Critical edges in Schrijver graphs

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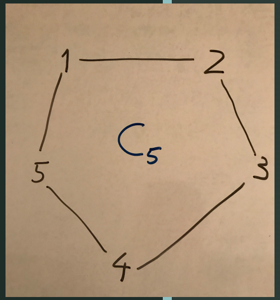
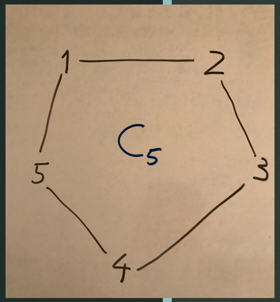


Kneser / Schrijver graphs

- n, k positive integers, $n \geq 2k$
- cycle C_n - vertices: $[n] = \{1,2,\dots,n\}$
  - edges: $\{(1,2),(2,3),\dots,(n-1,n),(n,1)\}$
- Kneser graph $KG(n,k)$ - vertices: $\binom{[n]}{k} = \{k\text{-subsets of [n]}\}$
  - edges between *disjoint* subsets
- Schrijver graph $SG(n,k)$ subgraph of $KG(n,k)$ induced by the vertices that are *independent sets* in C_n



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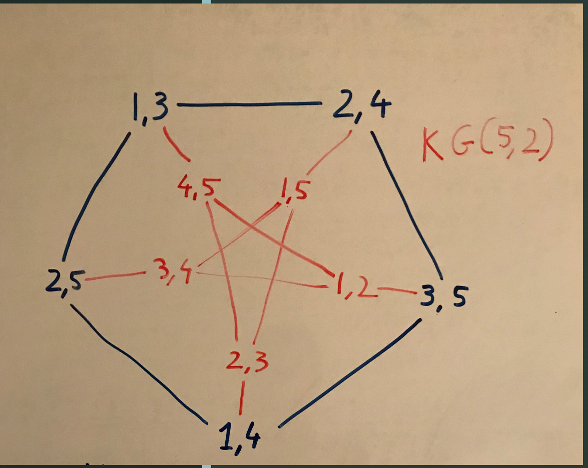

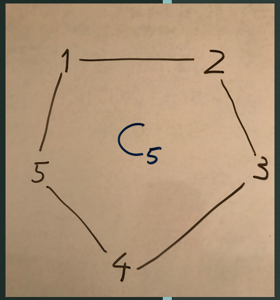

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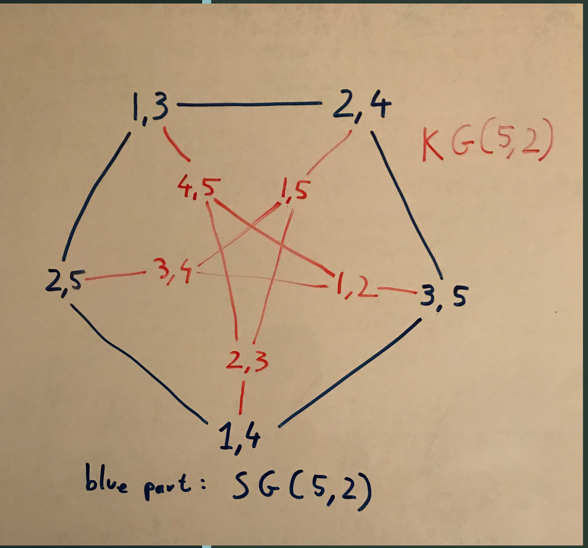

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Kneser-Lovász / Schrijver theorems

Kneser, 1955

Theorem: $\chi(KG(n,k)) \leq n-2k+2$

Conjecture: $=$

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Thm.: $\chi(KG(n,k)) = n - 2k + 2$

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Schrijver, 1978
Thm.: $\chi(SG(n,k)) = n-2k+2$
$SG(n,k)$ is *vertex-critical*, that is, for any vertex A:
$$\chi(SG(n,k) \setminus \{A\}) = n-2k+1$$

