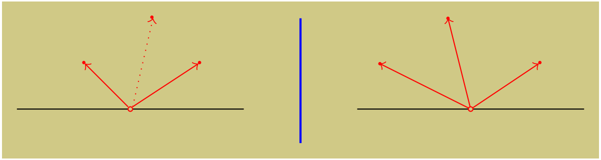
Geometry from Donaldson-Thomas invariants

Tom Bridgeland

University of Sheffield



Geometry from DT invariants

The aim is to use the DT invariants of a CY_3 triangulated category to define a geometric structure on its space of stability conditions.

There is a nice conceptual framework, and lots of interesting calculations in particular examples, but no general results yet.

The main idea is that the DT invariants define Stokes data for a family of meromorphic connections on $\mathbb{P}^1$, parameterised by $\mathrm{Stab}(\mathcal{D})$, with values in the group of Poisson automorphisms of $(\mathbb{C}^*)^n$.

The analytic input is a class of Riemann-Hilbert problems for piecewise holomorphic maps $X \colon \mathbb{C}^* \to (\mathbb{C}^*)^n$ with jumps across rays prescribed by the DT invariants, and fixed asymptotics at $0, \infty$.

Very similar ideas appeared in the work of Gaiotto, Moore and Neitzke 10 years ago: our story is the "conformal limit" of theirs.

Update on progress

- Joint work with Ian Strachan gives a geometric description of the genus 0 output of the RH problem: a complex hyperkähler structure on the total space of the tangent bundle $\mathcal{T}\,\mathrm{Stab}(\mathcal{D})$.
- Solutions to the RH problems defined by the DT theory of coherent sheaves on a CY_3 without compact divisors. This leads to a non-perturbative topological string partition function: a particular analytic function whose asymptotic expansion gives the genus expansion in GW theory.
- Solution to the RH problem for the DT theory of the A2 quiver. This involves the isomonodromic system for Painlevé I, and the Fock-Goncharov cluster structure on the character variety. Work in progress aims to generalise this to quivers associated to triangulated surfaces (rank 1 theories of class S).

Plan of the talk

1. Stokes data and isomonodromy in the case $G = \mathrm{GL}_n(\mathbb{C})$.
2. Review of the wall-crossing formula in DT theory.
3. The expected geometric structure on $\mathrm{Stab}(\mathcal{D})$.
4. The Riemann-Hilbert problem: A_1 case.
5. (The geometric case: coherent sheaves on a CY_3.)
6. (The A_2 case after Gaiotto-Moore-Neitzke.)

1. Isomondromy and Stokes data.

Finite-dimensional setting

We first consider Stokes data in the finite-dimensional setting

$$\mathfrak{g} = \mathfrak{g}l_n(\mathbb{C}), \qquad G = \mathrm{GL}_n(\mathbb{C}).$$

We use the standard root decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{g}^{\mathrm{od}}, \qquad \mathfrak{g}^{\mathrm{od}} = \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha,$$

with root system $\Phi = \{e_i^* - e_j^*\} \subset \mathfrak{h}^*$.

Later we will replace these with infinite-dimensional objects

$$\mathfrak{g} = \mathrm{Vect}_{\{-,-\}}(\mathbb{C}^*)^n, \qquad G = \mathrm{Aut}_{\{-,-\}}(\mathbb{C}^*)^n,$$

where $\{-,-\}$ is an invariant Poisson structure on $(\mathbb{C}^*)^n$.

An irregular singularity

Consider an equation for $Y\colon \mathbb{C}^* \to G = \mathrm{GL}_n(\mathbb{C})$ of the form

$$\frac{d}{dt}Y(t) = \left(\frac{U}{t^2} + \frac{V}{t}\right)Y(t)$$

with constant matrices $U, V \in \mathfrak{g} = \mathfrak{g}l_n(\mathbb{C})$, such that

(I) $U = \mathrm{diag}(u_1, \cdots, u_n) \in \mathfrak{h}^{\mathrm{reg}}$ is diagonal with $u_i \neq u_j$,

(II) $V \in \mathfrak{g}^{\mathrm{od}}$ has zeroes on the diagonal.

The Stokes rays of the equation at $t = 0$ are the rays

$$\mathbb{R}_{>0} \cdot (u_i - u_j) = \mathbb{R}_{>0} \cdot U(\alpha).$$

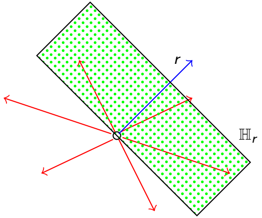
This is the simplest example of an irregular singularity; when $V = 0$ the solution is $Y(t) = \exp(-U/t)$.

