7 Летняя Школа по Геометрическим Методам Математической Физики

Дискретные Интегрируемые Системы и Уравнения Пенлеве

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References:

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* K. Kajiwara, M. Noumi, Y. Yamada, Geometric aspects of Painlevé equations, J. Phys. A: Math, Theor, **50** (2017) arXiv:1509,08186[nlin.SI]
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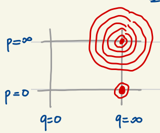
Аффинные группы Вейля и Симметрии

$P_{II}(\alpha) \rightsquigarrow H_{II}(\overset{\alpha+1/2}{\beta}) \rightsquigarrow$ Пространство НУ Окамото $\quad \mathbb{C}\times\mathbb{C} \rightsquigarrow \mathbb{P}^1\times\mathbb{P}^1 \xleftarrow{Bl_{p_1-p_8}} \mathcal{X} \supset \mathcal{Z}_{\mathcal{X}}$ вирт. листы

$\mathbb{P}^2$

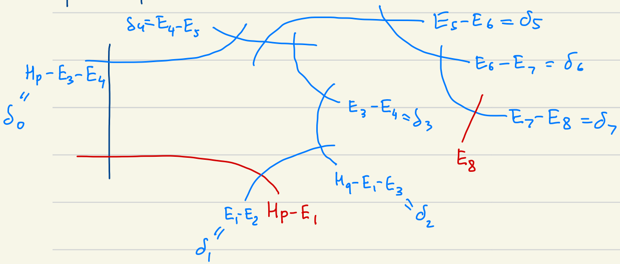
$Pic(\mathcal{X}) = Span_{\mathbb{Z}}\{\mathcal{H}_q, \mathcal{H}_p, \mathcal{E}_1, \dots, \mathcal{E}_8\} \qquad \mathcal{H}_p\cdot\mathcal{H}_q = 1 = -\mathcal{E}_i^2$

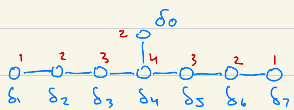


$p_3(\infty,\infty) \leftarrow p_4(u_3=0, v_3=0) \leftarrow p_5(u_4=0, v_4=\frac{1}{2}) \leftarrow p_6(u_5=0, v_5=0)$

$\leftarrow p_7(u_6=0, v_6=-\frac{t}{4}) \leftarrow p_8(u_7=0, v_7=\frac{1-\beta}{2})$

$p_1(\infty,0) \leftarrow p_2(u_1=0, v_1=-\beta)$





$\delta = -K_{\mathcal{X}} = 2\mathcal{H}_q + 2\mathcal{H}_p - \mathcal{E}_1 - \dots - \mathcal{E}_8$

$= 2\delta_0 + \delta_1 + 2\delta_2 + 3\delta_3 + 4\delta_4 + 3\delta_5 + 2\delta_6 + \delta_7$

$R = \{\delta_0, \dots, \delta_7\} \subset Pic(\mathcal{X})$

$\Pi(R) = Span_{\mathbb{Z}}\{\delta_i\} \lhd Pic(\mathcal{X}) \quad \leftarrow$ геометрия $\qquad \mathcal{E}_i^2 = -1$

$\alpha: \quad \alpha_i^2 = -2 \qquad \boxed{\alpha_i \cdot \delta_j = 0} \qquad \alpha = c_q\mathcal{H}_q + c_p\mathcal{H}_p - c_1\mathcal{E}_1 - \dots - c_8\mathcal{E}_8$

$\alpha\cdot\delta_0 = c_q - c_3 - c_4 = 0 \qquad \alpha\cdot\delta_1 = c_1 - c_2 = 0 \qquad c_1 = c_2$

$\alpha\cdot\delta_2 = c_p - c_1 - c_3 = 0 \qquad c_3 = c_4 = \dots = c_8 \qquad \delta_2 = \mathcal{H}_q - \mathcal{E}_1 - \mathcal{E}_3 \qquad \delta_1 = \mathcal{E}_1 - \mathcal{E}_2 \qquad \delta_3 = \mathcal{E}_3 - \mathcal{E}_4, \ \delta_4 = \mathcal{E}_4 - \mathcal{E}_5 \ \dots$

$\alpha = (2c_3)\mathcal{H}_q + (c_1 + c_3)\mathcal{H}_p - c_1(\mathcal{E}_1 + \mathcal{E}_2) - c_3(\mathcal{E}_3 + \dots + \mathcal{E}_8)$

