

# On the relations between principal eigenvalue and torsional rigidity

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- (numerical experiments by Ginevra Biondi, Università di Pisa, Master thesis).

Our goal is to present some relations between two important quantities that arise in the study of elliptic equations. We always consider the Laplace operator  $-\Delta$  with Dirichlet boundary conditions; other elliptic operator can be considered, while considering other boundary conditions (Neumann or Robin) adds to the problem severe extra difficulties, essentially due to the fact that in the Dirichlet case functions in  $H_0^1(\Omega)$  can be easily extended to  $\mathbf{R}^d$  while this is not in general true in the other cases.

To better understand the two quantities we deal with, let us make the following two **measurements**.

- Take in  $\Omega$  an uniform heat source ( $f = 1$ ), fix an initial temperature  $u_0(x)$ , wait a long time, and measure the **average temperature** in  $\Omega$ .
- Consider in  $\Omega$  no heat source ( $f = 0$ ), fix an initial temperature  $u_0(x)$ , and measure the decay rate to zero of the temperature in  $\Omega$ .

The first quantity is usually called **torsional rigidity** and is defined as

$$T(\Omega) = \int_{\Omega} u \, dx$$

where  $u$  is the solution of

$$-\Delta u = 1 \text{ in } \Omega, \quad u \in H_0^1(\Omega).$$

In the thermal diffusion model  $T(\Omega)/|\Omega|$  is the **average temperature** of a conducting medium  $\Omega$  with **uniformly distributed heat sources** ( $f = 1$ ).

The second quantity is the **first eigenvalue** of the Dirichlet Laplacian

$$\lambda(\Omega) = \min \left\{ \frac{\int_{\Omega} |\nabla u|^2 dx}{\int_{\Omega} u^2 dx} : u \in H_0^1(\Omega) \setminus \{0\} \right\}$$

In the thermal diffusion model, by the **Fourier analysis**,

$$u(t, x) = \sum_{k \geq 1} e^{-\lambda_k t} \langle u_0, u_k \rangle u_k(x),$$

so  $\lambda(\Omega)$  represents the **decay rate** in time of the temperature when an initial temperature is given and no heat sources are present.

If we want, under the **measure constraint**  $|\Omega| = m$ , the highest average temperature, or the slowest decay rate, the optimal  $\Omega$  is the same and is the ball of measure  $m$ . Also, **it seems consistent** to expect a slow (resp. fast) heat decay related to a high (resp. low) temperature. We then want to study if

$$\lambda(\Omega) \sim T^{-1}(\Omega),$$

or more generally

$$\lambda(\Omega) \sim T^{-q}(\Omega),$$

where by  $A(\Omega) \sim B(\Omega)$  we mean

$$0 < c_1 \leq A(\Omega)/B(\Omega) \leq c_2 < +\infty \quad \text{for all } \Omega.$$

We further aim to study the so-called **Blaschke-Santaló** diagram for the quantities  $\lambda(\Omega)$  and  $T(\Omega)$ . This consists in identifying the set  $E \subset \mathbf{R}^2$

$$E = \left\{ (x, y) : x = T(\Omega), y = \lambda(\Omega) \right\}$$

where  $\Omega$  runs among the **admissible sets**. In this way, minimizing a quantity like

$$F\left(T(\Omega), \lambda(\Omega)\right)$$

is reduced to the optimization problem in  $\mathbf{R}^2$

$$\min \left\{ F(x, y) : (x, y) \in E \right\}.$$



The **difficulty** consists in the fact that characterizing the set  $E$  is **hard**. Here we only give some **bounds** by studying the inf and sup of

$$\lambda^\alpha(\Omega)T^\beta(\Omega)$$

when  $|\Omega| = m$ .

Since the two quantities scale as:

$$T(t\Omega) = t^{d+2}T(\Omega), \quad \lambda(t\Omega) = t^{-2}\lambda(\Omega)$$

it is **not restrictive** to reduce ourselves to the case  $|\Omega| = 1$ , which **simplifies** a lot the presentation.

For the relations between  $T(\Omega)$  and  $\lambda(\Omega)$ :

- Kohler-Jobin ZAMP 1978
- van den Berg, B., Velichkov in Birkhäuser 2015
- van den Berg, Ferone, Nitsch, Trombetti  
Integral Equations Operator Theory 2016
- Lucardesi, Zucco preprint

The Blaschke-Santaló diagram has been studied for other pairs of quantities:

- for  $\lambda_1(\Omega)$  and  $\lambda_2(\Omega)$  by D. Bucur, G.B., I. Figueiredo (SIAM J. Math. Anal. 1999);
- for  $\lambda_1(\Omega)$  and  $\text{Per}(\Omega)$  by M. Dambrine, I. Ftouhi, A. Henrot, J. Lamboley (on HAL);
- for  $T(\Omega)$  and  $\text{cap}(\Omega)$  by M. van den Berg, G.B. (on arxiv and cvgmt);
- for  $T(\Omega)$  and  $\text{Per}(\Omega)$  by L. Briani, G.B., F. Prinari (on arxiv and cvgmt).