Examples of Schrijver graphs

- 2-chromatic: $SG(2k, k) \cong K_2$
- 3-chromatic: $SG(2k+1, k) \cong C_{2k+1}$
- 4-chromatic: $SG(2k+2, k)$ mostly grid like

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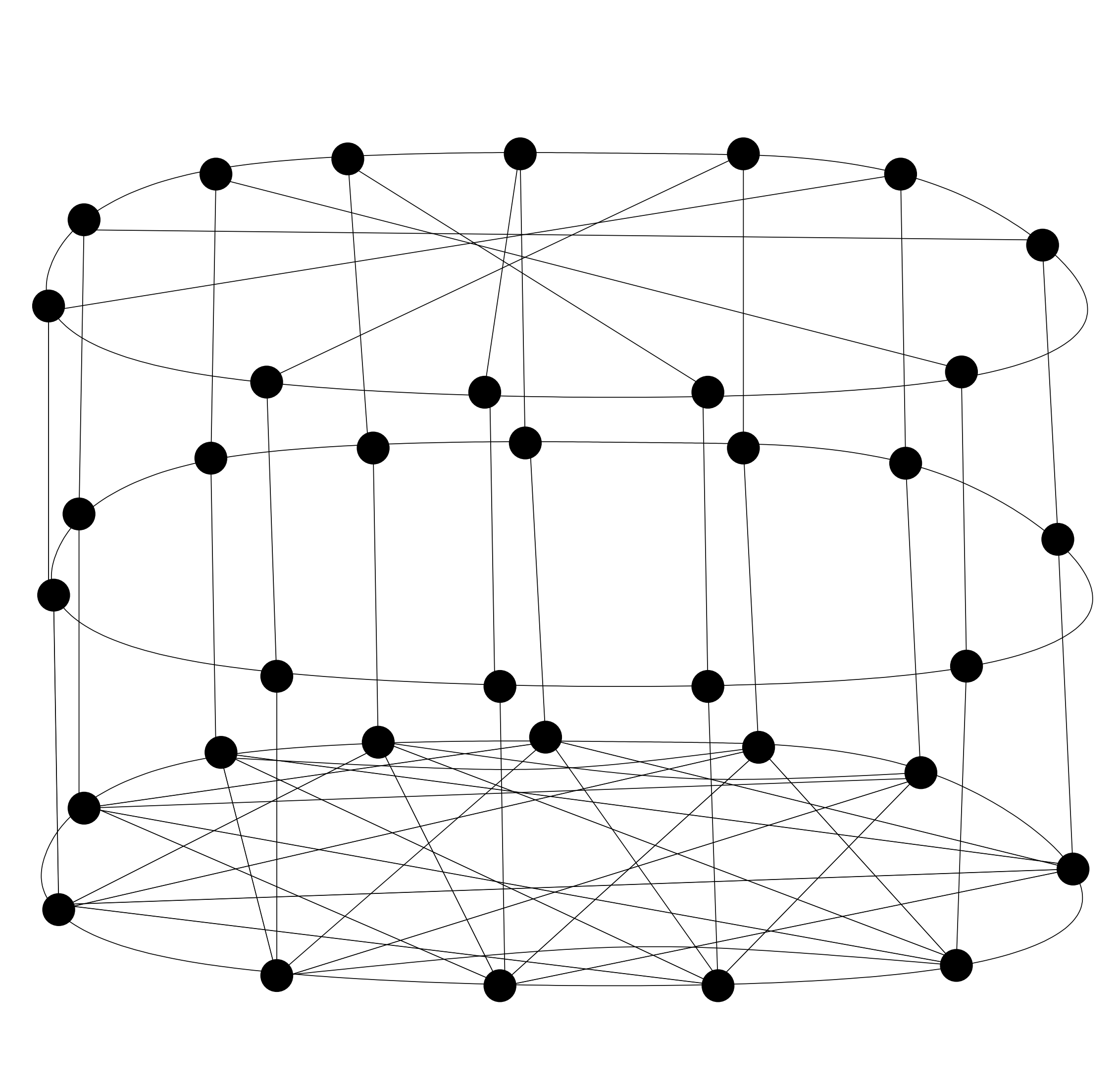