Canonical solutions in half-planes

Theorem (Balser, Jurkat, Lutz)

For any half-plane $\mathbb{H}_r \subset \mathbb{C}^*$ centered on a non-Stokes ray $r \subset \mathbb{C}^*$, there is a unique solution $Y_r \colon \mathbb{H}_r \to G$ such that

$$Y_r(t) \cdot \exp(U/t) \to 1 \text{ as } t \to 0.$$

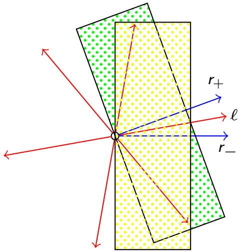


Stokes factors

The Stokes factor $\mathbb{S}(\ell)$ associated to a Stokes ray ℓ is defined by

$$Y_{r_+}(t) = Y_{r_-}(t) \cdot \mathbb{S}(\ell), \qquad \mathbb{S}(\ell) \in \exp\Big(\bigoplus_{U(\alpha)\in\ell} \mathfrak{g}_\alpha \Big) \subset G,$$

where $r_\pm$ are small perturbations of ℓ.



Note that there is the same amount of data in V and $\{\mathbb{S}(\ell)\}$: both determine by an element of $\mathfrak{g}^{od}$.

Isomonodromic deformations

Recall our matrix differential equation

$$\frac{d}{dt}Y(t) = \left(\frac{U}{t^2} + \frac{V}{t}\right)Y(t), \qquad U \in \mathfrak{h}^{\mathrm{reg}},\ V \in \mathfrak{g}^{\mathrm{od}}.$$

If we vary the leading term $U \in \mathfrak{h}^{\mathrm{reg}}$, we can uniquely deform the matrix $V = V(U)$ so that the Stokes factors $\mathbb{S}(\ell)$ remain constant.

This isomonodromy condition is equivalent to the p.d.e.

$$d \log V_\gamma(U) = \sum_{\alpha+\beta=\gamma} [V_\alpha, V_\beta] \cdot d \log U(\beta), \qquad V = \sum_{\gamma \in \Phi} V_\gamma.$$

This is the irregular version of the Schlesinger equations; generic solutions give rise to semi-simple Frobenius manifolds.

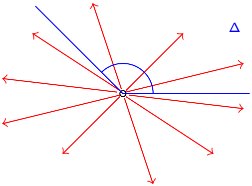
Precise statement of isomonodromy

Note that as $U \in \mathfrak{h}^{reg}$ varies, the Stokes rays $\mathbb{R}_{>0} \cdot U(\alpha)$ may cross.

For any convex sector $\Delta \subset \mathbb{C}^*$, the clockwise product over rays

$$\mathbb{S}(\Delta) = \overset{\curvearrowright}{\prod_{\ell \in \Delta}} \mathbb{S}(\ell) \in G,$$

should be constant, providing no Stokes ray crosses $\partial\Delta$.



This is the same form as the wall-crossing formula in DT theory.

Reconstruction of V from Stokes data

Suppose given the Stokes factors $\mathbb{S}(\ell)$ as a function of $U \in \mathfrak{h}^{\mathrm{reg}}$. To reconstruct V we first try to construct the half-plane solutions $Y_r(t)$.

Riemann-Hilbert problem

For each non-Stokes ray $r \subset \mathbb{C}^*$ find $Y_r \colon \mathbb{H}_r \to G$ such that

$$Y_r(t) \cdot \exp(U/t) \to 1 \text{ as } t \to 0 \text{ in } \mathbb{H}_r,$$

$$|t|^{-k} < \|Y_r(t)\| < |t|^k \text{ as } t \to \infty \text{ in } \mathbb{H}_r,$$

and if $\Delta \subset \mathbb{C}^*$ is a convex sector with $\partial\Delta = \{r_+\} \cup \{r_-\}$ then

$$Y_{r+}(t) = Y_{r_-}(t) \cdot \mathbb{S}(\Delta) \text{ for } t \in \mathbb{H}_{r_+} \cap \mathbb{H}_{r_-}.$$

Note that even in this finite-dimensional setting, existence and uniqueness of solutions is subtle (cf. Hilbert's 21st problem).