For the inf/sup of

$$\lambda^\alpha(\Omega)T^\beta(\Omega)$$

the case  $\beta = 0$  is well-known and reduces to the **Faber-Krahn** result ( $B$  ball with  $|B| = 1$ )

$$\min \left\{ \lambda(\Omega) : |\Omega| = 1 \right\} = \lambda(B),$$

while

$$\sup \left\{ \lambda(\Omega) : |\Omega| = 1 \right\} = +\infty$$

(take many **small balls** or a **long thin rectangle**).

Similarly, the case  $\alpha = 0$  is also well-known through a **symmetrization argument** (**Saint-Venant** inequality):

$$\max \{T(\Omega) : |\Omega| = 1\} = T(B),$$

while

$$\inf \{T(\Omega) : |\Omega| = 1\} = 0$$

(take many **small balls** or a **long thin rectangle**).

The case when  $\alpha$  and  $\beta$  have a different sign is also easy, since  $T(\Omega)$  is increasing for the set inclusion, while  $\lambda(\Omega)$  is decreasing.

So we can reduce the study to the case

$$\lambda(\Omega)T^q(\Omega)$$

with  $q > 0$ . If we want to remove the constraint  $|\Omega| = 1$  the corresponding **scaling free** shape functional is

$$F_q(\Omega) = \frac{\lambda(\Omega)T^q(\Omega)}{|\Omega|^{(dq+2q-2)/d}}$$

that we consider on various classes of **admissible domains**.

We start by considering the class of **all domains** (with  $|\Omega| = 1$ ). The known cases are:

- $q = 2/(d + 2)$  in which the minimum of  $\lambda(\Omega)T^q(\Omega)$  is reached when  $\Omega$  is a ball (**Kohler-Jobin** ZAMP 1978);
- $q = 1$  in which (**Pólya** inequality)

$$0 < \lambda(\Omega)T(\Omega) < 1.$$

**When  $0 < q \leq 2/(d+2)$ :**

$$\begin{cases} \min \lambda(\Omega)T^q(\Omega) = \lambda(B)T^q(B) \\ \sup \lambda(\Omega)T^q(\Omega) = +\infty. \end{cases}$$

For the minimum

$$\begin{aligned} \lambda(\Omega)T^q(\Omega) &= \lambda(\Omega)T(\Omega)^{2/(d+2)}T(\Omega)^{q-2/(d+2)} \\ &\geq \lambda(B)T(B)^{2/(d+2)}T(B)^{q-2/(d+2)} \\ &= \lambda(B)T^q(B), \end{aligned}$$

by Kohler-Jobin and Saint-Venant inequalities.

For the sup take  $\Omega = N$  disjoint small balls.



**When  $2/(d+2) < q < 1$ :**

$$\begin{cases} \inf \lambda(\Omega) T^q(\Omega) = 0 \\ \sup \lambda(\Omega) T^q(\Omega) = +\infty. \end{cases}$$

For the sup take again  $\Omega = N$  disjoint balls.

For the inf take as  $\Omega$  the union of a fixed ball  $B_R$  and of  $N$  disjoint balls of radius  $\varepsilon$ .

We have

$$\lambda(\Omega) T^q(\Omega) = R^{-2} \lambda(B_1) T^q(B_1) \left( R^{d+2} + N \varepsilon^{d+2} \right)^q$$

and choosing first  $\varepsilon \rightarrow 0$  and then  $R \rightarrow 0$  we have that  $\lambda(\Omega) T^q(\Omega)$  vanishes.