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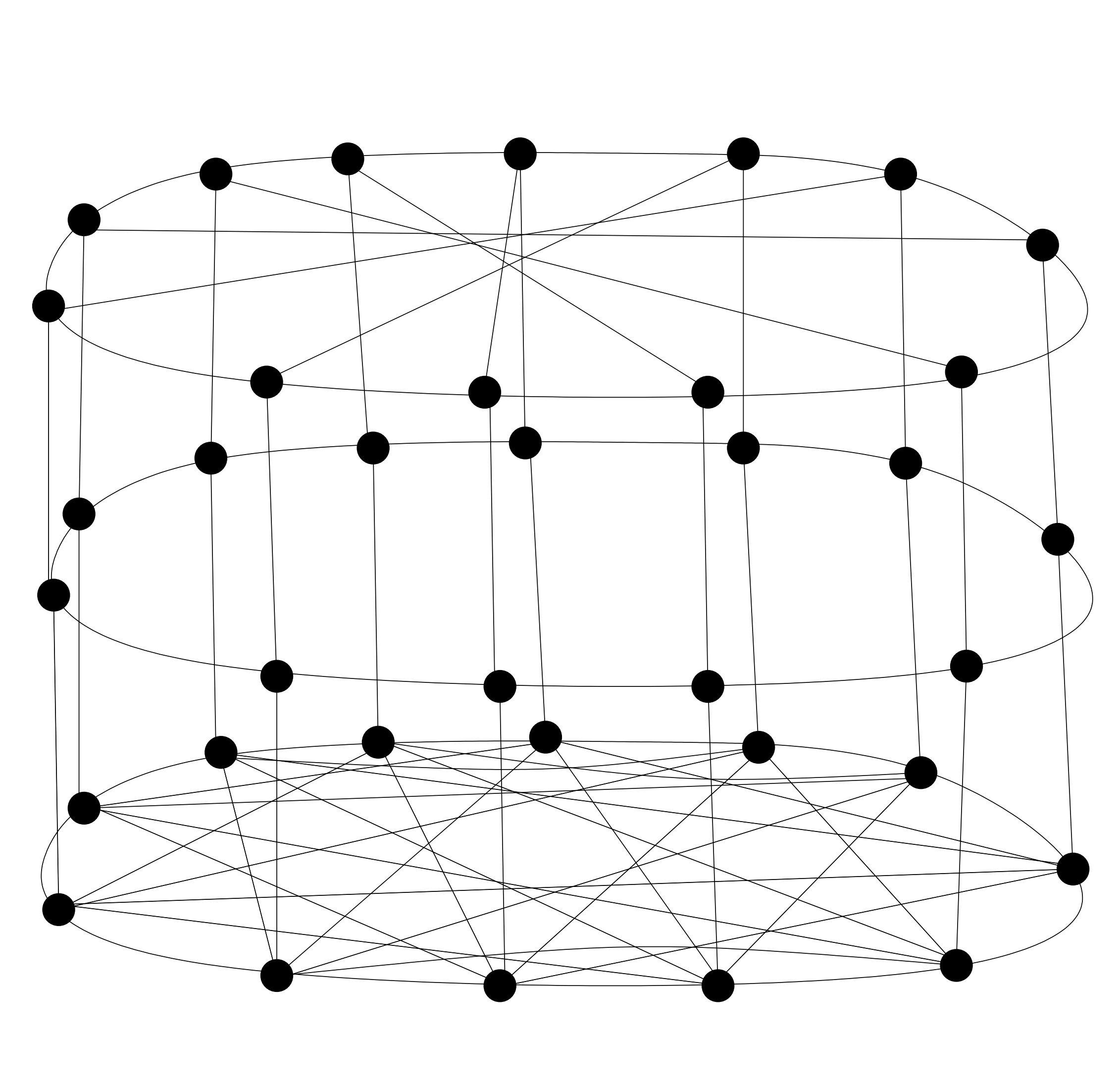

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Bottom: complete bipartite. Remove any one diagonal: χ does not change $\rightarrow SG(12,5)$ is NOT edge-critical

Interlacing edges in Schrijver graphs

- DEF: If $1 \leq a_1 < b_1 < a_2 < b_2 < \cdots < a_k < b_k \leq n$, then $(\{a_1, a_2, \ldots, a_k\}, \{b_1, b_2, \ldots, b_k\})$ is an edge of $SG(n, k)$ – an *interlacing* edge.

