Topology of integrable systems on 4-manifolds

Beijing–Moscow Mathematics Colloquium 2020

May 15, 2020

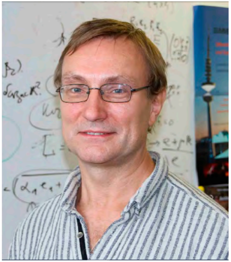
Elena Kudryavtseva

Moscow State University

May 15, 2020

Classification of 4D Integrable systems and singularities: researchers in MSU

A. Fomenko



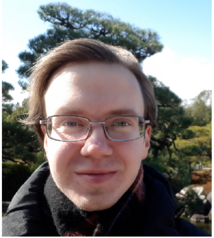
A. Bolsinov

E.Kudryavtseva

A.Oshemkov

I. Kozlov

S. Nikolaenko



V. Kibkalo

Integrable systems and their topological (and/or symplectic) invariants

An *integrable system* on a symplectic $2n$-manifold (M^{2n}, ω) is defined by n functions $f_1, \ldots, f_n$ satisfying two properties:

- they pairwise Poisson commute: $\{f_i, f_j\} = 0$, $\{f, g\} := \omega(X_f, X_g)$, $\omega(\cdot, X_f) = df$;
- they are functionally independent on M^{2n} almost everywhere.

Integral (momentum) map $F = (f_1, \ldots, f_n) : M^{2n} \to \mathbb{R}^n$.

Singular Lagrangian fibration (Liouville fibration) on M, whose regular fibres are invariant tori with quasi-periodic dynamics.

The fibres are connected components $\mathcal{L}_a$ of integral surfaces $F^{-1}(a)$.

Singular set S $= \{x \in M^{2n} \mid \operatorname{rank} dF(x) < n\}$.
Bifurcation diagram $\Sigma = F(S) \subset \mathbb{R}^n$.

Assume that all fibres are compact.
Then F generates the Hamiltonian action of $\mathbb{R}^n$ on M^{2n} by F-preserving symplectomorphisms $\phi^{t_1}_{X_{f_1}} \circ \cdots \circ \phi^{t_n}_{X_{f_n}} : M \to M$, $(t_1, \ldots, t_n) \in \mathbb{R}^n$. $\mathcal{L} = \bigcup_i \mathcal{O}_i$.

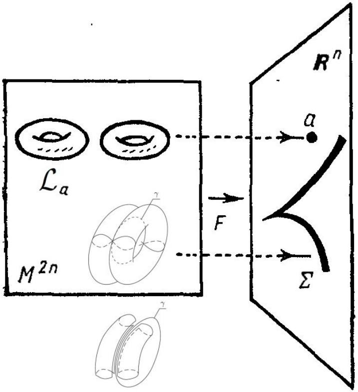
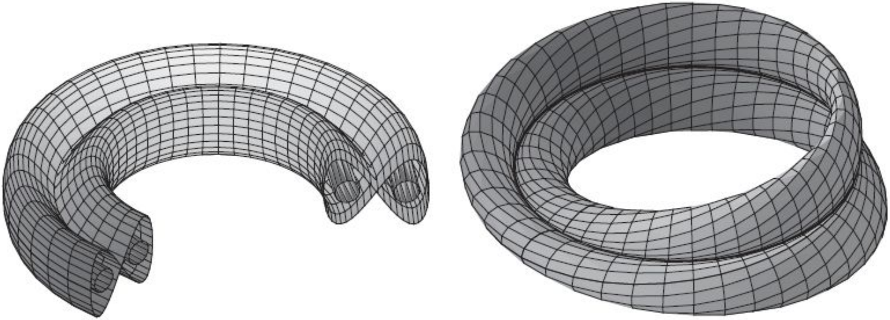


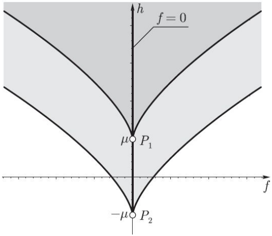
Illustration: Liouville fibration

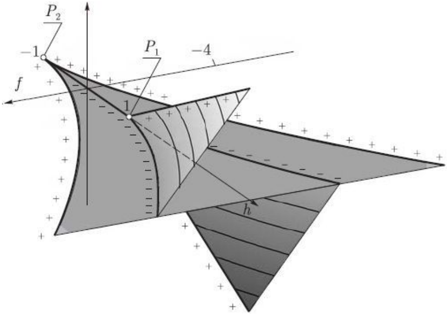
For simplicity: two degrees of freedom.
(M^4, ω, F), $F = (H, K) : M^4 \to \mathbb{R}^2$, $F(x) = (H(x), K(x))$.



Bifurcation Diagram and Bifurcation Complex. Goryachev-Chaplygin system

Take a regular point $a = (h, k) \in F(M) \setminus \Sigma$ in the image of the integral map F. Its preimage $\mathcal{L}_a$, i.e., the corresponding integral manifold, may contain more than one connected component (torus). We may think of this as different two-dimensional "leaves" over a regular region in $F(M)$. Different leaves are glued together only along branches of the bifurcation diagram Σ. Informally speaking, this collection of glued-together leaves and curves is called the bifurcation complex [A.Fomenko, 1986].



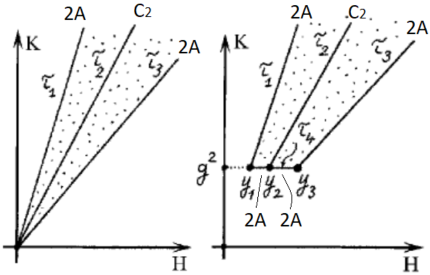


Definition

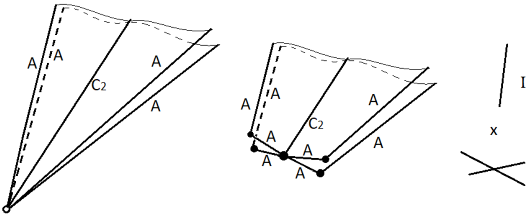
The bifurcation complex B is the topological space whose points are defined to be the fibres with the natural quotient topology [A.Fomenko, 1986] (i.e. B = the base).