**When  $q = 1$ :**

$$\inf \lambda(\Omega)T(\Omega) = 0, \quad \sup \lambda(\Omega)T(\Omega) = 1.$$

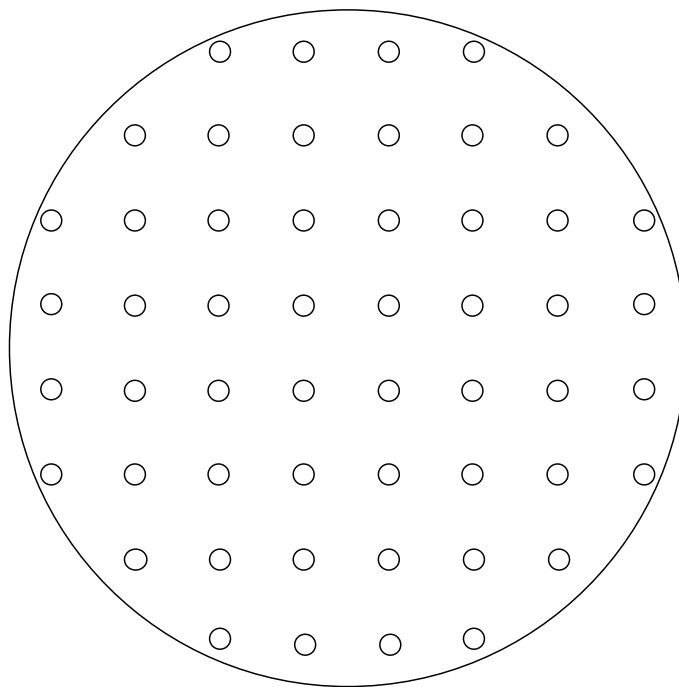
For the inf take as  $\Omega$  the union of a fixed ball  $B_R$  and of  $N$  disjoint balls of radius  $\varepsilon$ , as above.

The sup equality, taking  $\Omega$  a finely perforated domain, was shown by van den Berg, Ferone, Nitsch, Trombetti [Integral Equations Operator Theory 2016]. A shorter proof can be given using the theory of capacity measures.

The finely perforated domains:

$\varepsilon$  = distance between holes     $r_\varepsilon$  = radius of a hole

$r_\varepsilon \sim \varepsilon^{d/(d-2)}$  if  $d > 2$ ,     $r_\varepsilon \sim e^{-1/\varepsilon^2}$  if  $d = 2$ .



**When  $q > 1$ :**

$$\inf \lambda(\Omega) T^q(\Omega) = 0, \quad \sup \lambda(\Omega) T^q(\Omega) < +\infty.$$

For the inf take as  $\Omega$  the union of a fixed ball  $B_R$  and of  $N$  disjoint balls of radius  $\varepsilon$ , as above.

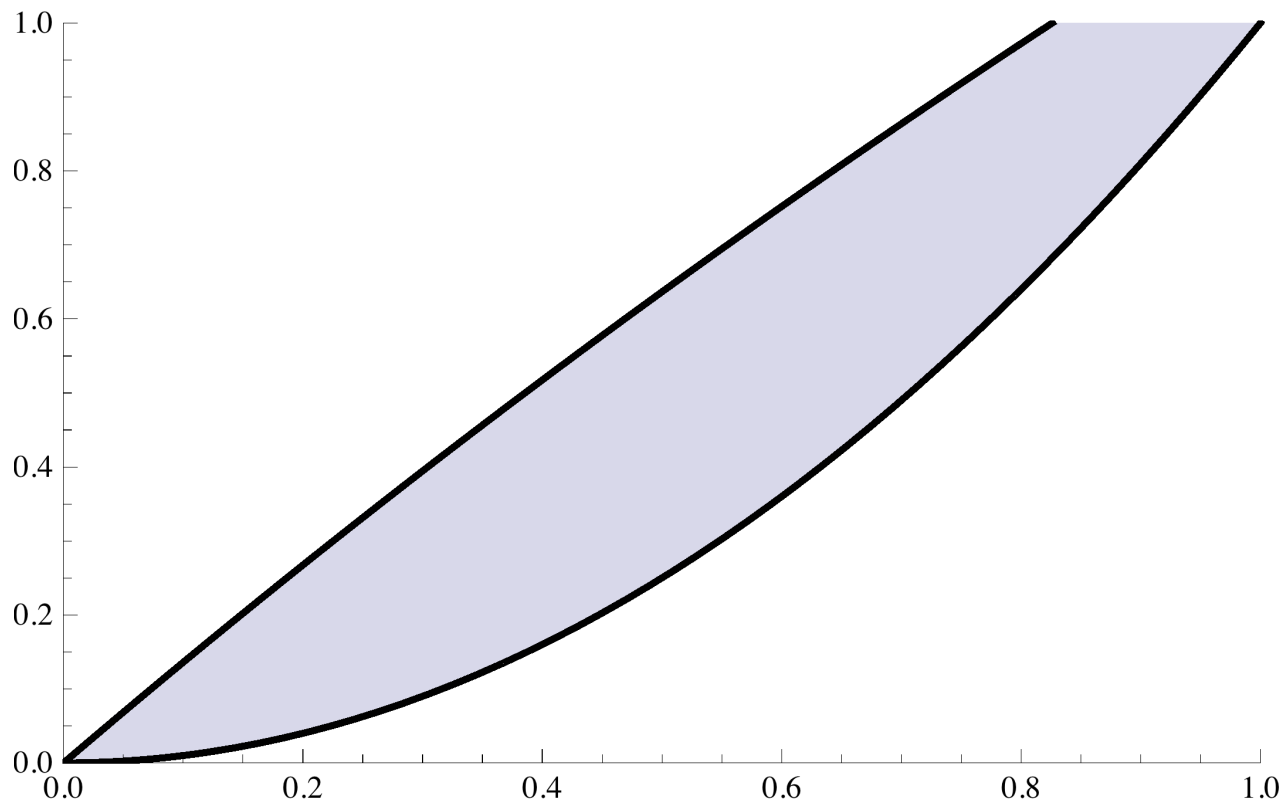
For the sup (using **Pólya** and **Saint-Venant**):

$$\begin{aligned} \lambda(\Omega) T^q(\Omega) &= \lambda(\Omega) T(\Omega) T^{q-1}(\Omega) \\ &\leq T^{q-1}(\Omega) \leq T^{q-1}(B) \end{aligned}$$