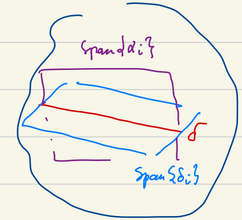
$\alpha_0 = 2\mathcal{H}_q + \mathcal{H}_p - \mathcal{E}_3 - \dots - \mathcal{E}_8 \qquad R^\perp = \{\alpha_0, \alpha_1\} \qquad \underset{\alpha_0}{O} = \underset{\alpha_1}{O} \qquad A_1^{(1)}$

$\alpha_1 = \mathcal{H}_p - \mathcal{E}_1 - \mathcal{E}_2 \qquad Q = Span_{\mathbb{Z}}\{\alpha_0, \alpha_1\} \lhd Pic(\mathcal{X})$

$\alpha_0 + \alpha_1 = 2\mathcal{H}_q + 2\mathcal{H}_p - \mathcal{E}_1 - \dots - \mathcal{E}_8 = \delta$

$\underset{rk=10}{Pic(\mathcal{X})} \rhd \underset{rk=9}{(-K_X)^\perp} \rhd Span\{\alpha_i\}$

$\qquad\qquad \lhd Span\{\delta_j\} \quad \cap = Span_{\mathbb{Z}}\{\delta\}$

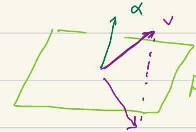


$\delta^\perp \qquad \delta^2 = 0$

$\underset{\alpha_0}{O} = \underset{\alpha_1}{O} \rightsquigarrow W(A_1^{(1)}) = \left\langle s_0, s_1 \;\middle|\; \begin{array}{l} s_i^2 = id \\ (s_i s_j)^\infty = id \end{array} \right\rangle$

$A_1^{(1)}$ аффинная группа Вейля

Есть представление W отражениями в $Pic(\mathcal{X})$ $\quad (\alpha_i^2 = -2)$



$H_\alpha = (\alpha)^\perp$

$r_{\alpha_i}: Pic(\mathcal{X}) \to Pic(\mathcal{X})$

$C \longmapsto C + (\alpha_i \cdot C)\alpha_i$

$r_\alpha(v) = v - 2\, proj_\alpha(v) = v - 2\frac{v\cdot\alpha}{\alpha\cdot\alpha}\alpha \qquad \underline{r_\alpha^2 = id}$

Утв. • r_{α_i} сохраняет δ_j $\quad$ ($\therefore$ симметрия семейства $\mathcal{X}_\beta$) (в $Pic(\mathcal{X})$)

• Можно реализовать $r_{\alpha_i}: Pic(\mathcal{X}) \to Pic(\mathcal{X})$ с помощью отображения $w_i: \mathcal{X}_\beta \xrightarrow{\simeq} \mathcal{X}_{\bar\beta}$

$\underset{r_{\alpha_i}}{\overset{\shortparallel}{(w_i)_*}}: \underset{Pic(\mathcal{X})}{\overset{\shortparallel}{Pic(\mathcal{X}_\beta)}} \longrightarrow \overset{\shortparallel}{Pic(\mathcal{X}_{\bar\beta})}$

$\Bigg\}$ бирациональное представление W

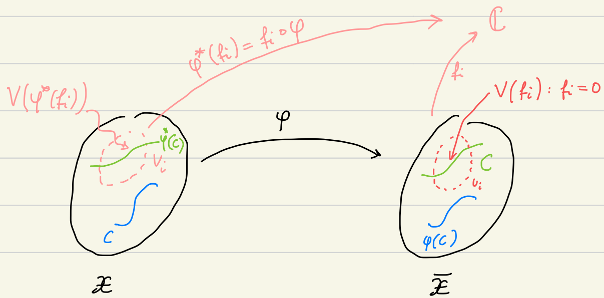
$$\begin{tikzcd} \mathcal{X}_\beta \arrow[r] \arrow[d] & \mathcal{X}_{\bar\beta} \arrow[d] \\ \mathbb{P}^1\times\mathbb{P}^1 \arrow[r, dashed] & \mathbb{P}^1\times\mathbb{P}^1 \end{tikzcd}$$

$\alpha_1 = \mathcal{H}_p - \mathcal{E}_1 - \mathcal{E}_2 \qquad w_1 = r_{\alpha_1} : C \longmapsto C + (C\cdot\alpha_1)\,\alpha_1 \qquad w_1^2 = id$

$\mathcal{H}_q \longleftrightarrow \mathcal{H}_q + \mathcal{H}_p - \mathcal{E}_1 - \mathcal{E}_2 \qquad \mathcal{H}_p \longleftrightarrow \mathcal{H}_p$