- Examples:
  $(\{1,3,5,7,9\}, \{2,6,8,10,12\})$ is a non-interlacing edge of $SG(12,5)$
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- Thm.: All non-interlacing edges of a Schrijver graph are non *non-critical*, that is
  $$\chi(SG(n,k) \setminus \{e\}) = n - 2k + 2$$
  if e is a non-interlacing edge

Bárány's proof of the Kneser-Lovász thm = Borsuk thm + Gale lemma

- DEF: Vertices of the *Borsuk graph* $B(d, \varepsilon)$: points of the unit sphere S^d. Edges connect *near-opposite* points: (p, q) is an edge iff $dist(p, -q) < \varepsilon$
- Note: the Borsuk graph is infinite

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- Borsuk Thm. (equivalent form):
  $$\chi(B(d, \varepsilon)) \geq d + 2$$
- Gale Lemma: $\exists\ d + 2k$ points in S^d such that all *open hemispheres* contain $\geq k$ of them.

Bárány's proof of the Kneser-Lovász thm

- $d = n - 2k$
- Gale lemma: $p_1, p_2, \ldots, p_n \in S^d$ with $\geq k$ of them in each open hemisphere
- continuity: $\exists \varepsilon > 0\ \forall p \in S^d \quad \geq k$ of the points p_i are of distance $< \sqrt{2} - \varepsilon$ from p.
- graph homomorphism $B(d, \varepsilon) \to KG(n, k)$: For $p \in S^d$ choose image V with $dist(p, p_i) < \sqrt{2} - \varepsilon$ whenever $i \in V$.
- $\chi\big(B(d, \varepsilon)\big) \leq \chi\big(KG(n, k)\big)$
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Proof of Schrijver's thm

Gale Lemma+ (Schrijver): $\exists\, p_1, p_2, \ldots, p_{d+2k} \in S^d$ such that for all open hemisphere H exists vertex V of $SG(d+2k, k)$ such that $p_i \in H$ for all $i \in V$.

- In the homomorphism $B(d,\varepsilon) \to KG(n,k)$ of the previous proof we can ensure to only use vertices of $SG(n,k)$ as images.
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Proof non-criticality of non-interlacing edges

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- In the homomorphism $B(d, \varepsilon) \to SG(n, k)$ of the previous proof we can ensure that the image of any ONE EDGE is interlacing.
- Not possible to do it for ALL EDGES of $B(d, \varepsilon)$ at the same time.
- Still, exists homomorphism $B(d, \varepsilon) \to SG(n, k) \setminus \{e\}$ if e is non-interlacing. Q.E.D.
- $\exists$ homomorphism $B\,(d, \varepsilon) \to SG(n, k) \setminus$ {non-int. edges at a single vertex V}
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DEF: A vertex of $SG(n,k)$ is *regular* if it contains at least 1 out of each 3 consecutive vertices of C_n.