It would be interesting to **compute explicitly**  $\sup F_q(\Omega)$  for  $q > 1$  (**is it attained?**).

Summarizing: for **general domains** we have

	General domains $\Omega$	
$0 < q \leq 2/(d+2)$	$\min F_q(\Omega) = F_q(B)$	$\sup F_q(\Omega) = +\infty$
$2/(d+2) < q < 1$	$\inf F_q(\Omega) = 0$	$\sup F_q(\Omega) = +\infty$
$q = 1$	$\inf F_q(\Omega) = 0$	$\sup F_q(\Omega) = 1$
$q > 1$	$\inf F_q(\Omega) = 0$	$\sup F_q(\Omega) < +\infty$



The Blaschke-Santaló diagram with  $d = 2$ , for  $x = \lambda(B)/\lambda(\Omega)$  and  $y = T(\Omega)/T(B)$  is **contained** in the colored region.

If we limit ourselves to consider only domains  $\Omega$  that are union of disjoint disks of radii  $r_k$  we find

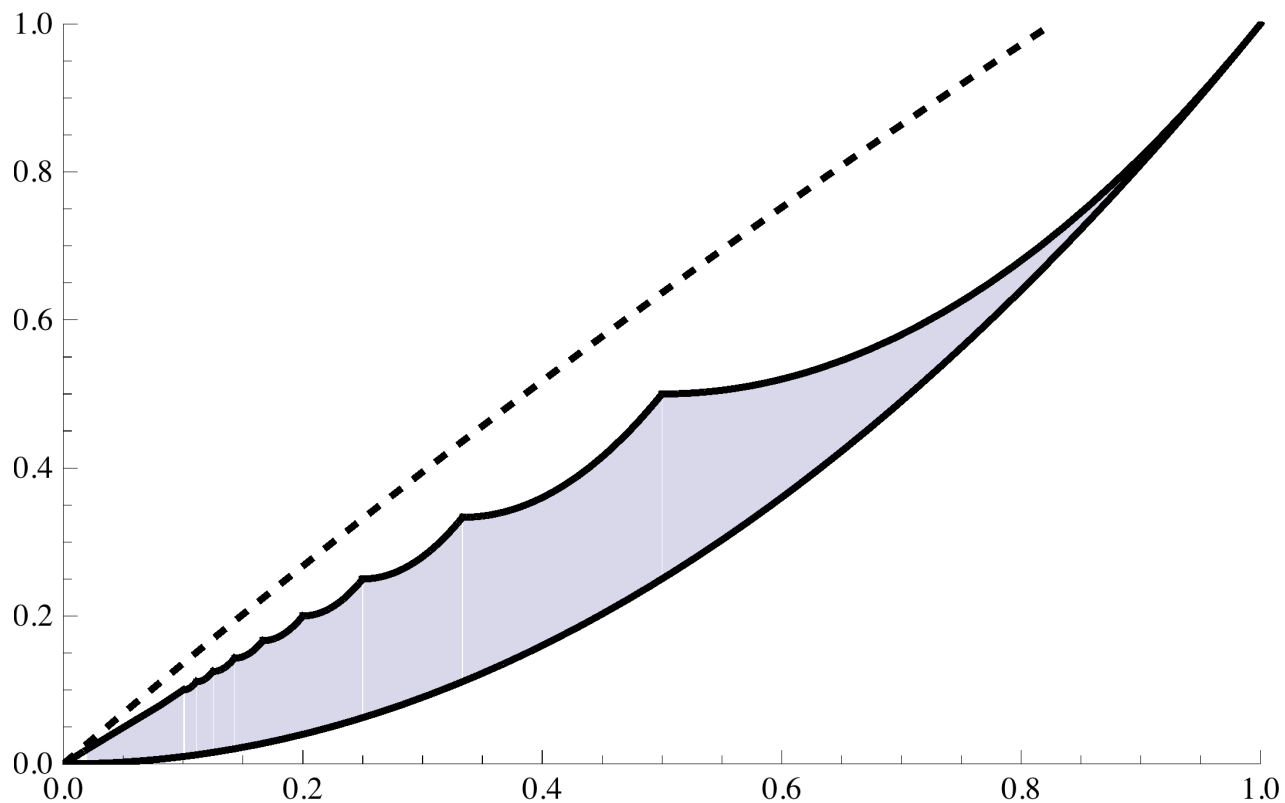
$$x = \frac{\max_k r_k^2}{\sum_k r_k^2}, \quad y = \frac{\sum_k r_k^4}{\left(\sum_k r_k^2\right)^2}.$$

It is not difficult to show that in this case we obtain the region

$$x^2 \leq y \leq x^2[1/x] + \left(1 - x[1/x]\right)^2$$

where  $[s]$  is the **integer part** of  $s$ .