There are natural projection maps $\tilde{F} : M \to B$ and $\pi : B \to F(M)$ such that $F = \pi \circ \tilde{F}$.

Bifurcation Diagram and Bifurcation Complex. Euler integrable case



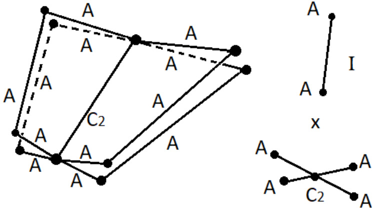
Bifurcation diagram (for zero and nonzero angular momentum values)

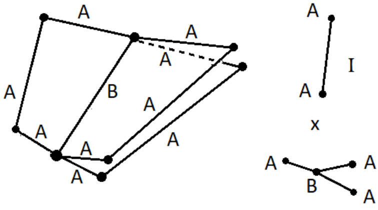


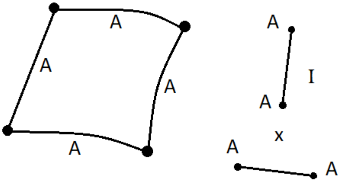
Bifurcation complex (for zero and nonzero angular momentum values)

Open problems

Problem: describe all singular Lagrangian fibrations (up to topological equivalence) having such a bifurcation complex: $X \times I$, $Y \times I$.
How many are them? What is their structure?







Singular Lagrangian fibrations

We are interested in the properties of the momentum map and, in some sense, "ignore" the dynamics. In particular,

- we are not going to solve this Hamiltonian system;
- we do not choose any distinguished Hamiltonian function among $f_1, \dots, f_n$;
- we do not fix these functions $f_1, \dots, f_n$ either allowing any kind of invertible transformations $(f_1, \dots, f_n) \mapsto (\tilde{f}_1, \dots, \tilde{f}_n)$.

In this view, the object we want to study is just a singular Lagrangian fibration

$$M^{2n} \to B^n,$$

which locally can be given by commuting functions.

It is more convenient to replace the image $F(M^{2n})$ of the momentum map by the set of fibres B^n which, in general, is not a smooth manifold.
However in all interesting examples, B has a structure of a stratified n-manifold with good topological properties, with integer affine connection.

Equivalent integrable systems

Given two integrable systems $F : M^{2n} \to B$ and $\tilde{F} : \tilde{M}^{2n} \to \tilde{B}$ (singular Lagrangian fibrations), we want to find/discuss/study conditions for the existence of *fibrewise maps* Φ between them:

$$\begin{array}{ccc} M^{2n} & \xrightarrow{\Phi} & \tilde{M}^{2n} \\ \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle \tilde{F}} \\ B & \xrightarrow{\phi} & \tilde{B}. \end{array}$$

Does **diffeomorphic** $\Longrightarrow$ **symplectomorphic**?

If not, what are additional **symplectic invariants**?

3 options for M and $\tilde{M}$:

- local (neighbourhood of a singular point or a singular orbit);
- semilocal (neighbourhood of a singular fibre);
- global (whole manifold M).

Two options for Φ (fibrewise map between M and $\tilde{M}$):

- topological;
- symplectic.

Allowed types of local singularities:

- non-degenerate singularities (direct products of ellyptic, hyperbolic, focus-focus);
- more general singularities, e.g. structurally stable ones (parabolic, integrable Hopf bifurcation etc.).

What is known about topological/symplectic invariants?

- Local
  - J. Vey 1978, H. Eliasson 1990:
    Non-degenerate singularities: no local symlectic invariants,
  - E. Miranda and N.T. Zung, 2004:
    Equivariant version of this result (near a non-degenerate orbit).
- Semi-local
  - A. Fomenko and H. Zischang, 1990: Topology of hyperbolic corank 1 singularities (2 d.f.),
    N.T. Zung, 1996: Topology of nondegenerate singularities,
    N.T. Zung, 2000: Topology of degenerate corank 1 singularities,
    A.S. Lermontova, 2005: Topology of hyperbolic corank 1 singularities,
  - J.-P. Dufour, P. Molino and A. Toulet, 1994: Hyperbolic singularities (one d.f.),
  - S. Vu Ngoc: Focus-focus singularities (two d.f., pinched torus),
  - H. Dullin and S. Vu Ngoc: Hyperbolic (saddle-saddle) singularities (2 d.f.),
  - A. Bolsinov and S. Vu Ngoc (2005, unpublished): Non-degenerate singularities.
- Global
  - A. Fomenko and H.Zieschang, 1990: Topology of fibrations on 3D isoenergy manifolds (2.d.f.),
  - J. Duistermaat, 1987: Regular case (no singular fibres), Mishachev, 1996 (2 d.f.),
  - T. Delzant, 1988: Toric actions,
  - A. Pelayo, S. Vu Ngoc, 2009: Semitoric manifolds (2 d.f.),
  - N.T. Zung, 2003: Very general case (topological and symplectic classifications).

Non-degenerate singularities

Definition

A singular point $x \in M^{2n}$ of rank 0 is called non-degenerate if

- the linear operators A_{f_i}, which are the linearizations of X_{f_i} at x, are linearly independent, i.e. generate an n-dimensional subalgebra of $sp(T_x M) \approx sp(\mathbb{R}, 2n)$,
- there exists a linear combination $\sum \lambda_i A_{f_i}$, $\lambda_i \in \mathbb{R}$, having only simple eigenvalues.

