Geometric quantization

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June, 2020, w/ L. Ioos, L. Polterovich

Let (X, ω) be a symplectic C^∞- manifold of dimension $2d$ without boundary. We denote by $\{,\} : C^\infty(X) \times C^\infty(X) \to C^\infty(X)$ the Poisson bracket $\{f, g\} := \omega(dg, df)$.

There are two different notions of a *quantization* of (X, ω): the **algebraic** and the **geometric** quantizations.

A formal (algebraic) quantization of a pair (X, ω) is an associative ring $(\hat{A}, \star)$ which is isomorphic as an $\mathbb{C}[[h]]$-module to $C^\infty(X)[[h]]$ and such that the product $(f, g) \to f \star g$ is of the form $f \star g = fg + \sum_{j>0} c_j(f, g)h^i$ where c_i are bidifferential operators and

$$\boxed{c_1(f, g) - c_1(g, f) = i\{f, g\}}$$

By definition a formal quantization is characterized by the family $\mathcal{C} = \{c_j\}, j \geq 1$ of bidifferential operators.

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In the case of a geometric quantization in addition to the deformation of the multiplication of $C^\infty(X)$ to an non-commutative algebra structure on $C^\infty(X)[[h]]$ we deform the *square root* of the natural action μ of the ring $C^\infty(X)$ on the Hilbert space $L^2(X, \omega^d)$.

A naive definition of a geometric quantization of (X, ω) would be a pair (A, ρ) where $A \subset \hat{A}$ is a subalgebra and $\rho : A \to End(\mathcal{H})$ a representation of A such that corresponding action $\tilde{\rho}$ of A on $\tilde{\mathcal{H}}$ is a quantization of μ where $\tilde{\mathcal{H}}$ is the Hilbert space of Hilbert-Schmidt operators on $\mathcal{H}$.

In the representation theory the geometric quantization (known as the *orbit method*) is applied to cases when (X, ω) is a homogeneous G-variety and provides a construction of irreducible representation of G in a Hilbert space $\mathcal{H}$. The constructions of the space $\mathcal{H}$ uses the existence of a polarization $\mathcal{P}$ (real or complex) on X. The Hilbert space $\mathcal{H}$ consists of functions constant along fibers of $\mathcal{P}$ and is a *square root* of $L^2(X, \omega^d)$. On the other hand non-homogeneous symplectic manifolds usually do not admit any polarization.

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Felix Berezin had an ingenious idea to replace a representation $\rho : A \to End(\mathcal{H})$ by an approximating family of maps $\rho_n : C^\infty(X) \to End(\mathcal{H}_n)$. In my talk I only consider the case of compact manifolds X.

Definition

Let $(\hat{A}, \star)$ be a formal quantization of (X, ω).
A *geometric quantization* of $(\hat{A}, \star)$ is

1. a subset $\Lambda \subset \mathbb{R}_{>0}$ having 0 as the only limit point ;
2. a finite-dimensional complex Hilbert space $\mathcal{H}_h$ for each $h \in \Lambda$;
3. a collection of $\mathbb{R}$-linear surjective maps T_h from $C^\infty(X)$ to the space $\tilde{H}_h$ of Hermitian endomorphisms of $\mathcal{H}_h$ with $T_h(1) = Id$ satisfying the following conditions as $h \to 0$:

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Definition (continued)

(P1) $\|T_h(f)\| = \|f\|_\infty + O(h)$ where $\|T_h(f)\|$ is the operator norm and $\|f\|_\infty$ is equal to $max_{x\in X}|f(x)|$;

(P2) For any $s > 0$ and $f, g \in C^\infty(X)$ we have

$$\|T_k(f)T_k(g) - (T_k(fg) + \sum_{j=1}^{s} T_k(c_j(f,g)h^j)\| = o(h^s)$$

(P3)

$$tr(T_h(f)) - (2\pi h)^{-d} \int_X f\omega^d/d! = O(h^{-d+1})$$

A geometric quantization T_h is a *Berezin-Toeplitz* quantization if

(P4) $T_h(f)$ is positive definite for $f > 0$.

Boutet de Monvel-Guillemin, Bordemann, Meinrenken and Schlichenmaier

Example

Let $X = S^1 \times S^1, \omega = du \wedge dv/(2\pi)^2, \Lambda = \{1/n\}, n \geq 1$. The algebra $C^\infty(X)$ is the algebra of functions $f(u,v)$ of the form

$$f(u,v) = \sum_{p,q \in \mathbb{Z}} \alpha_{a,b} u^a v^b, \alpha_{p,q} \in \mathbb{C}, \bar{\alpha}_{p,q} = \alpha_{-p,-q}$$

such that the coefficients $\alpha_{p,q}$ are rapidly decreasing for $(p,q) \to \infty$.