2. Donaldson-Thomas invariants and wall-crossing.

Recap on stability space

Let $\mathcal{D}$ be a Δ-category and assume that $\Gamma := K_0(\mathcal{D}) \cong \mathbb{Z}^{\oplus n}$.

The space of stability conditions $\mathrm{Stab}(\mathcal{D})$ parameterises pairs

(I) a group homomorphism $Z \colon \Gamma \to \mathbb{C}$,

(II) an $\mathbb{R}$-graded full subcategory $\mathcal{P} = \cup_{\phi \in \mathbb{R}} \mathcal{P}(\phi) \subset \mathcal{D}$,

satisfying some axioms.

The map Z is called the central charge, and the objects of $\mathcal{P}(\phi)$ are said to be semistable of phase ϕ.

Theorem

The forgetful map $\sigma = (Z, \mathcal{P}) \mapsto Z$ defines a local isomorphism

$$\pi \colon \mathrm{Stab}(\mathcal{D}) \to \mathrm{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{C}) \cong \mathbb{C}^n.$$

Varying the stability condition σ is locally the same as varying Z (but the map π is not usually a regular covering of its image).

DT invariants

Given further assumptions on $\mathcal{D}$ one can define DT invariants

$$\mathrm{DT}_\sigma(\gamma) = \text{“}\, e(\mathcal{M}^{\sigma-ss}(\gamma))\text{''} \in \mathbb{Q}, \qquad \gamma \in \Gamma,$$

depending on a choice of stability condition $\sigma \in \mathrm{Stab}(\mathcal{D})$.

For a fixed class $\gamma \in \Gamma$ there is a wall-and-chamber decomposition of the space $\mathrm{Stab}(\mathcal{D})$, such that $\mathrm{DT}_\sigma(\gamma)$ is constant in each chamber.

The variation of $\mathrm{DT}_\sigma(\gamma)$ with $\sigma \in \mathrm{Stab}(\mathcal{D})$ is controlled by the wall-crossing formula: knowledge of all the invariants $\Omega_\sigma(\gamma)$ at one stability condition determines them at all nearby stability conditions.

There are equivalent invariants $\Omega_\sigma(\gamma)$ defined by

$$\mathrm{DT}_\sigma(\alpha) = \sum_{\alpha = k\cdot\beta} \frac{1}{k^2} \cdot \Omega_\sigma(\beta).$$

Example: quadratic differentials

Theorem (Ivan Smith, tb)

Fix $g \geq 0$, and $m = (m_1, \cdots, m_d)$, with $d \geq 1$ and $m_i \geq 3$. Then there is a CY_3 triangulated category $\mathcal{D} = \mathcal{D}(g, m)$ such that

$$\operatorname{Stab} \mathcal{D}(g, m) / \operatorname{Aut} \mathcal{D}(g, m) \cong \operatorname{Quad}(g, m),$$

where $\operatorname{Quad}(g, m)$ parameterizes pairs (S, q), with S a compact Riemann surface of genus g, and $q = q(x)dx^{\otimes 2}$ a quadratic differential on S having d poles of order m_i, and simple zeroes.

The DT invariants $\Omega_\sigma(\gamma)$ for a given stability condition σ count finite-length horizontal trajectories of the corresponding pair (S, q).

Charge torus and Poisson structure

Introduce the algebraic torus

$$\mathbb{T} = \mathrm{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{C}^*) \cong (\mathbb{C}^*)^n, \qquad \mathbb{C}[\mathbb{T}] = \bigoplus_{\gamma \in \Gamma} \mathbb{C} \cdot x_\gamma.$$

Assume now that $\mathcal{D}$ has the CY_3 property. Then the Euler form

$$\langle [E], [F] \rangle = \sum_{i \in \mathbb{Z}} (-1)^i \dim_{\mathbb{C}} \mathrm{Hom}^i_{\mathcal{D}}(E, F[i]) \colon \Gamma \times \Gamma \to \mathbb{Z}$$

is skew-symmetric. This gives an invariant Poisson structure on $\mathbb{T}$

$$\{x_\alpha, x_\beta\} = \langle \alpha, \beta \rangle \cdot x_{\alpha+\beta}.$$

We sometimes assume that the form $\langle -, - \rangle$ is non-degenerate. We also ignore quadratic refinements, twisted torus etc.