Example: in dimension 4, there are 4 conjugacy classes of such subalgebras:

центр-центр
$$\begin{pmatrix} 0 & 0 & -A & 0 \\ 0 & 0 & 0 & -B \\ A & 0 & 0 & 0 \\ 0 & B & 0 & 0 \end{pmatrix}$$

центр-седло
$$\begin{pmatrix} -A & 0 & 0 & 0 \\ 0 & 0 & 0 & -B \\ 0 & 0 & A & 0 \\ 0 & B & 0 & 0 \end{pmatrix}.$$

седло-седло
$$\begin{pmatrix} -A & 0 & 0 & 0 \\ 0 & -B & 0 & 0 \\ 0 & 0 & A & 0 \\ 0 & 0 & 0 & B \end{pmatrix}$$

фокус-фокус
$$\begin{pmatrix} -A & -B & 0 & 0 \\ B & -A & 0 & 0 \\ 0 & 0 & A & -B \\ 0 & 0 & B & A \end{pmatrix}$$

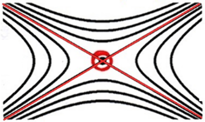
Example

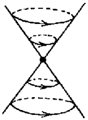
The canonical foliation L_{can} of the Williamson type (k_e, k_h, k_f) and rank r is given by the following quadratic functions on $(\mathbb{R}^{2n}, \omega = \sum dp_i \wedge dq_i)$:
1) $P_i^{ell} = p_i^2 + q_i^2$ (elliptic type), $1 \le i \le k_e$,
2) $P_i^{hyp} = p_i q_i$ (hyperbolic type), $k_e + 1 \le i \le k_e + k_h$,
3) $P_i^{foc} = p_i q_i + p_{i+1} q_{i+1}$, $P_{i+1}^{foc} = p_i q_{i+1} - p_{i+1} q_i$ (focus-focus type), $i = k_e + k_h + 2j - 1$, $1 \le j \le k_f$,
4) $P_i^{reg} = p_i$ (regular type), $k_e + k_h + 2k_f + 1 \le i \le n$. Here $k_e + k_h + 2k_f + r = n$.

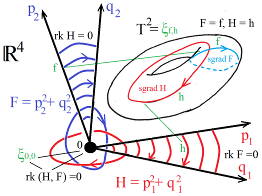
Local non-degenerate singularities = symplectic direct products

So, the canonical (local) foliation L_{can} is a direct product of basic foliations:
1) elliptic: $p_i^2 + q_i^2$,
2) hyperbolic: $p_i q_i$,
3) focus-focus: $p_j q_j + p_{j+1} q_{j+1} = \mathfrak{R}(p_j - i p_{j+1})(q_j + i q_{j+1})$,
$p_j q_{j+1} - p_{j+1} q_j = \mathfrak{I}(p_j - i p_{j+1})(q_j + i q_{j+1})$,
4) regular: p_i, where $k_e + k_h + 2k_f + r = n$.


центр


седло


фокус-фокус



Theorem (J. Vey 1978, H. Eliasson 1990, local symplectic classification)

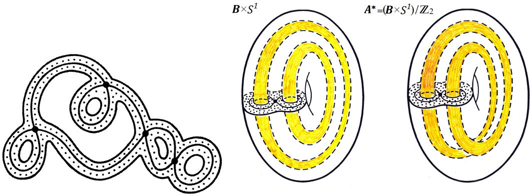
The Liouville foliation in a neighborhood of a non-degenerate singular point of rank r is locally symplectomorphic to a canonical foliation L_{can}, which is the direct product of basic foliations: elliptic, hyperbolic, focus-focus, and regular ones.

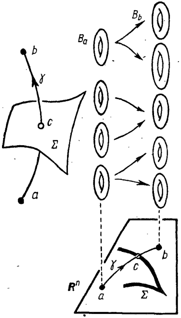
E. Miranda, N.T. Zung, 2004:
Equivariant version of this result near a nondegenerate orbit.

Semilocal topological classification of non-degenerate singularities

Theorem (A. Fomenko, H. Zieschang, 1990, semi-local topological classification of corank 1 non-degenerate singularities)

In dimension 4, the hyperbolic corank 1 singularities that satisfy the non-splitting condition can be of the following two topological types:
1) direct products $V_{hyp} \times V_{reg}$;
2) almost direct products $(V_{hyp} \times V_{reg})/\mathbb{Z}_2$ *with the action of the group* $\mathbb{Z}_2$ *defined by* $(x, s, \varphi) \mapsto (\tau(x), s, \varphi + \pi)$, *where* $x \in V_{hyp}$, (s, φ) *are action-angle variables on* $V_{reg} = D^1 \times S^1$, *and* τ *is an involution* $V_{hyp} \to V_{hyp}$ *whose fixed points are some vertices of the hyperbolic atom* V_{hyp}.





A.S. Lermontova, 2005: corank 1 hyperbolic $\sim (V_{hyp} \times V_{reg}^{n-1})/(\mathbb{Z}_2)^{n-1}$.
N.T. Zung, 2000: corank 1 degenerate $\sim (V \rtimes V_{reg}^{n-1})/(\oplus_{i=1}^{n-1}\mathbb{Z}_{k_i})$.