I define the formal quantization $\hat{A}$ as the algebras of series

$$\sum_{p,q \in \mathbb{Z}} \alpha_{p,q} U^p V^q, \alpha_{p,q} \in \mathbb{C}[[h]]$$

with analogous conditions on coefficients $\alpha_{p,q}$ and such that $UV = \exp(2\pi i h) VU$. We denote by $A' \subset \hat{A}$ the subalgebra of finite sums of element of the form $h^j U^p V^q$ and by $A \subset \hat{A}$ the subalgebra of elements of the form $\sum_{j=0}^r f_j h^j, f_j \in C^\infty(X)$.

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Example, continued

Let $\Lambda = \{1/n\}, n > 1$. For any $n \geq 1$ we define $\mathcal{H}_{1/n}$ as the space of complex valued functions F on the group $\mathbb{Z}_n$.

Claim

1. *For any $h = 1/n$ there exists a homomorphism $\kappa_n : A' \to End(\mathcal{H}_{1/n})$ such that $\kappa_n(h) = 1/n$ and*

$$\kappa_n(U)(F)(t) := F(t+1), \kappa_n(V)(F)(t) := \exp(2\pi i t/n)F(t), F \in \mathcal{H}_{1/n}, t \in \mathbb{Z}_n$$

2. *The homomorphism κ_n extends to a continuos map $\tilde{T}_{1/n} : A \to End(\mathcal{H}_{1/n})$.*

Let $i : C^\infty(X) \hookrightarrow A$ be the map given by

$$i(\sum_{p,q \in \mathbb{Z}} \alpha_{a,b} u^a v^b) = \sum_{p,q \in \mathbb{Z}} \alpha_{a,b} U^a V^b$$

and $T_{1/n} := \tilde{T}_{1/n} \circ i$.

Question

Does the family $\{T_{1/n}\}$ constitute a geometric quantization?

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Question

Does the family $\{T_{1/n}\}$ constitute a geometric quantization?

The existence of a geometric quantization is shown only in the case when the symplectic manifold (X, ω) admits a prequantization which is a complex line bundle L on X such that $[\omega] \in H^2(X, \mathbb{R})$ is the first Chern class of L. It is clear that (X, ω) admits a prequantization iff the class $[\omega] \in H^2(X, \mathbb{R})$ of the form ω is in the image of $(2\pi) \cdot H^2(X, \mathbb{Z})$. From now on I will assume that X is equipped with a prequantization.

Let $\|\|$ be a Hermitian metric on L such that the corresponding Chern curvature is equal to $-i\omega$. Let V_n^∞ the space of smooth sections of L^n. We denote by $V_n \supset V_n^\infty$ the Hilbert space completion in respect to the Hermitian form $\|s\|_{V_n}^2 := \int_X \|s\|^2 \omega^d$.

Definition

A compatible almost complex structure on a symplectic manifold (X, ω) is an automorphism J of the tangent bundle T_X such that

1. $J^2 = -Id$;
2. ω is J-invariant;
3. The form κ_J on $T^\star(X)$ is positive where $\kappa_J(v) := \omega(v, J(v))$.

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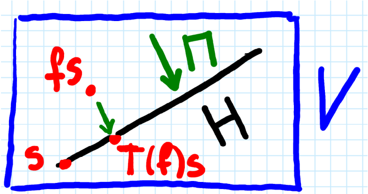
Berezin constructed a geometric quantization of (X, L) in the case when X is supplied with a compatible complex structure J. In this case L admits a canonical complex structure and the corresponding line bundle $\mathcal{L}$ on X is ample. He took $\Lambda := \{1/n\}, n \geq 1, \mathcal{H}_n := \Gamma(X, \mathcal{L}^n) \subset V_n$ and defined operators $T_h^J(f) \in End(\mathcal{H}_n), h = 1/n$ as the composition $T_h^J(f)(s) := \Pi_n(fs), s \in \mathcal{H}_n$ where $\Pi_n : V_n \to \mathcal{H}_n$ is the orthogonal projection.



As was shown later this construction can be applied to any pair (X, L) by supplying X with a compatible almost complex structure J and that the families $\{T_h^J(f)\}$ constitute geometric BT quantizations.

Remark

Since the construction of the geometric quantization from an almost complex structure uses Toeplitz operators, geometric quantizations satisfying the condition $(P4)$ are called a Berezin-Toeplitz (BT) quantization.

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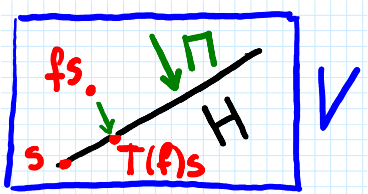

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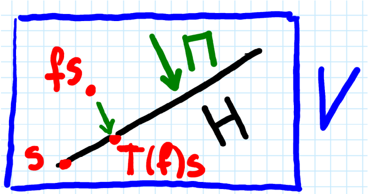

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The condition $(P4)$ of positivity implies stronger conditions on bidifferential operators c_j in the formal quantization.