$\mathcal{E}_1 \longleftrightarrow \mathcal{H}_p - \mathcal{E}_2 \qquad \mathcal{E}_2 \longleftrightarrow \mathcal{H}_p - \mathcal{E}_1 \qquad \mathcal{E}_i \leftrightarrow \mathcal{E}_i \quad i > 2$

Алгебро-геометрическое отступление



"Дивизоры Вейля"

$\varphi_* : Pic(\mathcal{X}) \longrightarrow Pic(\bar{\mathcal{X}})$

$[C] \longmapsto [\varphi(C)]$

(push-forward)

"Дивизоры Картье"

$\varphi^* : Pic(\bar{\mathcal{X}}) \longrightarrow Pic(\mathcal{X})$

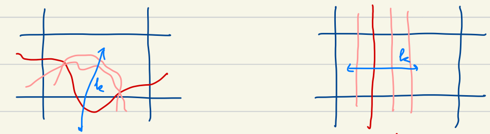
(pull-back)

В хорошем случае : $\varphi^* = \varphi_*^{-1}$

$$\begin{tikzcd} w_1 : & \mathcal{X}_\beta \arrow[r] \arrow[d] & \bar{\mathcal{X}}_{\bar\beta} \arrow[d] \\ w_1 : & \mathbb{P}^1 \times \mathbb{P}^1 \arrow[r, dashed] & \mathbb{P}^1 \times \mathbb{P}^1 \\ & (q,p) & (\bar q, \bar p) \end{tikzcd}$$

$(w_1)_* = r_{\alpha_1}$ в $Pic(\mathcal{X})$

$$\bar q = \frac{f(q,p)}{g(q,p)}$$



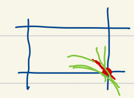
$\bar q = k \sim \bar H_q \sim \bar{\mathcal{H}}_q$

$f(q,p) - k\,g(q,p) = 0$

семейство (пучок) кривых

$\varphi^*(\bar H_{\bar q = k}) \sim \dfrac{f(q,p)}{g(q,p)} = k$

$\mathcal{H}_q \longleftrightarrow \mathcal{H}_q + \mathcal{H}_p - \mathcal{E}_1 - \mathcal{E}_2$



$w_1^*(\bar{\mathcal{H}}_q) = \mathcal{H}_q + \mathcal{H}_p - \mathcal{E}_1 - \mathcal{E}_2 \in Pic(\mathcal{X})$

кривые проходят через p_1 и p_2

$(H_q + H_p - E_1 - E_2) \quad \overset{\psi}{?}$ уравнения степени 1 по q и по p

$Aqp + Bq + Cp + D = 0 \qquad A, \dots, D$ – некоторые коэффициенты

$p_1(\infty, 0) \leftarrow p_2\,(u_1 = 0,\ v_1 = -\beta) \qquad q = \dfrac{1}{Q} \qquad Q = u_1 \quad p = u_1 v_1$

$Q\left(A\frac{1}{Q}p + B\frac{1}{Q} + Cp + D\right) = Ap + B + CpQ + DQ = 0$

$p_1 : (Q = 0, p = 0) \implies \boxed{B = 0} \qquad Au_1v_1 + Cu_1^2v_1 + Du_1 = 0$

$\boxed{D = A\beta}$

$u_1\,(\underbrace{Av_1 + Cu_1v_1 + D}_{H_q + H_p - E_1}) = 0$

$u_1 = 0$

E_1

$-A\beta + D = 0$

$p_2 : u_1 = 0 \quad v_1 = -\beta$

$$w_1^*(\bar H_q) = \{ Aqp + Cp + A\beta = 0 \}$$

$\mathbb{P}^1$ - пучок ← автоморфизмы преобр. Мёбиуса $PGL_2(\mathbb{C})$

$$A(qp+\beta) + Cp = 0 \quad \text{<~ координата ~} \quad \frac{C}{A} \qquad [A:C]$$

$$(qp+\beta) + \frac{C}{A}p = 0 \qquad -k = \frac{qp+\beta}{p} = \bar q$$

$$f(q,p) - kg(q,p) = 0$$

$$\bar q \sim \frac{A\bar q + B}{C\bar q + D}$$

$$\begin{cases} \bar q = \dfrac{qp+\beta}{p} \\ \bar p = p \end{cases} \rightsquigarrow \begin{cases} \bar q = \dfrac{A\frac{qp+\beta}{p} + B}{C\frac{qp+\beta}{p} + D} \\ \bar p = \dfrac{Kp+L}{Mp+N} \end{cases}$$

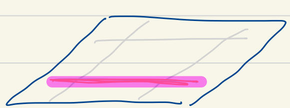
$A, \dots, N$ - коэффициенты, которые надо найти.