Theorem (Nguyen Tien Zung, decomposition Theorem for semi-local non-degenerate singularities, 1996)

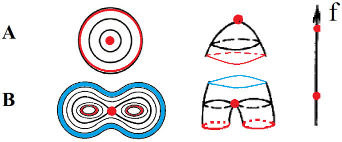
Each non-degenerate singularity (satisfying the non-splitting condition) is topologically equivalent to a singularity of almost direct product type $(V_1 \times \cdots \times V_k)/G$ *whose factors have one of the following four types:*

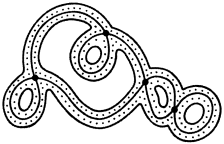
1) an elliptic singularity V_{ell} *with one degree of freedom (i.e., 2-atom A);*

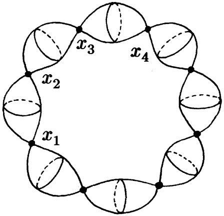
2) a hyperbolic singularity V_{hyp} *with one degree of freedom (2-atom);*

3) a focus-focus singularity V_{foc} *with two degrees of freedom;*

4) a trivial Liouville foliation $V_{reg} = D^1 \times S^1$ *without singularities with 1 degree of fr.*







Semilocal topological classification of non-degenerate singularities

The complexity of a semilocal singularity (= a singular fibre) of rank r is the number of singular orbits of rank r lying in this fibre.

- A.V. Bolsinov, A.A. Oshemkov, I.K. Kozlov: topological classification of semilocal singularities of rank 0 (that satisfy the non-splitting condition) having small complexity (≤ 3), in dimensions 4 and 6.
  E.g. for complexity 1: $B \times B$, $(B \times C_2)/\mathbb{Z}_2$, $(B \times D_1)/\mathbb{Z}_2$, $(C_2 \times C_2)/(\mathbb{Z}_2 \times \mathbb{Z}_2)$.
- Topological stability problem for singularities.

A (local/semilocal) singulariry is called structurally stable if the topology of the fibration is preserved after any (small enough) integrable perturbation of the system.

Examples of stable singularities:

- all local non-degenerate singularities (from Eliasson-Vey theorem),
- any semilocal non-degenerate singularity of complexity one, topologically a direct product (from Eliasson-Vey theorem),
- some degenerate singularities: parabolic circles (Lerman, Umanskii, 1987), parabolic circles with resonances (Kalashnikov, 1998, K., work in progress).

Problem

Describe all stable non-degenerate (semilocal) singularities.
Describe stable degenerate (local, semilocal) singularities.

Global symplectic invariant: Actions = integer affine structure on B

Theorem (Liouville theorem)

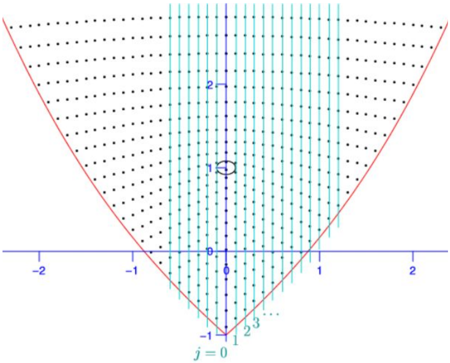
Let $\mathcal{L}$ be a regular compact fibre of a Lagrangian fibration. Then, in some neighbourhood $U(\mathcal{L})$ (i.e., semilocally) this fibration is fibrewise symplectomorphic to the standard model: $F : T^n \times D^n \to D^n$, (here T^n is a torus and D^n is a disc) and $\omega = \sum_{i=1}^{n} dI_i \wedge d\varphi_i$, where $\varphi_1, \dots, \varphi_n$ (angles) are 2π-periodic coordinates on T^n (fibre) and $I_1, \dots, I_n$ (actions) are coordinates on D^n (base).

Important properties: (i) explicit formula for action variables: $2\pi I_i = \oint_{\gamma_i} \alpha = \iint_{C\gamma_i} \omega$ where $d\alpha = \omega$; (ii) the actions are defined modulo $\mathbb{R}^n \rtimes GL(n, \mathbb{Z})$.

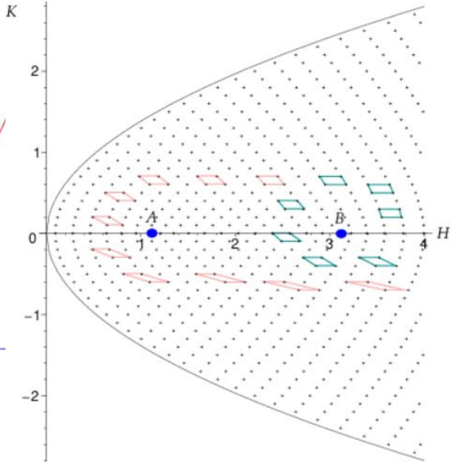
Conclusion: action variables = integer affine structure on $B_{\mathrm{reg}} \subseteq B$.

Problem 2: Let $\phi : B \to \tilde{B}$ be an affine C^∞-equivalence. Can it be lifted up to a fibrewise symplectomorphism $\Phi : M \to \tilde{M}$?

Yes: Liouville (regular case), Delzant (toric), Vũ Ngọc & Pelayo (semitoric 2 d.f.), Toulet & Dufour (generic hyperbolic 1 d.f.), Oshemkov & Rembovskaya (non-generic case), Kirillov (boundary case), Dullin & Vũ Ngọc (hyperbolic 2 d.f.), Vũ Ngọc (focus-focus 2 d.f.).
No: Oshemkov & Rembovskaya (non-generic hyperbolic 1 d.f.), Guglielmi ("too degenerate" 1 d.f.).

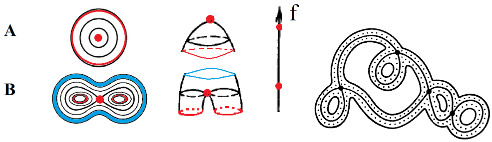


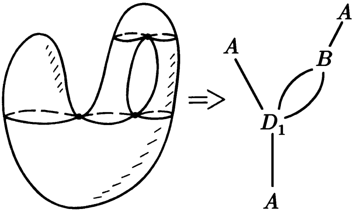
Spherical pendulum



Quadratic spherical pendulum

One degree of freedom





Liouville fibration $\mapsto$ the molecule W.