Theorem

Assume that the formal quantization $\{c_j\}, j \geq 1$ on (X, ω) admits a BT quantization. Then

1. *There exist a metric G on $T^\star(X)$ such that $c_1^+(f,g) =: -\frac{1}{2}G(df, dg)$ where $c_1^+(f,g) := \frac{c_1(f,g)+c_1(g,f)}{2}$.*
2. *There exists unique compatible almost complex structure J on (X, ω) such that G is J-invariant and $G = \kappa_J + \rho$ where ρ is a non-negative symmetric bilinear form on $T^\star X$.*
3. *$G = \kappa_J$ if $T_h = T_h(J)$.*

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A BT quantization is *minimal* if $G = \kappa_J$.

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Almost all recent results in symplectic geometry and in the mirror symmetry are based on the study of compatible almost complex structures. Will be very interested to see whether our result is an indication for a possibility to bring the concept of BT quantizations into the game.

Definition

1. Two quantizations T_h and T'_h of a formal quantization with families of Hilbert spaces $\{H_h\}$ and $\{H'_h\}$, $h \in \Lambda$ of the same dimension are *s-equivalent* if there exists a sequence of unitary operators $U_h : H_h \to H'_h$ such that for all $f \in C^\infty(M)$,

$$\|U_h T_h(f) U_h^{-1} - T'_h(f)\| = O(h^s)$$

2. We say that quantizations T_h and T'_h are *semi-classically* equivalent when they are 1-equivalent.
3. The quantizations T_h and T'_h are equivalent if they are s-equivalent for all $s \geq 1$.

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Remark

A bidifferential operator c_1^+ depends only on a 2-equivalence class of a geometric quantization.

Claim (Charles)

Geometric quantizations $T^J_{1/n}$ and $T^{J'}_{1/n}$ are semi-classically equivalent for any two compatible complex structures J, J' on (X, ω, L).

Question

Does there exist a pair of non-equivalent geometric quantizations with the same formal quantization?

From now on I will only consider quantizations of (S^2, ω) where ω is the invariant form such that $\int_{S^2} \omega = 2\pi$.

Theorem

All minimal $SU(2)$-equivariant BT- quantizations of S^2 are 2-equivalent.

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From now on I will only consider quantizations of (S^2, ω) where ω is the invariant form such that $\int_{S^2} \omega = 2\pi$.

Theorem

All minimal $SU(2)$-equivariant BT- quantizations of S^2 are 2-equivalent.

Remark

A bidifferential operator c_1^+ depends only on a 2-equivalence class of a geometric quantization.

Claim (Charles)

Geometric quantizations $T^J_{1/n}$ and $T^{J'}_{1/n}$ are semi-classically equivalent for any two compatible complex structures J, J' on (X, ω, L).

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Let $T_k : C^\infty(S^2) \to End(H_k)$, $k \geq 1$, *be a geometric quantization of the sphere such that*

$$\limsup_{k \to +\infty} \dim H_k / k < 2 \ . \qquad (1)$$

Then there exists an integer $m \in \mathbb{Z}$ *such that*
$\dim H_k = k + m$ *for* $k >> 1$.
Furthermore, any other geometric quantization $Q_k : C^\infty(S^2) \to End(H'_k)$, $k \geq 1$, *with* $\dim H'_k = k + m$ *for all* $k >> 1$ *is semi-classically equivalent to* T_k.

Corollary

Under the dimension assumption (1), every geometric quantization of the sphere is semi-classically equivalent to a Berezin-Toeplitz quantization.

Remark

Since our proof of the last result is based on the following result in the theory of almost representations of Lie algebras we can not extend it to other surfaces.

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For every $c \in \mathbb{R}$ and $r > 0$, there exist $k_0 \in \mathbb{N}$ and $C > 0$ such that the following holds. Let H be a finite-dimensional Hilbert space, and assume that there exist $k \in \mathbb{N}$ with $k \geq k_0$ and a triple of operators $x_i \in \mathfrak{su}(H), i \in \mathbb{Z}/3\mathbb{Z}$, such that

(R1) $\left\| x_1^2 + x_2^2 + x_3^2 + \left(\frac{k^2}{4} + \frac{kc}{2}\right) Id \right\|_{op} \leq r$;

(R2) $\| [x_j, x_{j+1}] - x_{j+2} \|_{op} \leq r/k$ *for* $j \in \mathbb{Z}/3\mathbb{Z}$.

Then

(I) $c \in \mathbb{Z}$;

(II) $k/2 - C \leq \|x_j\|_{op} \leq k/2 + C$ *for all* $j \in \mathbb{Z}/3\mathbb{Z}$.

If in addition $\dim H < 2(k + c)$, *then*

(III) $\dim H = k + c$;

(IV) *there exists an irreducible representation* $\rho : \mathfrak{su}(2) \to \mathfrak{su}(H)$ *such that* $\|x_j - \rho(L_j)\|_{op} \leq C$.

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