is a functorial increasing exhaustive filtration Fil_*^{conj} on $dR_{R/k}$ in $D(R^{(1)})$ equipped with canonical identifications $gr_i^{conj} dR_{R/k} \simeq \Lambda^i L_{R(1)/k}[-i]$.

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Proof.

Left Kan extensions commute with colimit preserving functors such as ev_i, gr_i and U. Using the classical Cartier isomorphism and functoriality of Frobenius, this gives the isomorphisms on derived functors of the graded pieces in question. By functoriality of taking graded pieces with respect to connecting maps and the property of left Kan extensions to commute with gr_i and U, we see that the left handsides of the isomorphism glue to the correct derived de Rham complex.

Smooth algebras

By a similar argument, we obtain

Corollary

If R is a smooth k-algebra, then there is a canonical isomorphism $dR_{R/k} \simeq \Omega^*_{R/k}$ identifying the conjugate filtration on either side.

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Also note that if it was not for Cartier isomorphism the whole construction could and would be rather stupid. For example, in char 0 one obtains a constant functor.

Regular semiperfect rings

Let k be a perfect ring. Let S be a k-algebra of the form R/I where R is a perfect k-algebra and $I \subset R$ is an ideal generated by a regular sequence; we call such rings regular semiperfect. Then $L_{S/k} \simeq L_{S/R}$ by the transitivity triangle since $L_R/k \simeq 0$ (Frobenius acts by 1=0). As I is generated by a regular sequence, we have $L_{S/R} \simeq I/I^2[1]$, with I/I^2 being a finite projective S-module. A standard lemma with derived exterior powers then shows that $\Lambda^i L_{S/R} \simeq \Gamma^i_R(I/I^2)[i]$. In particular, for any $i \geq 0$, the complex $\Lambda^i L_{S/k}[-i]$ has homology only in degree 0. The conjugate filtration on $dR_{S/k}$ then shows that $dR_{S/k}$ also has homology only in degree 0, and can thus be identified with a classical commutative k-algebra (it can be shown to coincide with $D_I(R)$).

Hodge-Tate comparison

Fix a bounded prism (A, I). From now on, R will be an algebra over A/I.

Remind that in Dima's talk the following theorem was proved.

Theorem

The Hodge-Tate comparison map $\eta_R^* : (\Omega^*_{R/(A/I)}, d_{dR}) \to (\mathrm{H}^*(\bar{\Delta}_{R/A})\{*\}, \beta_I)$ *is an isomorphism on p-completely smooth algebras, where* $\{*\} := \otimes_{A/I}(I/I^2)^{\otimes n}$ *stands for the Breuil-Kisin twist. In particular, we have* $\Omega^i_{R/(A/I)} \simeq \mathrm{H}^i(\bar{\Delta}_{R/A})\{i\}$ *for all* i*. [*Ω *stands for p-completed* Ω*!]*

Having this analogue of the Cartier isomorphism in our arsenal, we can upgrade the derived de Rham business to prismatic level.

Derived prismatic cohomology

Denote by D_{comp} the derived category of (p, I)- / p-complete complexes.

Definition

The derived prismatic cohomology functor $L\Delta_{-/A} : \mathrm{CAlg}_A/I \longrightarrow D_{comp}(A)$ is the left derived functor of the functor $\mathrm{Poly}_{A/I} \longrightarrow D_{comp}(A)$ given by $R \mapsto \Delta_{R_p^\wedge/A}$. The derived Hodge-Tate cohomology functor $L\Delta_{-/A} : \mathrm{CAlg}_{A/I} \longrightarrow D_{comp}(A/I)^{\mathbb{N}}$ is the left derived functor of the functor $R \mapsto \{F_n(\bar{\Delta}_{R_p^\wedge/A})\}$. We set $L\Delta_{-/A} := UL\Delta_{-/A} \in D_{comp}(A/I)$.

Derived Hodge-Tate comparison

By checking on polynomial algebras, we verify $L\Delta_{R/A} \otimes_A^L A/I \simeq L\bar{\Delta}_{R/A}$. $L\Delta_{R/A}$ comes equipped with a ϕ_A-semilinear "Frobenius" endomorphism ϕ_R.

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Proposition

For any $R \in \mathrm{CAlg}_{A/I}$, the complex $L\overline{\Delta}_{R/A}$ comes equipped with a functorial increasing exhaustive filtration Fil^{HT}_* and canonical identifications $gr^{HT}_i(L\overline{\Delta}_{R/A}) \simeq \Lambda^i L_{R/(A/I)}\{-i\}[-i]^\wedge_p$.

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Note that on the lhs above we replace R by its p-completion first then apply $\bar{\Delta}$, while on the rhs we consider p-completed differential forms. The following lemma ensures some comfort about this.

HT proof

Lemma

Let A be a p-adically complete commutative ring with bounded p_∞-torsion. Let B be a flat A-algebra, so B also has bounded p_∞-torsion. Then the cotangent complex of the map $B \to B_p^\wedge$ vanishes after derived p-completion.

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Proof.

By boundedness of p_∞-torsion, the classical and derived completions coincide. Then, by derived Nakayama, can check the claim for $L_{B_p^\wedge/B} \otimes_B^L (B/p)$. By base change for L, this is 0. □

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Proof of Hodge-Tate comparison.

Checking on graded pieces with the help of the Lemma above and using functoriality a lot.

p-completely smooth algebras

Corollary

If R is a p-completely smooth A/I-algebra, then $L{\mathbb{\Delta}}_{R/A} \simeq {\mathbb{\Delta}}_{R/A}$ compatibly with all the structure.

Proof.