Theorem (Bolsinov, Fomenko, 2004)

Two (non-degenerate) integrable systems with one degree of freedom are topologically equivalent if and only if their molecules are identical.

Theorem (J.-P. Dufour, P. Molino, A. Toulet, 1994)

The graph W endowed with an integer affine C^∞-structure is a complete symplectic invariant of a one-dimensional Liouville fibration with singularities of types A and B only. In other words, two Liouville fibrations L_1 and L_2 are fibrewise symplectomorphic if and only if there exists a C^∞-isomorphism $\psi : W_1 \to W_2$ which preserves the integer affine structure on the edges.

Global topological invariants: Fomenko-Zieschang marked molecule

Two degrees of freedom
$Q^3 = \{H = \text{const}\}$, a Bott function $K|_{Q^3} \mapsto$ marked molecule W^* (a very simple object).
Examples of a marked molecule W^*: $A \frac{\varepsilon=1}{r=0} A$, $A \frac{\varepsilon=1}{r=0} C_1 \frac{\varepsilon=-1}{r=\infty} A$.

Theorem (A.T. Fomenko, H. Zieschang, 1990)

Marked molecule W^* classifies non-degenerate integrable Hamiltonian systems on isoenergy 3-surfaces up to the topological equivalence. In other words, two fibrations (Q_1^3, L_1) and (Q_2^3, L_2) are fibrewise homeomorphic if and only if the corresponding marked molecules W_1^* and W_2^* are identical.

Theorem (A.T. Fomenko, H. Zieschang, 1987)

An orientable 3-manifold M^3 admits a Liouville fibration with non-degenerate singularities if and only if M is a graph-manifold (i.e., can be glued from pieces of two types: solid tori $D^2 \times S^1$ and "pants" $N^2 \times S^1$).

Generalizations:

Loop molecule W^* classifies fibrations on $Q^3 = \{(H - h_0)^2 + (K - k_0)^2 = \varepsilon\}$.

Marked net classifies fibrations on their maximal invariant domains having only nondegenerate corank-1 singularities (for n degrees of freedom) [A.Fomenko, 1991].

Theorem (Duistermaat, 1987)

A complete set of symplectic invariants for non-singular Lagrangian fibrations over B^n:
1) the integer affine structure on the base B;
2) the topological structure of the fibration (which determines an obstruction to the existence of a global section of the fibration);
3) "Lagrangian Chern class" (which determines, roughly speaking, obstruction to the existence of a global Lagrangian section of the fibration).

Open problem: describe the complete list of Liouville fibrations without singularities.

Theorem (Mishachev, 1996)

All Lagrangian T^2-bundles over $B = T^2$ can be obtained from Lagrangian bundles admitting a global Lagrangian section (described in item 1 below) by applying operations 2 and 3:
1) changes the affine structure on B^2: $(T^*B^2)/Z$, $\omega = dx \wedge dp_x + dy \wedge dp_y$, where x, y are angle coordinates on the base B^2, Z is a family of lattices generated by 1-forms α_1, α_2, either $\alpha_1 = adx + bdy$ and $\alpha_2 = cdx + d\,dy$, $a, b, c, d \in \mathbb{R}$, $ad - bc \neq 0$, or $\alpha_1 = bdy$ and $\alpha_2 = adx + bky\,dy$, $a, b \in \mathbb{R}$, $k \in \mathbb{Z}_+$,
2) some surgery of T^2-bundle (cut a small disk from B^2 together with fibres over it, and then glue it back),
3) changes the symplectic structure, while preserving both the topology of the fibration and the affine structure on the base ($\omega \mapsto \omega + \pi^*\sigma$ where σ is a 2-form on B^2).

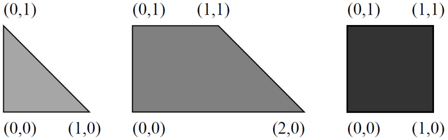
Hamiltonian torus action

Theorem (M.F. Atiyah 1982, V. Guillemin, S. Sternberg 1982)

Let first integrals $f_1, \dots, f_n$ of an integrable Hamiltonian system on a closed symplectic manifold (M^{2n}, ω) be 2π-periodic, i.e. generate a Hamiltonian T^n-action. Then $F(M)$ is a convex polytop.

Theorem (T. Delzant 1988)

If $F_1(M_1) = F_2(M_2)$, then there exists a T^n-equivariant symplectomorphism $\Phi : M_1 \to M_2$ such that $F_2 \circ \Phi = F_1$.
A polytop $P \subset \mathbb{R}^n$ can be the image of the momentum mapping $F : M^{2n} \to \mathbb{R}^n$ corresponding to a Hamiltonian effective n-torus action if and only if P is convex and each vertex O of the polytop P is incident to exactly n edges $v_1, \dots, v_n$ and there exist $\lambda_i \in \mathbb{R}$ such that $\lambda_1 v_1, \dots, \lambda_n v_n$ is a basis of the integer lattice $\mathbb{Z}^n \subset \mathbb{R}^n$.



Example: momentum polytope of $\mathbb{C}P^2$ (left), a Hirzebruch surface (center) and $(\mathbb{C}P^1)^2$ (right), all of which determine the isomorphism type of the fibration.

Remark. Hamiltonian torus action $\Longrightarrow$ only nondegenerate elliptic singularities.

"$\Longleftarrow$" is not true: almost toric fibrations (= no hyperbolic components) $S^2 \times T^2 \to S^1 \times D^1$ of type (ii) below does not admit any global torus action.