Poisson vector fields on $\mathbb{T}$

Consider the Lie algebra of algebraic vector fields on $\mathbb{T}$ whose flows preserve the Poisson bracket $\{-,-\}$. There is a decomposition

$$\mathfrak{g} = \mathsf{vect}_{\{-,-\}}(\mathbb{T}) = \mathfrak{h} \oplus \mathfrak{g}^{\mathrm{od}}.$$

- the Cartan subalgebra $\mathfrak{h}$ consists of translation-invariant vector fields on $\mathbb{T}$, and can be identified with $\mathrm{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{C})$.
- the subalgebra $\mathfrak{g}^{\mathrm{od}}$ consists of Hamiltonian vector fields, and is the Poisson algebra of non-constant algebraic functions on $\mathbb{T}$

$$\mathfrak{g}^{\mathrm{od}} = \bigoplus_{\gamma \in \Gamma^\times} \mathfrak{g}_\alpha = \bigoplus_{\gamma \in \Gamma^\times} \mathbb{C} \cdot x_\gamma.$$

Note the lack of a well-defined exponential map (time 1 flow)

$$\exp\colon \mathfrak{g} = \mathsf{vect}_{\{-,-\}}(\mathbb{T}) \dashrightarrow G = \mathrm{Aut}_{\{-,-\}}(\mathbb{T})$$

DT automorphisms

Let $\mathcal{D}$ be a CY_3 Δ-category, with a stability condition $\sigma = (Z, \mathcal{P})$.

Try to define, for each ray $\ell \subset \mathbb{C}^*$, an automorphism $\mathbb{S}(\ell) \in G$ by

$$\mathbb{S}(\ell)^*(x_\beta) = \exp\Big\{ \sum_{Z(\gamma)\in\ell} \mathrm{DT}(\gamma)\cdot x_\gamma, - \Big\}(x_\beta) = x_\beta \prod_{Z(\gamma)\in\ell} (1 - x_\gamma)^{\Omega(\gamma)\cdot\langle\beta,\gamma\rangle}$$

Work required to make rigorous sense of this. Possible approaches are

- formal approach: replace $\mathrm{Aut}\,\mathbb{C}[x_i^{\pm 1}]$ with $\mathrm{Aut}\,\mathbb{C}[[x_i]]$,
- work with birational automorphisms of $\mathbb{T}$,
- use automorphisms defined on analytic open subsets of $\mathbb{T}$.

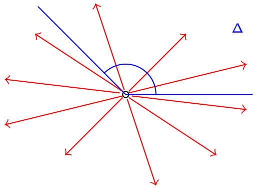
The wall-crossing formula

Controls the variation of the DT invariants as $\sigma \in \mathsf{Stab}(\mathcal{D})$ varies.

For any convex sector $\Delta \subset \mathbb{C}^*$, the clockwise ordered product

$$\mathbb{S}_\sigma(\Delta) = \prod_{\ell \in \Delta}^{\curvearrowright} \mathbb{S}_\sigma(\ell) \in \mathsf{Aut}(\mathbb{T})$$

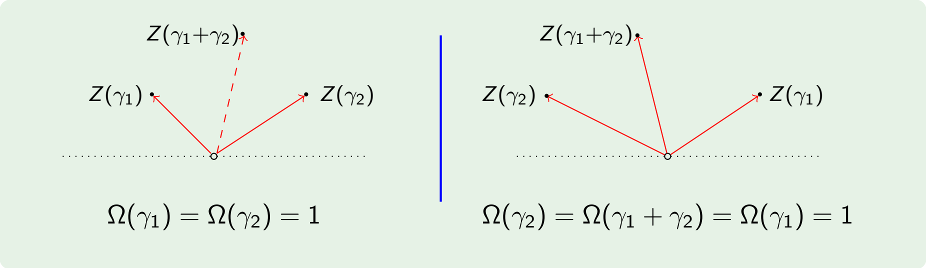
is constant, providing no Stokes ray crosses $\partial\Delta$.



The Stokes rays are spanned by the points $Z(\gamma)$ with $\mathrm{DT}(\gamma) \neq 0$; the formula makes sense in the completed torus algebra.

Example: A_2 case

Here $\Gamma = \mathbb{Z}\gamma_1 \oplus \mathbb{Z}\gamma_2$ with $\langle \gamma_1, \gamma_2 \rangle = 1$, and $\mathbb{C}[\mathbb{T}] = \mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}]$.