In this way in the **Blaschke-Santaló** diagram we can reach the colored region in the picture below.



In the Blaschke-Santaló diagram with  $d = 2$ , the colored region can be reached by domains  $\Omega$  made by union of disjoint disks.



## The case $d = 1$

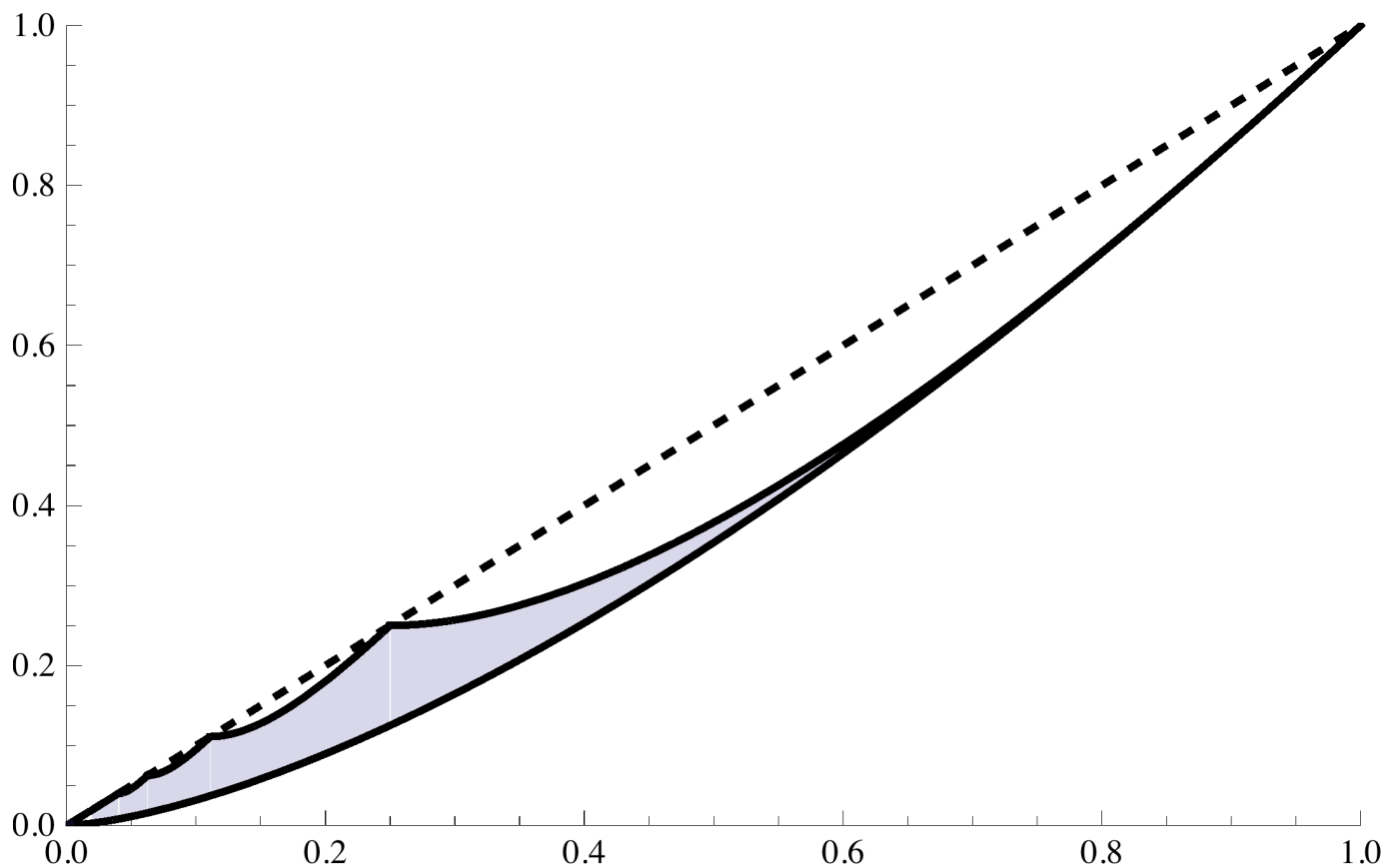
In the one-dimensional case every domain  $\Omega$  is the union of disjoint intervals of half-length  $r_k$ , so that we have

$$x = \frac{\max_k r_k^2}{\left(\sum_k r_k\right)^2}, \quad y = \frac{\sum_k r_k^3}{\left(\sum_k r_k\right)^3}$$

and we deduce that the **full** Blaschke-Santaló set is given by the region

$$x^{3/2} \leq y \leq x^{3/2}[x^{-1/2}] + \left(1 - x^{1/2}[x^{-1/2}]\right)^3$$

where  $[s]$  is the **integer part** of  $s$ .