Semitoric fibrations

An integrable system (M, ω, F) is semitoric if M is a connected 4D manifold, $F = (J, H) : M \to \mathbb{R}^2$ has only non-degenerate singularities, without hyperbolic blocks, and J is a proper momentum map for a Hamiltonian circle action on M.

Assumption: values $J(x_i)$ at focus-focus points $x_i \in M$ are pairwise different.

Two semitoric systems are isomorphic if there exists a symplectomorphism $\Phi : M_1 \to M_2$ s.t. $\Phi^*(J_2, H_2) = (J_1, f(J_1, H_1))$ for some smooth function f, $\partial f / \partial H_1 \neq 0$.

Theorem (A. Pelayo, S. Vu Ngoc, 2009)

Two 4-dimensional semitoric integrable systems $(M_i, \omega_i, (J_i, H_i))$, $i = 1, 2$, are isomorphic if and only if they have the same list of invariants (i)–(v):

(i) the number of singularities invariant: the number $0 \leq m_f < \infty$ of focus-focus pts;
(ii) the singularity type invariant: the m_f-tuple $((S_i)^\infty)_{i=1}^{m_f}$ characterizing singularities up to semilocal fibrewise symplectomorphism;
(iii) the polygon invariant: a family of weighted rational convex polygons $\Delta_{weight} := (\Delta, (\ell_j)_{j=1}^{m_f}, (\varepsilon_j)_{j=1}^{m_f})$;
(iv) the volume invariant: the m_f-tuple $(h_i)_{i=1}^{m_f}$, where $h_i > 0$ is the height ($=$ volume);
(v) the twisting index invariant: the m_f-tuple of integers $(k_i)_{i=1}^{m_f}$ (for each Δ_{weight} from (iii)) measuring how twisted the system is around singularities.

One could say that (i) and (ii) are analytical invariants, (iii) is a combinatorial/group-theoretic invariant, and (iv), (v) are geometric invariants.
Remark: (v) determines (iii).

A singular Lagrangian fibration $\pi : (M^{2n}, \omega) \to B^n$ is almost toric if M^{2n} is closed, any critical point is non-degenerate and has no hyperbolic components.

Assumption: $\dim M = 4$, and the values $\pi(x_i) \in B$ (called nodes) at focus-focus points $x_i \in M$ are pairwise different.

Theorem (N.C. Leung, M. Symington, 2010)

The symplectic classification of such almost toric fibrations is given by the following list:

(i) $B = D^2$ with $n \geq 0$ nodes and $k \geq \max(0, 3-n)$ vertices, $M = \mathbb{C}P^2 \# (n+k-3)\overline{\mathbb{C}P}^2$ or $M = S^2 \times S^2$ (if $n + k = 4$);
(ii) $B = S^1 \times D^1$ with $n \geq 0$ nodes and no vertices, $M = (S^2 \times T^2) \# n \overline{\mathbb{C}P}^2$ or $M = (S^2 \tilde{\times} T^2) \# n \overline{\mathbb{C}P}^2$;
(iii) $B = S^1 \tilde{\times} D^1$ with $n \geq 0$ nodes and no vertices, M is the same as in (ii);
(iv) $B = S^2$ with 24 nodes and no vertices, M is a K3 surface;
(v) $B = \mathbb{R}P^2$ with 12 nodes and no vertices, M is an Enriques surface;
(vi) M is a T^2-bundle over $B = T^2$ with monodromy $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \right\}$;
(vii) M is a T^2-bundle over $B = S^1 \tilde{\times} S^1$ with monodromy $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \right\}$.

Global topological/symplectic classifications by Zung

Two integrable Hamiltonian systems (= singular Lagrangian fibrations) are said to be roughly topologically (resp. roughly symplectically) equivalent if there exists a homeomorphism between the bases $\phi : B_1 \to B_2$ which locally in a neighborhood of each point can be lifted up to a fibrewise homeomorphism (resp. symplectomorphism). Besides, it is required that on intersections the lifted homeomorphisms are homotopic to the identity on each orbit and induce the same map on $H_1(\mathcal{L}_a, \mathbb{Z})$ for each fibre $\mathcal{L}_a$.

So, the following invariants should coincide for roughly equivalent systems:
1) the base B^n,
2) all types of singularities,
3) "homological monodromy" (= affine monodromy sheaf R, $\approx$ sheaf of local system-preserving symplectic S^1-actions, $\approx$ marked molecules), which establishes the relationship between the fundamental (homological) groups of different leaves.

What additional invariants are sufficient for a complete solution of our problems?

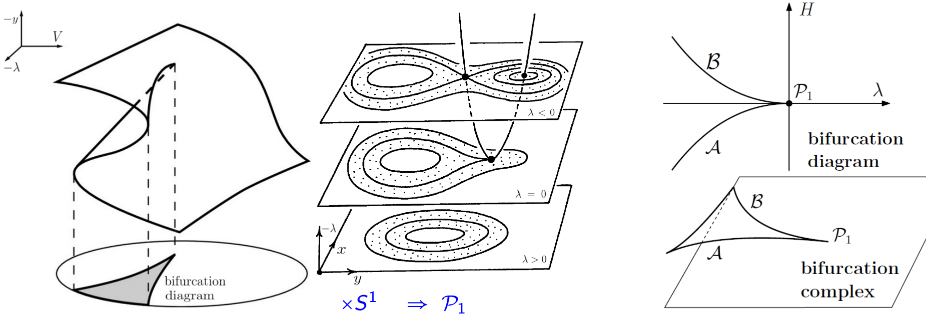
Theorem (N.T. Zung, 2003)

Two roughly topologically (resp., roughly symplectically) equivalent systems are topologically (resp., symplectically) equivalent if and only if the corresponding characteristic Chern (resp., Lagrangian) classes coincide.