The wall-crossing formula is the cluster pentagon identity

$$C_{\gamma_1} \circ C_{\gamma_2} = C_{\gamma_2} \circ C_{\gamma_1+\gamma_2} \circ C_{\gamma_1},$$

$$C_\alpha \colon x_\beta \mapsto x_\beta \cdot (1 + x_\alpha)^{\langle \alpha, \beta \rangle}.$$

3. Geometric structures on stability space.

See: “Complex hyperkähler structures defined by Donaldson-Thomas invariants”, arxiv.

DT invariants as Stokes data

Let $\mathcal{D}$ be a CY_3 Δ-category, and take as before

$$\mathfrak{g} = \mathrm{Vect}_{\{-,-\}}(\mathbb{T}), \qquad G = \mathrm{Aut}_{\{-,-\}}(\mathbb{T}).$$

The wall-crossing formula is the isomonodromy condition for a family of meromorphic connections on the trivial G-bundle over $\mathbb{P}^1$:

$$\nabla_\sigma = d - \left(\frac{Z}{t^2} + \frac{\mathrm{Ham}_F}{t} \right) dt,$$

parameterised by the points of $\mathrm{Stab}(\mathcal{D})$, where

(I) $Z \in \mathfrak{h}$ is the central charge $Z \colon \Gamma \to \mathbb{C}$,

(II) $F = \sum_{\gamma \in \Gamma^\times} F_\gamma \cdot x_\gamma \in \mathfrak{g}^{\mathrm{od}}$ is a function on $\mathbb{T}$.

The Fourier coefficients $F_\gamma = F_\gamma(Z)$ are holomorphic functions on $\mathrm{Stab}(\mathcal{D})$ defined implicitly by the DT invariants.

Isomonodromy equation

The isomonodromy p.d.e. is

$$dF_\gamma(Z) = \sum_{\alpha+\beta=\gamma} \langle \alpha, \beta \rangle \cdot F_\alpha F_\beta \cdot d \log Z(\beta).$$

Define the Joyce function

$$J(Z,\theta) = \sum_{\gamma \in \Gamma^\times} F_\gamma(Z) \cdot \frac{e^{\theta(\gamma)}}{Z(\gamma)} : \mathcal{T}\,\mathrm{Stab}(\mathcal{D}) \to \mathbb{C}.$$

Note that tangent vectors to $\mathrm{Stab}(\mathcal{D})$ are deformations of the central charge, i.e. linear maps $\theta \colon \Gamma \to \mathbb{C}$.

The isomonodromy p.d.e. becomes

$$\frac{\partial^2 J}{\partial \theta_i \partial z_j} - \frac{\partial^2 J}{\partial \theta_j \partial z_i} = \sum_{p,q} \eta^{pq} \cdot \frac{\partial^2 J}{\partial \theta_i \partial \theta_p} \cdot \frac{\partial^2 J}{\partial \theta_j \partial \theta_q}.$$

We took a basis $(\gamma_1, \cdots, \gamma_n) \subset \Gamma$ and set $z_i = Z(\gamma_i)$, $\theta_i = \theta(\gamma_i)$ and $\eta^{ij} = \langle \gamma_i, \gamma_j \rangle$.

Complex hyperkähler structure

Ian Strachan pointed out that when $n = 2$ the previous equation is Plebanski's second heavenly equation.

The isomonodromy p.d.e. is equivalent to the statement that

$$g = \sum_{i,j} \eta_{ij} \cdot (dz_i \otimes d\theta_j + d\theta_j \otimes dz_i) + \sum_{i,j} \frac{\partial^2 J}{\partial\theta_i \partial\theta_j} \cdot dz_i \otimes dz_j$$

is a complex hyperkähler metric on $\mathcal{T}\,\mathrm{Stab}(\mathcal{D})$.

The relation between hyperkähler geometry and isomondromy goes back to Mason and Newman in the late 1980s.

The operators I, J, K are constant block matrices in the basis

$$h_i = \frac{\partial}{\partial z_i} + \sum_{p,q} \eta^{pq} \cdot \frac{\partial^2 J}{\partial\theta_i \partial\theta_p} \cdot \frac{\partial}{\partial\theta_q}, \qquad v_i = \frac{\partial}{\partial\theta_i}.$$

The solutions to the RH problem are annhilated by the vector fields $\frac{v_i}{t} + h_i$.

Joyce structures

Define a Joyce structure on a complex manifold M to be the analogue of a Frobenius structure when $\mathfrak{gl}_n(\mathbb{C})$ is replaced by $\mathrm{vect}_{\{-,-\}}(\mathbb{C}^*)^n$.