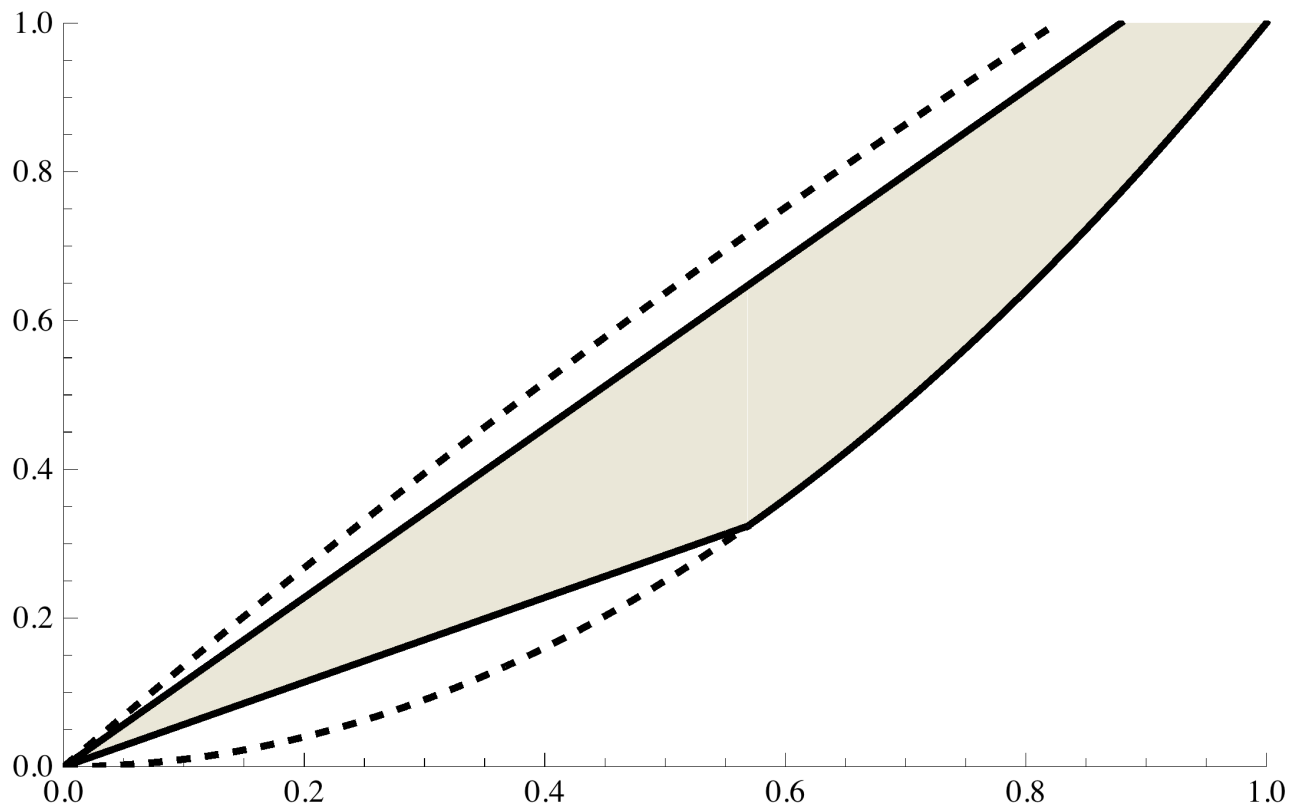
The **full** Blaschke-Santaló diagram in the case  $d = 1$ ,  
 where  $x = \pi^2/\lambda(\Omega)$  and  $y = 12 T(\Omega)$ .

## The case $\Omega$ convex

If we consider only **convex** domains  $\Omega$ , the Blaschke-Santaló diagram is clearly smaller. For the dimension  $d = 2$  the **conjecture** is

$$\frac{\pi^2}{24} \leq \frac{\lambda(\Omega)T(\Omega)}{|\Omega|} \leq \frac{\pi^2}{12} \quad \text{for all } \Omega$$

where the left side corresponds to  $\Omega$  a **thin triangle** and the right side to  $\Omega$  a **thin rectangle**.