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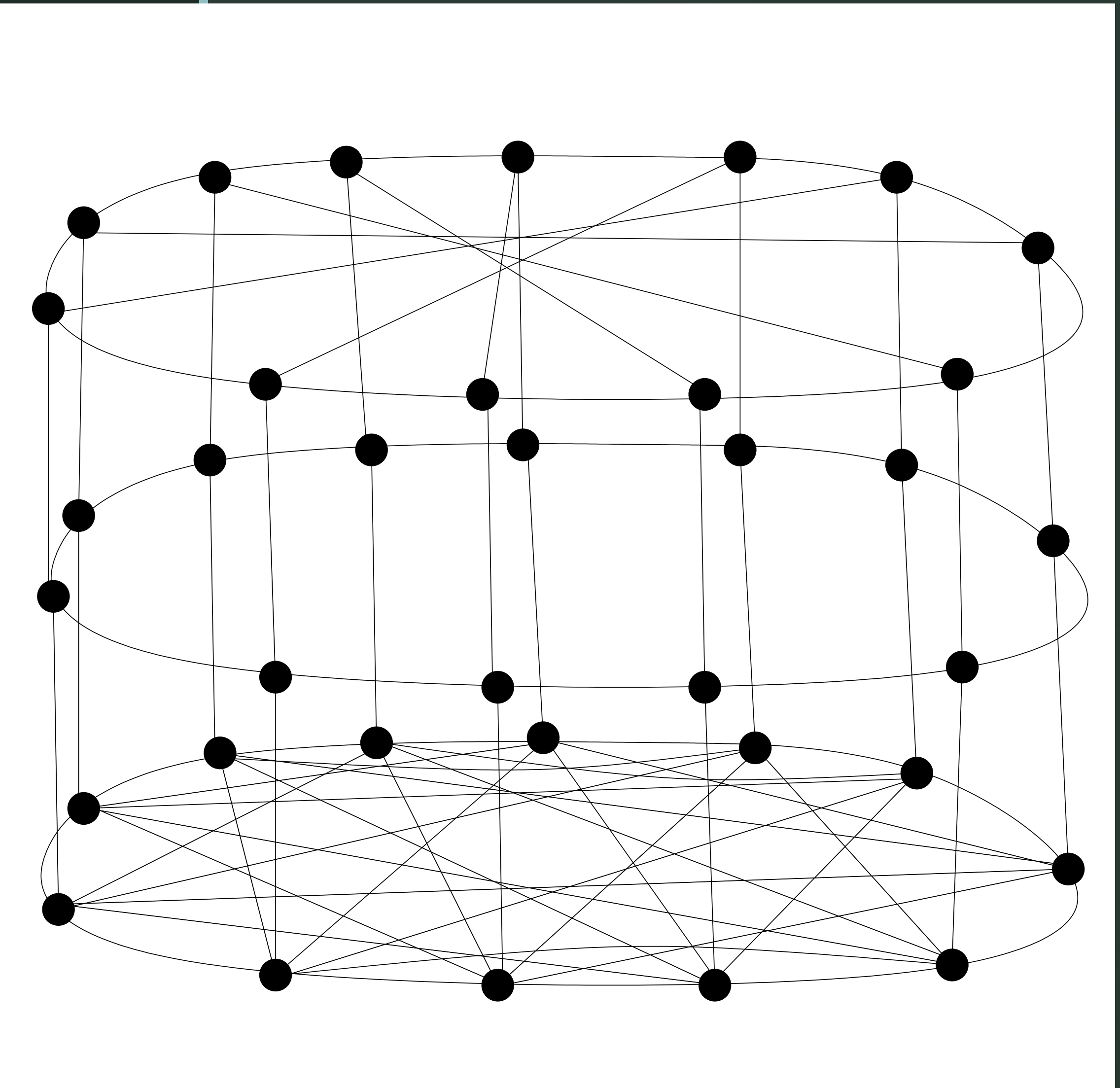
Thm.: Each edge of $SG(n,k)$ between two regular vertices is critical.