The main ingredient is a complex hyperkähler structure on the total space of the tangent bundle $\pi\colon \mathcal{T}(M) \to M$ satisfying

$$\Omega_- = \pi^*(\omega) \quad \text{where} \quad \Omega_-(v,w) = g(v,(J-iK)(w)).$$

Also require the existence of an Euler vector field, periodicity with respect to a lattice, and invariance under -1 acting in the fibres of π.

Can we rigorously define the Stokes data of a Joyce structure, as we can define the Stokes matrices of a semi-simple Frobenius manifold?

The linear Joyce connection

Let M be a complex manifold with a Joyce structure.

The Levi-Civita connection on the tangent bundle of $\mathcal{T}(M)$ preserves tangents to the zero-section $M \subset \mathcal{T}(M)$, and induces a torsion-free, flat connection on the tangent bundle of M.

$$\nabla^J_{\frac{\partial}{\partial z_i}}\Big(\frac{\partial}{\partial z_j}\Big) = -\sum_{p,q} \eta^{pq} \cdot \frac{\partial^3 J}{\partial\theta_i\,\partial\theta_j\,\partial\theta_p}\Big|_{\theta=0} \cdot \frac{\partial}{\partial z_q}.$$

Work in progress: flat submanifolds for this connection give a way to define "physicist's slices": half-dimensional Lagrangian submanifolds of Stab($\mathcal{D}$) equipped with special Kähler structures.

4. The Riemann-Hilbert problem: the A_1 case.

See: "Riemann-Hilbert problems from Donaldson-Thomas theory", arxiv.

The DT Riemann-Hilbert problem

Fix BPS data $(\Gamma, \langle -, - \rangle, Z, \Omega)$ and a point $\xi = \exp(\theta) \in \mathbb{T}$.

For each non-Stokes ray $r \subset \mathbb{C}^*$ find $X_r \colon \mathbb{H}_r \to \mathbb{T}$ such that

$$X_r(t) \cdot \exp(Z/t) \to \xi \in \mathbb{T} \text{ as } t \to 0 \text{ in } \mathbb{H}_r,$$

$$|t|^{-k} < \|X_r(t)\| < |t|^k \text{ as } t \to \infty \text{ in } \mathbb{H}_r,$$

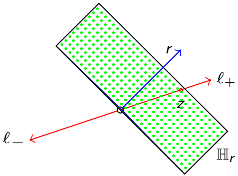
and if $\Delta \subset \mathbb{C}^*$ is a convex sector with $\partial\Delta = \{r_+\} \cup \{r_-\}$ then

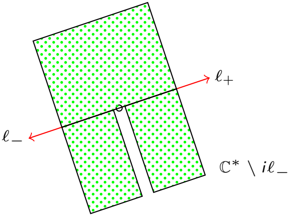
$$X_{r+}(t) = X_{r_-}(t) \cdot \mathbb{S}(\Delta) \text{ for } t \in \mathbb{H}_{r_+} \cap \mathbb{H}_{r_-}.$$

We are taking $G = \mathrm{Aut}_{\{-,-\}}(\mathbb{T})$, and composing $Y \colon \mathbb{C}^* \to G$ with the evaluation map $\mathrm{ev}_\xi \colon G \to \mathbb{T}$.

EXAMPLE: DOUBLED A_1 (WITH $\xi = 1$)

Fix $z \in \mathbb{C}^*$ and $\hat{z} \in \mathbb{C}$, and define the rays $\ell_\pm = \pm\mathbb{R}_{>0} \cdot z$.





RH problem: Find functions $X_\pm, \hat{X}_\pm \colon \mathbb{C}^* \setminus i\ell_\pm \to \mathbb{C}^*$ such that:

$$X_+(t) = X_-(t) \qquad \hat{X}_+(t) = \begin{cases} \hat{X}_-(t) \cdot (1 - X(t)^{-1}) & \mathrm{Re}(t/z) > 0 \\ \hat{X}_-(t) \cdot (1 - X(t)^{+1}) & \mathrm{Re}(t/z) < 0 \end{cases}$$

As $t \to 0$:

$$X_\pm(t) \cdot e^{z/t} \to 1 \quad \text{and} \quad \hat{X}_\pm(t) \cdot e^{\hat{z}/t} \to 1,$$