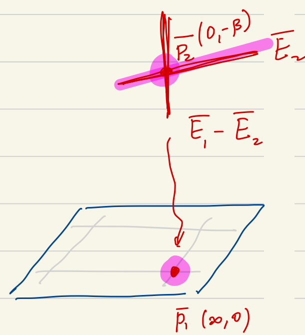
$$(w_1)_*: \quad \mathcal{H}_p - \mathcal{E}_1 \leftrightarrow \mathcal{E}_2 \qquad \mathcal{H}_p - \mathcal{E}_2 \leftrightarrow \mathcal{E}_1 \qquad \mathcal{E}_i \leftrightarrow \mathcal{E}_i \quad i > 2$$

$$\psi: \quad H_p - E_1 \to E_2$$

$$p = 0$$

$$E_1 - E_2, \ E_3 - E_4, \dots \longmapsto$$





$$\bar Q = \bar u_1 \qquad \bar u_1 = \bar Q$$

$$\bar P = \bar u_1 \bar v_1 \qquad \bar v_1 = \bar P / \bar Q = \bar p \bar q$$

Упр. Досчитать по каскаду в P_3

$$M = 0 \quad B = 0, \ A = 1, \ N = 1$$

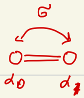
$$\boxed{\begin{aligned} \bar q &= \frac{qp+\beta}{p} = q + \frac{\beta}{p} \\ \bar p &= p \\ \bar \beta &= -\beta \end{aligned}}$$

$$\left.\begin{aligned} \bar q &= \frac{A(qp+\beta) + Bp}{C(qp+\beta) + Dp} \\ \bar p &= \frac{Kp+L}{Mp+N} \end{aligned}\right|_{p=0} \begin{aligned} &= \frac{A}{C} = \infty \quad \boxed{C = 0} \\ &= \frac{L}{N} = 0 \quad \boxed{L = 0} \end{aligned}$$

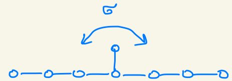
$$\bar q = \frac{A(qp+\beta) + Bp}{p}$$

$$\bar p = \frac{p}{Mp+N}$$

$$\bar v_1 = \bar p \bar q = \left.\frac{A(qp+\beta) + Bp}{Mp+N}\right|_{p=0} = \frac{A\beta}{N} = -\bar\beta$$



$A_1^{(1)}$



$$\tilde{W}(A_1^{(1)}) = \langle w_0, w_1, \sigma \mid \sigma w_0 \sigma = w_1 \rangle$$

$s \leftrightarrow w_1 \qquad r \leftrightarrow \sigma \qquad w_0 = \sigma w_1 \sigma = rsr$

$$\alpha_0 \xmapsto{r_{\alpha_1}} \alpha_0 + 2\alpha_1 = \alpha_1 + \delta \xmapsto{\sigma} \alpha_0 + \delta$$
$$\alpha_1 \longmapsto -\alpha_1 = \alpha_0 - \delta \longmapsto \alpha_1 - \delta$$

$\delta = \alpha_0 + \alpha_1 \qquad \langle \alpha_0, \alpha_1 \rangle \longmapsto \langle \alpha_0, \alpha_1 \rangle + \langle 1, -1 \rangle \delta$

элемент переноса в афф. гр.

"дискретная динамика на $\mathrm{Span}_{\mathbb{Z}}\{\alpha_i\}$"

$\parallel$

дискретное уравнение Пенлеве!

$\sigma \circ w_1 = r \circ s \qquad s: (q, p, t, \beta) \longmapsto \left(q + \frac{\beta}{p}, p, t, -\beta\right)$

$r: (q, p, t, \beta) \longmapsto (-q, -p + 2q^2 + t, t, 1 - \beta)$

$$r \circ s: (q, p, t, \beta) \longmapsto \left(-q + \frac{1-\beta}{2q^2 - p + t}, 2q^2 - p + t, t, \beta - 1\right)$$

$$\begin{cases} \bar{q} = -q + \dfrac{1-\beta}{2q^2 - p + t} \\ \bar{p} = 2q^2 - p + t \end{cases} \qquad \text{alt. } d\text{-}P_{\mathrm{I}}$$

$\longrightarrow P_{\mathrm{I}}$