Derived prismatic cohomology and perfectoidization, continued

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These are the slides for an expository talk at the Arithmetic geometry seminar in Moscow. This is the second talk – the slides for the previous talk will be available at http://www.mathnet.ru/php/seminars.phtml?option_lang=rus &presentid=27214 . We remind its most important statements and ideas here as well. The talk mainly follows B. Bhatt's lecture 8 available at http://www-personal.umich.edu/ bhattb/teaching/prismatic-columbia/ . Since the talk was given via Zoom, some slides were intentionally left blank to provide some space for answering questions.

Reminder: derived prismatic cohomology

Denote by D_{comp} the derived category of (p, I)- / p-completed complexes.

Definition

The derived prismatic cohomology functor $L\Delta_{-/A} : \mathrm{CAlg}_A/I \longrightarrow D_{comp}(A)$ is the left derived functor of the functor $\mathrm{Poly}_{A/I} \longrightarrow D_{comp}(A)$ given by $R \mapsto \Delta_{R_p^\wedge/A}$. The derived Hodge-Tate cohomology functor $L\bar{\Delta}_{-/A} : \mathrm{CAlg}_{A/I} \longrightarrow D_{comp}(A/I)^{\mathbb{N}}$ is the left derived functor of the functor $R \mapsto \{F_n(\bar{\Delta}_{R_p^\wedge/A})\}$. We set $L\bar{\Delta}_{-/A} := UL\bar{\Delta}_{-/A} \in D_{comp}(A/I)$.

Reminder: derived Hodge-Tate comparison

By checking on polynomial algebras, we verify $L\Delta_{R/A} \otimes_A^L A/I \simeq L\bar{\Delta}_{R/A}$. $L\Delta_{R/A}$ comes equipped with a ϕ_A-semilinear "Frobenius" endomorphism ϕ_R.

Proposition

For any $R \in \mathrm{CAlg}_{A/I}$, the complex $L\bar{\Delta}_{R/A}$ comes equipped with a functorial increasing exhaustive filtration Fil_*^{HT} and canonical identifications $gr_i^{HT}(L\bar{\Delta}_{R/A}) \simeq \Lambda^i L_{R/(A/I)}\{-i\}[-i]_p^\wedge$.

Corollary

If R is a p-completely smooth A/I-algebra, then $L\Delta_{R/A} \simeq \Delta_{R/A}$ compatibly with all the structure.

From now on, we denote $\Delta_{R/A} := L\Delta_{R/A}$, $\bar{\Delta}_{R/A} := L\bar{\Delta}_{R/A}$

Reminder: perfectoidizations

Fix a perfect prism $(A, I = (d))$. Analogously to the char p picture, we give the following definition.

Definition

For a p-complete A/I-algebra R,
(1) the perfection $\Delta_{R/A,\mathrm{perf}}$ is defined as the (p, I)-completed colimit $\Delta_{R/A,\mathrm{perf}} := \mathrm{colim}_{\phi_R} \Delta_{R,A} \in D_{comp}(A)$;
(2) the perfectoidization R_{perfd} is defined by $R_{\mathrm{perfd}} := \Delta_{R/A,\mathrm{perf}} \otimes^L_A A/I \in D_{comp}(R)$, where the R-linear structure is defined via the canonical map $R \to \bar{\Delta}_{R/A} \to R_{\mathrm{perfd}}$.

Lemma (Independence of the base; correctness of definition above)

Let $(A, I) \to (B, J)$ be a map of perfect prisms, and let S be a p-complete B/J-algebra. Then the natural map gives isomorphisms $\Delta_{S/A(,\mathrm{perf})} \simeq \Delta_{S/B(,\mathrm{perf})}$.

Examples

1) Let $I = (p)$. Then $R_{\mathrm{perfd}} \simeq R_{\mathrm{perf}}$, $\Delta_{R/A,\mathrm{perf}} \simeq W(R_{\mathrm{perf}})$. In particular, these are classical algebras.

$\pi/2$) Let (A, I) be the perfection of $(\mathbb{Z}[[q-1]], ([p]_q))$, so A is the $(p, [p]_q)$-completion of $\mathbb{Z}_p[q^{1/p^\infty}]$. Take $R = A/I[x^{\pm 1}]^\wedge$. In this case, $\mathrm{H}^1(\Delta_{R/A,\mathrm{perf}}) \neq 0$ and $\mathrm{H}^1(R_{\mathrm{perfd}}) \neq 0$ (computation with q-de Rham complex).

2) Let R be a perfectoid ring. Then $\Delta_{R/A} \simeq A_{inf}(R) \simeq \Delta_{R/A,\mathrm{perf}}$ and $R \simeq R_{\mathrm{perfd}}$. In particular, they are both concentrated in deg 0.