The Chern class is an element of $H^2(B, R)$. The Lagrangian class lies in $H^1(B, Z^1/R)$. Here Z^1 is the sheaf of local closed differential 1-forms on B, $R \subset Z^1$.

Example (parabolic orbit: Lerman & Umanskii'87)

on some neighbourhood $U \approx D^3_{(x,y,\lambda)} \times T^1_{(\varphi)}$ of a singular orbit $\mathcal{O} = \{0\} \times T^1$ (i.e., locally), the singular Lagrangian fibration has a ("semidirect product") structure: $F = (H, I) : U \to D^1 \times D^1$ and $\Omega = \pi^* \omega_1 + d\lambda \wedge d\varphi$, where $H = x^2 + y^3 + \lambda y$, $I = \lambda$. Here $\pi : U \to D^3_{(x,y,\lambda)}$, ω_1 is a closed 2-form on $D^3_{(x,y,\lambda)}$ ($\omega_1|_{\lambda=0} = g dx \wedge dy$, $g > 0$). A fibre $\mathcal{L} \supset \mathcal{O}$ with regular $\mathcal{L} \setminus \mathcal{O}$ is a *cuspidal torus*.



$V_\lambda(y) = y^3 + \lambda y$ yields a *Fold Catastrophe* for the map $(y, \lambda) \mapsto (V, \lambda)$ at $\lambda = y = 0$.

Properties: stable under integrable perturbations (Lerman & Umanskii'87).

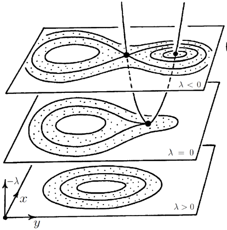
Example (parabolic orbit with resonance: Kalashnikov'98 + Wasserman'76/98)

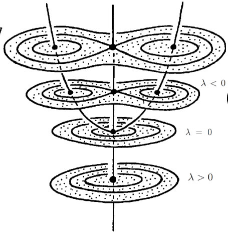
on some neighbourhood $U \approx (D^3_{(x,y,\lambda)} \times T^1_{(\varphi)})/G$ of $\mathcal{O} = (\{0\} \times T^1)/G$ (i.e., locally), the singular Lagrangian fibration has an ("almost-semidirect product") structure: $F = (H, I) : U \to D^1 \times D^1$ and $\Omega = \pi^* \omega_s + d\lambda \wedge d\varphi$, where $H = f_{\lambda,s}(x, y)$, $I = I(\lambda)$ $(= \lambda$ if $s \le 4)$, ω_s is a G-invariant closed 2-form on $D^3_{(x,y,\lambda)}$ $(\omega_s|_{\lambda=0} = g(x, y)dx \wedge dy$, $g > 0)$, $\pi : D^3_{x,y,\lambda} \times S^1_{(\varphi)} \to D^3_{(x,y,\lambda)}$. A generator of $G = \mathbb{Z}_s$ acts by $(z, \lambda, \varphi) \mapsto (e^{2\pi i\ell/s} z, \lambda, \varphi + \frac{2\pi}{s})$, (here $z = x + iy$),

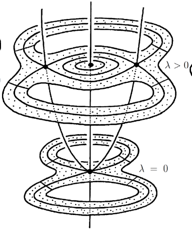
$f_{\lambda,1}(x, y) = x^2 + y^3 + \lambda y$,	semiloc: cuspidal torus (birth/death of $\mathcal{A}, \mathcal{B}$)
$f_{\lambda,2}(x, y) = x^2 \pm y^4 + \lambda y^2$,	semiloc: 3 period-doubling bif's
$f_{\lambda,3}(z) = Re(z^3) + \lambda\|z\|^2$,	semilocally: 2 period-tripling bif's
$f_{\lambda,4}(z) = Re(z^4) + a(\lambda)\|z\|^4 + \lambda\|z\|^2$,	$a^2 \ne 1$, semiloc: 3 period-quadrupling bif's
$f_{\lambda,s}(z) = Re(z^s) + \|z\|^4 + \lambda\|z\|^2$,	$s \ge 5$. semiloc: $\frac{\varphi(s)}{2}$ period-s-tupling bif's

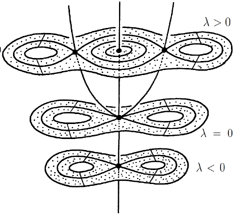
Properties: often appear in mechanics (Kovalevskaya top, ...), are typical rank-1 orbits (Kalashnikov'98 + Zung'00), naturally occur as "transition states" between non-degenerate singularities; parabolic singularities of a given resonance ℓ/s, $s \ne 4$, are (real-analytic locally, C^∞-semilocally) fibrewise diffeomorphic.

Kalashnikov's typical rank-1 singularities: vanishing cycles

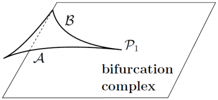


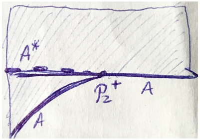


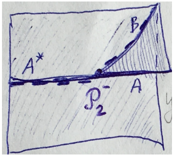


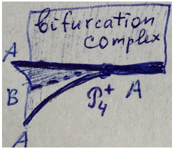


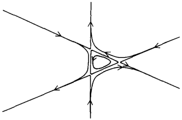
$\times S^1 \quad \Rightarrow \quad \mathcal{P}_1$ $\qquad$ $\times S^1/\mathbb{Z}_2 \quad \Rightarrow \quad \mathcal{P}_2^+$ $\qquad$ $\times S^1/\mathbb{Z}_2 \quad \Rightarrow \quad \mathcal{P}_2^-$ $\qquad$ $\times S^1/\mathbb{Z}_2 \quad \Rightarrow \quad \mathcal{P}_2^-$