Example: 4-chromatic Schrijver graphs



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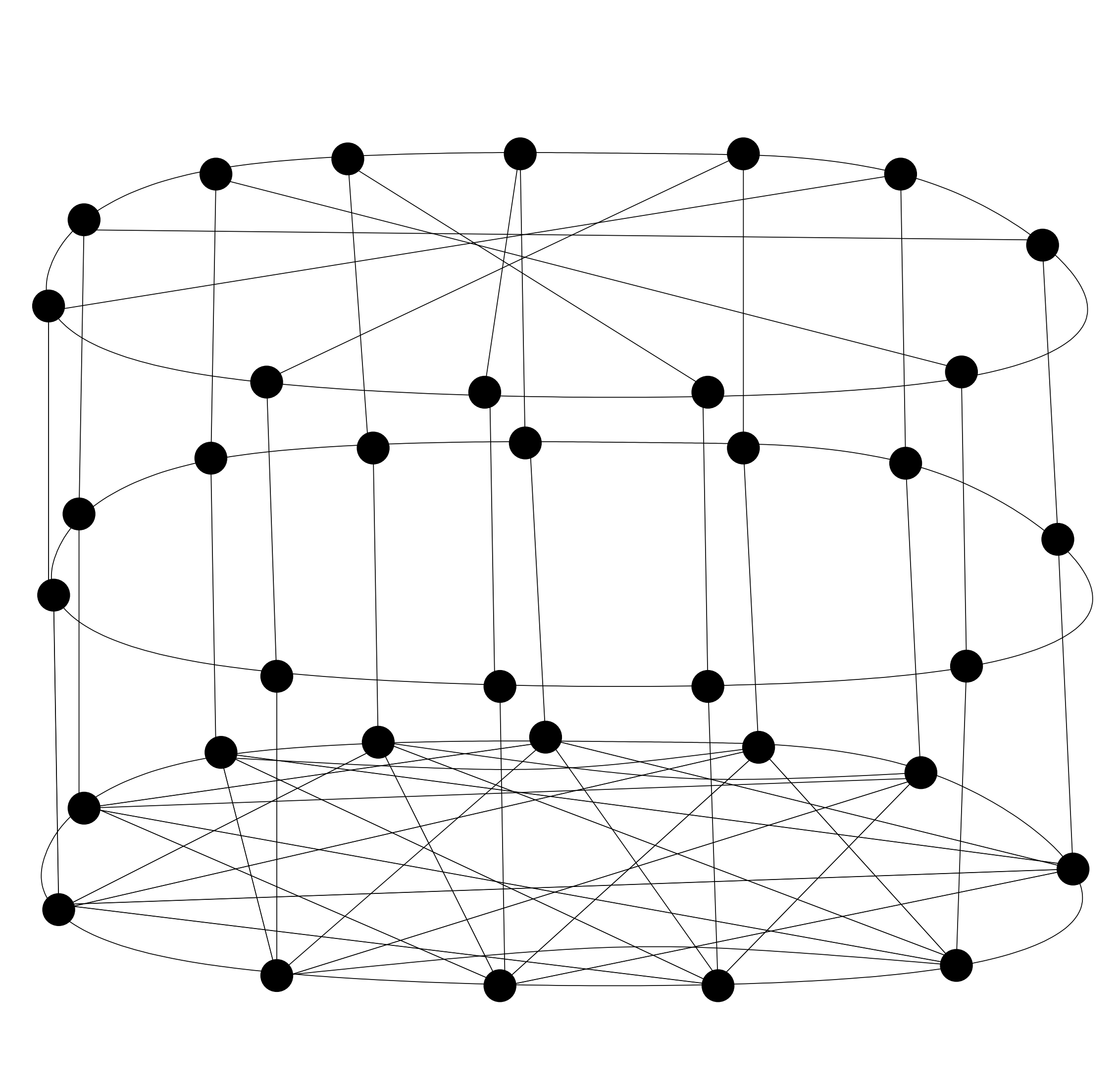

- non-interlacing edges = diagonals on bottom
- non-regular vertices = bottom layer