Regular semiperfectoid rings

Let S be an $A/(d)$-algebra of the form R/J, where R be a perfectoid $A/(d)$-algebra and $J \subset R$ is an ideal generated by a regular sequence; we will call such S regular semiperfectoid. Assume that $A/(d)$ is p-torsionfree and that S is p-completely flat over $A/(d)$. The Hodge-Tate comparison shows that $\overline{\Delta}_{S/A}$ admits an increasing exhaustive filtration with graded pieces $\Lambda^i L_{S/(A/d)}\{-i\}[-i]^\wedge \simeq \Lambda^i L_{S/R}\{-i\}[-i]^\wedge$. By the assumption on S, each of these pieces is a finite projective S-module. It follows that $\overline{\Delta}_{S/A}$ is concentrated in degree 0 and given by a p-completely flat S-algebra. Consequently, $\Delta_{S/A}$ is concentrated in degree 0 and given by a (p, d)-completely flat A-algebra. In fact, one can also show that the Frobenius ϕ_R on prismatic cohomology makes it into a δ-A-algebra, so $(\Delta_{S/A}, (d))$ is a flat prism over $(A, (d))$. For a general J, we only get $\Delta_{S/A} \in D^{\leq 0}(A)$. Yet, since ϕ acts by 0 on the negative cohomology, $\Delta_{S/A,\mathrm{perf}}$ lives only in deg 0 as well.

The fact about Frobenius used in the previous slide is written, e.g. in Lurie's DAG XIII, section 2.2 (available at https://www.math.ias.edu/ lurie/papers/DAG-XIII.pdf). Actually for a commutative algebra object A over fixed k, one can define the P^0 power operation, which, one can easily show, coincides with Frobenius on H^0, and is 0 on $H^{<0}$ (basically, a class in H^{-i} is represented by a map of modules $\eta : k[i] \to A$ and the map $P^0\eta : k[i] \to A$ by construction is presented by some composition $k[i] \to k[pi]_{hG} \to A$ where the first map is null-homotopic whenever $i > 0$ since $H^j(k[pi]_{hG}) = 0$ for $-j < pi$.

Andre's flatness lemma

Theorem

Let R be a perfectoid ring. For any set $\{f_s \in R\}_{s\in S}$ of elements of R, there exists a p-completely faithfully flat map $R \to R_\infty$ of perfectoid rings such that each f_s admits a compatible system of p-power roots in R_∞.

It will be used to deduce the following fact.

Andre's flatness lemma

Theorem

Let R be a perfectoid ring. For any set $\{f_s \in R\}_{s \in S}$ of elements of R, there exists a p-completely faithfully flat map $R \to R_\infty$ of perfectoid rings such that each f_s admits a compatible system of p-power roots in R_∞. (This implies the map $\sharp : R - tilt \to R$ is surjective locally for the p-completely flat topology)

It will be used to deduce the following fact.

Theorem (Zariski closed sets are strongly Zariski closed)

Let R be a perfectoid ring, and let $S = R/J$ be a p-complete quotient. Then there is a universal map $S \to S'$ with S' being a perfectoid ring. Moreover, this map is surjective.

Andre's lemma proof

1. p-completely faithfully flatness is stable under filtered colimits and coproducts, so set $\{f_s \in R\}_{s\in S} = \{f\}$.
2. Denote (A, I) the perfect prism, s.t. $A/I = R$. "Add roots": choose $R' := R[x^{1/p^\infty}]$ and S to be the p-adic completion of $R'/(x-f)$, so $R \to S$ is p-completely faithfully flat, and f has a compatible system of p-power roots in S. We shall show that S_{perfd} is the answer, i.e., that it is p-completely faithfully flat over R. Therefore, it is concentrated in degree 0 and is a perfectoid ring with $\Delta_{S/A,\mathrm{perf}}$ being its A_{inf}.
3. Using stability of completely faithfully flatness under filtered colimits (and Frobenius twists, since A is perfect), we reduce to showing p-completely faithfulness of $\bar{\Delta}_{S/A}$. By properties of Hodge-Tate filtration, it remains to show S-flatness of $\Lambda^i L_{S/R}[-i]^\wedge \simeq \Lambda^i L_{S/R'}[-i]^\wedge \simeq \Gamma^i_S(S)$, which is obvious.

Perfectoid quotients, proof

We claim that $S' := S_{\mathrm{perfd}}$ solves the problem. The discussion in the slides above, implies that $\bar{\Delta}_{S/A}$ and $\Delta_{S/A}$ are connective, hence $\Delta_{S/A,\mathrm{perf}}$ and S_{perfd} live in degree zero and the latter is a perfectoid ring. Universality follows by functoriality, so it remains to confirm surjectivity.

We remind that the image of $\sharp : R^\flat \to R$ is exactly elements of R, admitting a compatible system of p-power roots. By p-completely faithfully flat base change (justified in lemma below) and Andre's flatness lemma, we can assume that J is (up to p-completion) generated by such elements. By a direct computation, one verifies that, for J_∞ obtained from J by adding p-power roots, R/J_∞ is perfectoid and $S = R/J \to R/J_\infty$ is the universal map to a perfectoid ring.

Base change compatibility

Lemma

The functor $R \mapsto R_{\mathrm{perfd}}$ commutes with (p, I)-completely faithfully flat base change on the perfect prism (A, I).

Proof.

We only explain the proof when R is bounded. Say $(A, I) \to (B, IB)$ is a faithfully flat map of perfect prisms. Write $S := R \otimes^{\wedge, L}_{A/I} B/IB$. Then S is p-completely R-flat and thus concentrated in degree 0 as R is bounded.
We must show that $R_{\mathrm{perfd}} \otimes^{\wedge, L}_{A/I} B/IB \simeq S_{\mathrm{perfd}}$. By compatibility with filtered colimits, it is enough to show the equivalence for corresponding Hodge-Tate cohomology, which, by Hodge-Tate filtration, boils down to base change for the cotangent complex.

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