As $t \to \infty$:

$$|t|^{-k} < |X_\pm(t), \hat{X}_\pm(t)| < |t|^k.$$

The unique solution to the problem is

$$X_\pm(t) = e^{-z/t}, \qquad \hat{X}_\pm(t) = e^{-\hat{z}/t} \cdot \Lambda\Big(\frac{\pm z}{2\pi i t}\Big)^{\pm 1},$$

where $\Lambda(s)$ is the modified gamma function

$$\Lambda(s) = \frac{e^s \cdot \Gamma(s)}{\sqrt{2\pi} \cdot s^{s-\frac{1}{2}}} \sim \exp\Big(\sum_{g=1}^{\infty} \frac{B_{2g} \cdot s^{1-2g}}{2g(2g-1)}\Big).$$

A_1 continued: prepotential

The Joyce function is

$$J(z,\theta) = \frac{1}{2\pi i} \cdot \frac{\theta^3}{6z}.$$

The linear Joyce connection has flat co-ordinates

$$z, \qquad \hat{z} - \frac{1}{2\pi i}\left(z\log(z) + \frac{1}{2}z\right) = \frac{\partial \mathcal{F}}{\partial z},$$

where the prepotential is

$$\mathcal{F}(z) = z\hat{z} - \frac{1}{4\pi i} z^2 \log(z).$$

A_1 CONTINUED: DEFORMED PREPOTENTIAL

Writing $X = \exp(2\pi i x)$ and $\hat{X} = \exp(2\pi i \hat{x})$ we have

$$t \cdot \frac{\partial \hat{x}(z,t)}{\partial t} = \frac{\partial \hat{\mathcal{F}}(x)}{\partial x}$$

where the deformed prepotential $\hat{\mathcal{F}}$ is a modified Barnes G-function

$$\hat{\mathcal{F}}(x) = \frac{e^{-\zeta'(-1)}\, e^{\frac{3}{4}x^2}\, G(x+1)}{(2\pi)^{\frac{x}{2}}\, x^{\frac{x^2}{2}}} \sim \sum_{g\geq 1} \frac{B_{2g}\cdot x^{2-2g}}{2g(2g-2)}.$$

5. Geometric case: coherent sheaves on a CY_3.

See: "Riemann-Hilbert problems and the resolved conifold", arxiv.

Small Calabi-Yau threefolds

We can solve the RH problems for the DT theory associated to coherent sheaves on a non-compact CY_3 without compact divisors.

For simplicity we just consider the resolved conifold. Then $\Gamma = \mathbb{Z}^{\oplus 2}$ and $\langle -, - \rangle = 0$. The space of stability conditions is a cover of

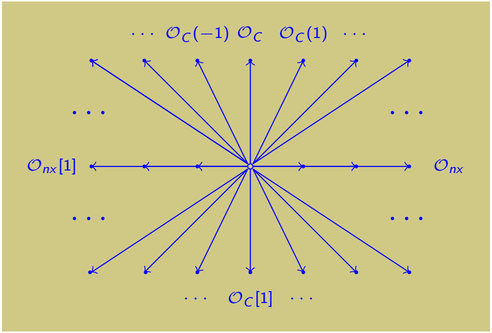
$$M = \left\{ (v, w) \in \mathbb{C}^2 : w \neq 0 \text{ and } v + dw \neq 0 \text{ for all } d \in \mathbb{Z} \right\} \subset \mathbb{C}^2$$

with central charge $Z(r, d) = rv + dw$.

The DT invariants are constant

$$\Omega(\gamma) = \begin{cases} 1 & \text{if } \gamma = \pm(1, d) \text{ for some } d \in \mathbb{Z}, \\ -2 & \text{if } \gamma = (0, d) \text{ for some } 0 \neq d \in \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases}$$

There is a unique solution to the RH problem which can be written explicitly using Barnes double and triple gamma functions.