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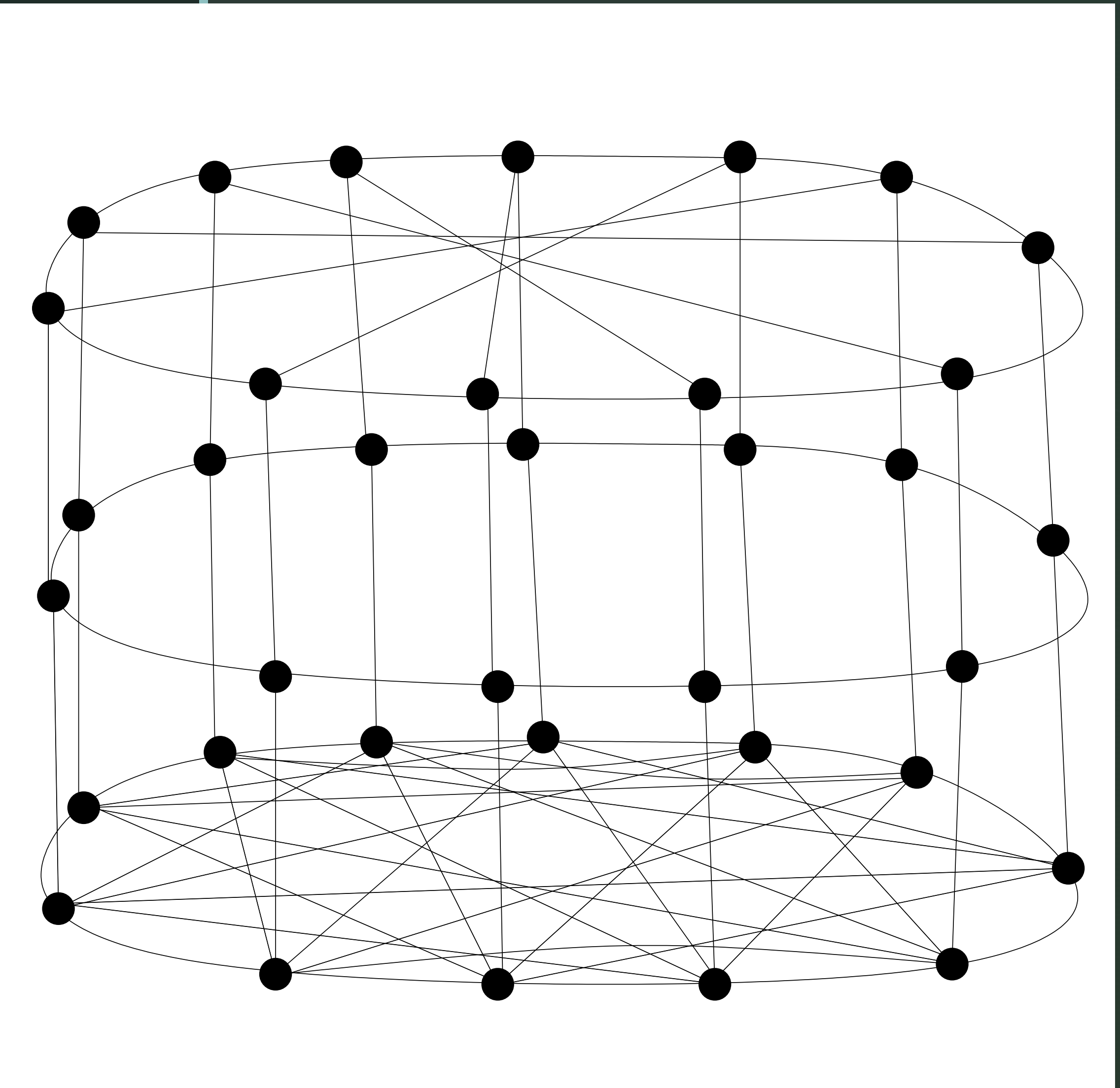

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Thm. All interlacing edges of $SG(2k+2, k)$ are critical

$SG(12,5)$

Bold conjecture

Conjecture: All interlacing edges of all Schrijver graphs are critical.

False ~~Bold~~ conjecture

Conjecture: All interlacing edges of all Schrijver graphs are critical.

$SG(n, 2)$ has many non-critical interlacing edges for $n \geq 8$.

Example: $SG(n, 2)$

- $SG(n, 2)$ = complement of edge graph of complement of C_n.
- vertices = diagonals of a convex n-gon
- edge = two diagonals with 4 separate end points
- interlacing edge = pair of crossing diagonals
- regular vertex: none if $n \geq 7$
- critical edge = two crossing diagonals,
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See also: Daneshpajouh, Meunier, Mizrahi, Colorings of complements of line graphs, 2020

Open problems

1. Are all edges incident to regular vertex in a Schrijver graph critical?

2. Is it true for any d and large enough k that all interlacing edges in $SG(2k+d,k)$ are critical?