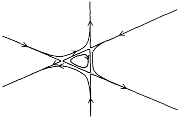


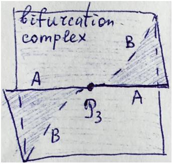


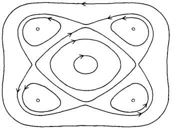


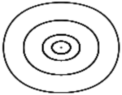












$\times S^1/\mathbb{Z}_3 \quad \Rightarrow \quad \mathcal{P}_3$ $\quad$ ($\mathcal{P}_4^-$ is similar) $\qquad$ $\times S^1/\mathbb{Z}_4 \quad \Rightarrow \quad \mathcal{P}_4^+$ $\quad$ ($\mathcal{P}_5, \ldots$ similar)

Symplectic invariants of (real-analytic) parabolic orbits with resonances

Consider two functions $H = f_{\lambda,s}(x,y)$ and $I = \lambda$ (e.g. $f_{\lambda,1} = x^2 + y^3 + \lambda y$ and $I = \lambda$) that commute simultaneously w.r.t. two real-analytic symplectic forms $\Omega = \pi^*\omega_s + dg(\lambda) \wedge d\varphi$ and $\tilde{\Omega} = \pi^*\tilde{\omega}_\lambda + d\tilde{g}(\lambda) \wedge d\varphi$ on $U \approx (D^3_{(x,y,\lambda)} \times T^1_{(\varphi)})/G$, where $\omega_\lambda = f(x,y,\lambda)dx \wedge dy$ and $\tilde{\omega}_\lambda = \tilde{f}(x,y,\lambda)dx \wedge dy$ and $f, g', \tilde{f}, \tilde{g}' > 0$.

Theorem 1

For such a parabolic orbit γ_0 with $\frac{\ell}{s}$ resonance, the following two statements are equivalent.

(i) *In a tubular neighborhood of γ_0, there is a (real-analytic) diffeomorphism Φ such that*

- Φ *preserves H and I;*
- $\Phi^*(\widetilde{\Omega}) = \Omega$ *(i.e. Φ is a symplectomorphism).*

(ii) *The actions (real-analytic on the "swallow-tail domain" corresponding to the family of "narrow" Liouville tori) corresponding to the family of vanishing cycles coincide, $I_\circ(H,I) = \widetilde{I}_\circ(H,I)$.*

Symplectic invariants of (real-analytic) parabolic orbits with resonances

Theorem 2

The following two statements are equivalent.

(i) *There exists a real-analytic fiberwise symplectomorphism* $\Phi : U(\gamma_0) \to \widetilde{U}(\widetilde{\gamma}_0)$ *between some tubular neighborhoods* $U(\gamma_0), \widetilde{U}(\widetilde{\gamma}_0)$ *of the parabolic orbits* $\gamma_0, \tilde{\gamma}_0$ *with resonances* $\frac{\ell}{s}, \frac{\tilde{\ell}}{\tilde{s}}$.

(ii) *The resonances coincide* $\frac{\ell}{s} = \frac{\tilde{\ell}}{\tilde{s}}$, *and the corresponding bases* B *and* $\tilde{B}$ *are affinely equivalent. This means that there exists a local real-analytic diffeomorphism* $\phi : B \to \tilde{B}$ *that*

- *respects the bifurcation diagrams together with their partitions into hyperbolic and elliptic branches:*

$$\phi(\Sigma) = \tilde{\Sigma}, \quad \text{moreover} \quad \phi(\Sigma_{\rm ell}) = \tilde{\Sigma}_{\rm ell} \quad \text{and} \quad \phi(\Sigma_{\rm hyp}) = \tilde{\Sigma}_{\rm hyp},$$

- *and preserves the action variables:* $I = \tilde{I} \circ \phi$ *and* $I_\circ = \tilde{I}_\circ \circ \phi$.

Symplectic invariants of (real-analytic) cuspidal tori with resonances

Suppose fibres $\mathcal{L}_0, \tilde{\mathcal{L}}_0$ contain parabolic orbits $\gamma_0, \tilde{\gamma}_0$ with resonances $\frac{\ell}{s}, \frac{\tilde{\ell}}{\tilde{s}}$. Suppose $\mathcal{L}_0 \setminus \gamma_0, \tilde{\mathcal{L}}_0 \setminus \tilde{\gamma}_0$ are regular.

Theorem 3
Suppose that there is a fiberwise diffeomorphism $\Psi : U(\mathcal{L}_0) \to \widetilde{U}(\widetilde{\mathcal{L}}_0)$ that preserves the actions in the sense that for every cycle $\tau \subset \mathcal{L}_{H,F}$ we have

$$\oint_{\Psi(\tau)} \tilde{\alpha} = \oint_{\tau} \alpha \qquad\qquad \left(\text{or } \iint_{\Psi(C\tau)} \tilde{\Omega} = \iint_{C\tau} \Omega\right)$$

for some 1-forms α and $\tilde{\alpha}$ satisfying $d\alpha = \Omega$, $d\tilde{\alpha} = \tilde{\Omega}$. Then there exists a fiberwise symplectomorphism $\Phi : U(\mathcal{L}_0) \to \widetilde{U}(\widetilde{\mathcal{L}}_0)$.

Bolsinov A.V., Guglielmi L., Kudryavtseva E.A. Symplectic invariants for parabolic orbits and cusp singularities of integrable systems with two degrees of freedom. Phyl. Trans. R. Soc. A **376** (2018), 20170424 (https://arxiv.org/abs/1802.09910).