The linear Joyce connection is described by the prepotential

$$\mathcal{F}(v, w) = \frac{w^2}{(2\pi i)^3} \cdot \mathrm{Li}_3(e^{2\pi i v/w}).$$

Deformed prepotential

$$H(v,w,t) = \int_{\mathbb{R}+i\epsilon} \frac{e^{vs}-1}{e^{ws}-1} \cdot \frac{-e^{ts}}{(e^{ts}-1)^2} \cdot \frac{ds}{s}$$

$$R(v,w,t) = \left(\frac{w}{2\pi i t}\right)^2 \left(\mathrm{Li}_3(e^{2\pi i v/w}) - \zeta(3)\right) + \frac{\pi i v}{12 w}$$

$$\hat{\mathcal{F}}(v,w,t) = \exp(H+R) \sim -\frac{1}{12}\log\left(\frac{-w}{t}\right) + \frac{\pi i v}{12 w}$$
$$+ \sum_{g\geq 1} \frac{B_{2g}}{2g\,(2g-2)!}\left(\mathrm{Li}_{3-2g}(e^{2\pi i v/w}) + \frac{B_{2g-2}}{2g-2}\right)\left(\frac{2\pi i t}{w}\right)^{2g-2}$$

The function H is a non-perturbative topological partition function.

Note: for a CY_3 without compact divisors this may coincide with the Nekrasov limit of the refined partition function?

6. The A_2 case (after Gaiotto-Moore-Neitzke).

See: "On the monodromy of the deformed cubic oscillator", arxiv.

Deformed cubic oscillator

Consider the second order o.d.e. ($\hbar = t$)

$$\hbar^2 \cdot y''(x) = Q(x,\hbar) \cdot y(x),$$

$$Q(x,\hbar) = Q_0(x) + \hbar \cdot Q_1(x) + \hbar^2 \cdot Q_2(x),$$

with potential terms

$$Q_0(x) = x^3 + ax + b, \qquad Q_1(x) = \frac{p}{x-q} + r,$$

$$Q_2(x) = \frac{3}{4(x-q)^2} + \frac{r}{2p(x-q)} + \frac{r^2}{4p^2}.$$

Parameter space

The above equation depends on $\hbar \in \mathbb{C}^*$ and a point in the space

$$M = \left\{ (a, b, q, p, r) \in \mathbb{C}^5 : p^2 = q^3 + aq + b \text{ and } \Delta \neq 0, p \neq 0 \right\},$$

where $\Delta = 4a^3 + 27b^2$. There is an obvious projection to

$$S = \left\{ (a, b) \in \mathbb{C}^2 : 4a^3 + 27b^2 \neq 0 \right\}.$$

Theorem (Sutherland)

Let $\mathcal{D}$ be the CY_3 Δ-category associated to the A_2 quiver. Then

$$S = \operatorname{Stab}(\mathcal{D}) / \operatorname{Sph}(\mathcal{D}).$$

Monodromy data

The Stokes data of the equation at $x = \infty$ defines a point of

$$V = \Big\{(\psi_0, \psi_1, \cdots, \psi_4) \in \mathbb{P}_1 : \psi_i \neq \psi_{i+1}\Big\} \Big/ \mathrm{PGL}_2(\mathbb{C}).$$

The space V has birational Fock-Goncharov co-ordinate systems

$$(x_1, x_2) \colon V \dashrightarrow (\mathbb{C}^*)^2$$

indexed by the triangulations of a pentagon.

The term $Q_2(x)$ in the potential is chosen to ensure that the equation has an apparent singularity at the point $x = q$. Analytically continuing solutions around this point changes their sign.

Asymptotics

As $\hbar \to 0$ the WKB approximation shows that

$$x_i(\hbar) \sim \exp(z_i/\hbar) \cdot \xi_i \cdot (1 + O(\hbar)),$$

where the leading terms are

$$z_i = \int_{\gamma_i} \sqrt{Q_0(x)}\, dx, \qquad \xi_i = \exp\left(\int_{\gamma_i} \frac{Q_1(x)\, dx}{2\sqrt{Q_0(x)}} \right),$$

providing we take the Fock-Goncharov co-ordinates for the triangulation determined by the horizontal trajectories of the quadratic differential

$$\hbar^{-2} \cdot Q_0(x) \cdot dx^{\otimes 2}.$$

Solution: the monodromy map

Each point $s = (a, b) \in S$ determines a stability condition up to autoequivalence, and an associated RH problem. Set

$$M_s = \left\{ (q, p, r) \in \mathbb{C}^3 : p^2 = q^3 + aq + b \right\} = \pi^{-1}(s) \subset M.$$

Theorem

The Riemann-Hilbert problem is solved by the composite map

$$\mathbb{T} \xleftarrow{(\xi_1,\xi_2)} M_s \xrightarrow{\text{Stokes data}} V \xrightarrow{(x_1(\hbar),x_2(\hbar))} \mathbb{T}\,.$$

The co-ordinates (a, b) are flat for the linear Joyce connection.