For $\bar{\mathbb{\Delta}}$ write down a map and use Hodge-Tate on both handsides.
For ${\mathbb{\Delta}}$ write down a map and use derived Nakayama.

□

From now on, we denote ${\mathbb{\Delta}}_{R/A} := L{\mathbb{\Delta}}_{R/A}$, $\bar{\mathbb{\Delta}}_{R/A} := L\bar{\mathbb{\Delta}}_{R/A}$

Semiperfectoid rings

Assume $(A, (d))$ is a perfect prism. Let S be an $A/(d)$-algebra of the form R/J, where R be a perfectoid $A/(d)$-algebra and $J \subset R$ is an ideal generated by a regular sequence; we will call such S regular semiperfectoid. Assume that $A/(d)$ is p-torsionfree and that S is p-completely flat over $A/(d)$. The Hodge-Tate comparison shows that $\overline{\Delta}_{S/A}$ admits an increasing exhaustive filtration with graded pieces $\Lambda^i L_{S/(A/d)}\{-i\}[-i]^\wedge \simeq \Lambda^i L_{S/R}\{-i\}[-i]^\wedge$. By the assumption on S, each of these pieces is a finite projective S-module. It follows that $\overline{\Delta}_{S/A}$ is concentrated in degree 0 and given by a p-completely flat S-algebra. Consequently, $\Delta_{S/A}$ is concentrated in degree 0 and given by a (p, d)-completely flat A-algebra. In fact, one can also show that the Frobenius ϕ_R on prismatic cohomology makes it into a δ-A-algebra, so $(\Delta_{S/A}, (d))$ is a flat prism over $(A, (d))$.
For a general J, we only/still get $\Delta_{S/A} \in D^{\leq 0}(A)$.

In the previous slide we used the following statement.

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Proof.

Choose perfect prisms (A, I) and (B, J) such that $D \simeq A/I$, $R \simeq B/J$. By rigidity and vanishing of Tors (Dima's talk), $J = IB$, $R \simeq D \otimes^{L}_{A} B$. By base change for cotangent complex, it remains to show $L^{\wedge}_{B/A} \simeq 0$. Nakayama lemma and vanishing of L for perfect $\mathbb{F}_p$-algebras finish the proof. □

Char p picture: perfection

From now on, k is a perfect field of char p.

Definition

For a k-algebra R, $R_{perf} := \mathrm{colim}_{\phi}(R \to \phi_* R \to \phi_*^2 R \to \cdots)$.

The maps in the diagram are k-linear, moreover, there is a map $R \to R_{perf}$ from the first term in the diagram.
From now on, we write such diagrams simply as $\mathrm{colim}_{\phi} R$.

Example: $(k[M])_{perf} \simeq k[M[p^{-1}]]$, in particular, $k[x]_{perf} \simeq k[x^{1/p^{\infty}}]$.

Char p: perfection via de Rham and Hodge-Tate

Proposition

The canonical map from $dR_{R/k,perf} := \mathrm{colim}_{\phi} dR_{R/k}$ to R_{perf} via the map $dR_{R/k} \to R$ on terms is an equivalence.

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Check on polynomials. Frobenius kills n-forms for $n > 0$. □

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Corollary

Let R be a k-algebra, and let $(A = W(k), (p))$ be the cristalline prism corresponding to k. Then the p-completed direct limit $\mathbb{\Delta}_{R/A,perf} := \mathrm{colim}_\phi (\mathbb{\Delta}_{R/A})^\wedge_p \in D(\mathbb{Z}_p)$ identifies with $W(R_{perf})$ as a $W(k)$-algebra. In particular, it is concentrated in degree 0 and $\overline{\mathbb{\Delta}}_{R/A,perf} \simeq R_{perf}$.

Mixed char: perfectoidizations

Fix a perfect prism $(A, I = (d))$. The previous discussion motivates the following definition.

Definition

For any p-complete A/I-algebra R,
(1) the perfection $\Delta_{R/A,perf}$ is defined as the (p, I)-completed colimit $\Delta_{R/A,perf} := \mathrm{colim}_{\phi_R} \Delta_{R,A} \in D_{comp}(A)$;
(2) the perfectoidization R_{perfd} is defined by $R_{perfd} := \Delta_{R/A,perf} \otimes^L_A A/I \in D_{comp}(R)$, where the R-linear structure is defined via the canonical map $R \to \bar{\Delta}_{R/A} \to R_{perfd}$.

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Lemma (Independence of the base; correctness of definition above)

Let $(A, I) \to (B, J)$ be a map of perfect prisms, and let S be a p-complete B/J-algebra. Then the natural map gives isomorphisms $\Delta_{S/A(,perf)} \simeq \Delta_{S/B(,perf)}$.

Proof of correctness

Proof.

By derived Nakayama, we reduce to Hodge-Tate complexes. Using Hodge-Tate filtration and exact triangle for L, we reduce to showing that the p-completion $L^{\wedge}_{(B/J)/(A/I)}$ is zero. This is the previous lemma.

□

Andre's flatness lemma

Next week, we will use perfectoidization to give simple proofs of the following two results (the former will be used in the proof of the latter).

Theorem

Let R be a perfectoid ring. For any set $\{f_s \in R\}_{s\in S}$ of elements of R, there exists a p-completely faithfully flat map $R \to R_\infty$ of perfectoid rings such that each f_s admits a compatible system of p-power roots in R_∞. (This implies the map $\sharp : R^\flat \to R$ is surjective locally for the p-completely flat topology)

Theorem

Let R be a perfectoid ring, and let $S = R/J$ be a p-complete quotient. Then there is a universal map $S \to S'$ with S' being a perfectoid ring. Moreover, this map is surjective.