If the **Conjecture** for convex domains is true, the Blaschke-Santaló diagram is **contained** in the colored region.

At present the only available inequalities are the ones of [BFNT2016]: for every  $\Omega \subset \mathbb{R}^2$  convex

$$0.2056 \approx \frac{\pi^2}{48} \leq \frac{\lambda(\Omega)T(\Omega)}{|\Omega|} \leq 0.9999$$

instead of the bounds provided by the conjecture, which are

$$\begin{cases} \pi^2/24 \approx 0.4112 & \text{from below} \\ \pi^2/12 \approx 0.8225 & \text{from above.} \end{cases}$$

In dimensions  $d \geq 3$  the **conjecture** is

$$\frac{\pi^2}{2(d+1)(d+2)} \leq \frac{\lambda(\Omega)T(\Omega)}{|\Omega|} \leq \frac{\pi^2}{12}$$

- the right side asymptotically reached by a **thin slab**

$$\Omega_\varepsilon = \left\{ (x', t) : 0 < t < \varepsilon \right\}$$

with  $x' \in A_\varepsilon$ , being  $A_\varepsilon$  a  $d - 1$  dimensional ball of measure  $1/\varepsilon$

- the left side asymptotically reached by a **thin cone** based on  $A_\varepsilon$  above and with height  $d\varepsilon$ .

The **convexity** assumption on the admissible domains provides a strong extra compactness that allows to prove the existence of optimal domains in the cases:

$$\begin{cases} \max \left\{ \lambda(\Omega) T^q(\Omega) : \Omega \text{ convex}, |\Omega| = 1 \right\} & \text{if } q > 1 \\ \min \left\{ \lambda(\Omega) T^q(\Omega) : \Omega \text{ convex}, |\Omega| = 1 \right\} & \text{if } q < 1. \end{cases}$$

This is obtained by showing that maximizing (resp. minimizing) sequences  $\Omega_n$  are not **too thin**, in the sense that

$$\frac{\text{inradius}(\Omega_n)}{\text{diameter}(\Omega_n)} \geq c_{d,q},$$

where  $c_{d,q} > 0$  depends only on  $d$  and  $q$ .

Summarizing: for **convex domains** we have

	Convex domains $\Omega$	
$q < 1$	$\min F_q(\Omega) > 0$	$\sup F_q(\Omega) = +\infty$
$q = 1$	$\inf F_1(\Omega) = C_d^- > 0$	$\sup F_1(\Omega) = C_d^+ < 1$
$q > 1$	$\inf F_q(\Omega) = 0$	$\max F_q(\Omega) < +\infty$



## Thin domains

We say that  $\Omega_\varepsilon \subset \mathbb{R}^2$  is a **thin domain** if

$$\Omega_\varepsilon = \left\{ (s, t) : s \in ]0, 1[, \varepsilon h_-(s) < t < \varepsilon h_+(s) \right\}$$

where  $\varepsilon$  is a small positive parameter and  $h_-, h_+$  are two given (smooth) functions. We denote by  $h(s)$  the **local thickness**

$$h(s) = h_+(s) - h_-(s)$$

and we assume that  $h(s) \geq 0$ .

The following **asymptotics** hold (as  $\varepsilon \rightarrow 0$ ):

$$\lambda(\Omega_\varepsilon) \approx \frac{\varepsilon^{-2}\pi^2}{\|h\|_{L^\infty}^2} \quad [\text{Borisov-Freitas 2010}]$$

$$T(\Omega_\varepsilon) \approx \frac{\varepsilon^3}{12} \int h^3(s) ds$$

$$|\Omega_\varepsilon| \approx \varepsilon \int h(s) ds.$$

Hence, for a thin domain  $\Omega_\varepsilon$  we have

$$\frac{\lambda(\Omega_\varepsilon)T(\Omega_\varepsilon)}{|\Omega_\varepsilon|} \approx \frac{\pi^2}{12} \frac{\int h^3(s) ds}{\|h\|_{L^\infty}^2 \int h ds}.$$

We are able to prove the [conjecture](#) above in the class of thin domains. More precisely:

- for every  $h$  we have

$$\frac{\int h^3(s) ds}{\|h\|_{L^\infty}^2 \int h ds} \leq 1 ;$$

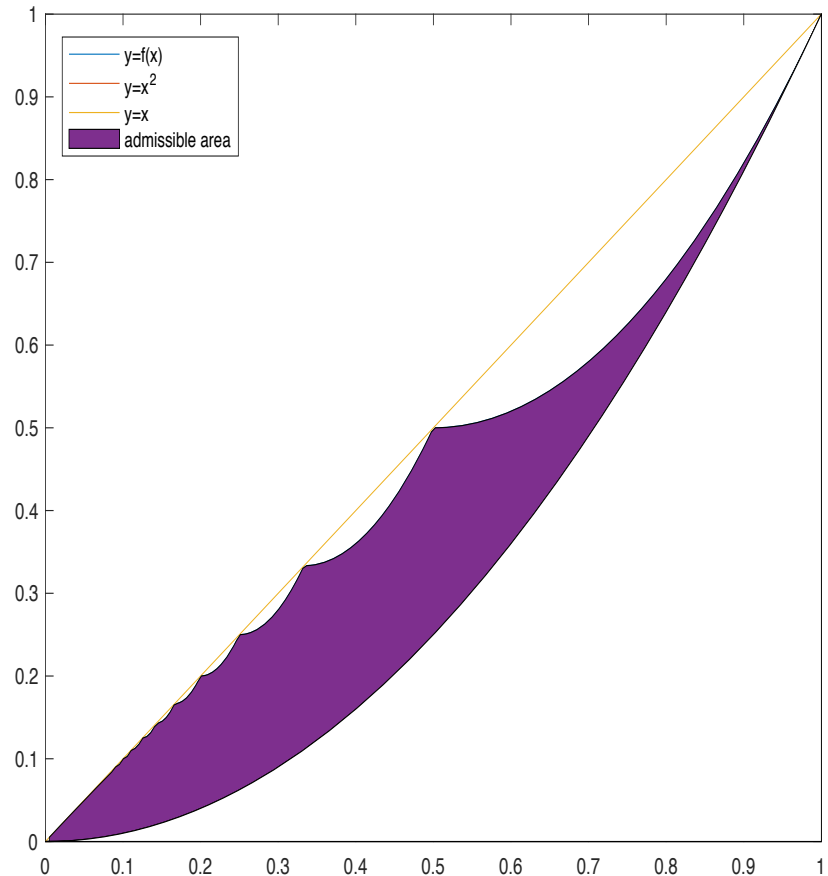
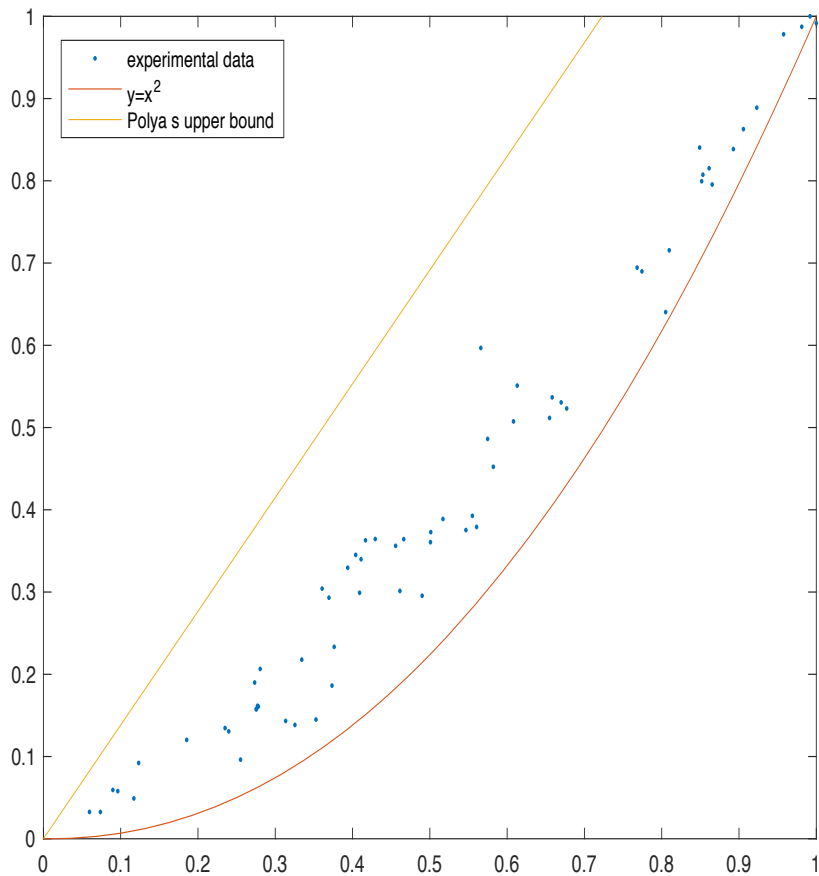
- for every  $h$  **concave** we have

$$\frac{\int h^3(s) ds}{\|h\|_{L^\infty}^2 \int h ds} \geq \frac{1}{2} .$$

Hence

$$\frac{\pi^2}{24} \leq \lim_{\varepsilon \rightarrow 0} \frac{\lambda(\Omega_\varepsilon) T(\Omega_\varepsilon)}{|\Omega_\varepsilon|} \leq \frac{\pi^2}{12}$$

where the right inequality holds for **all** thin domains, while the left inequality holds for **convex** thin domains.



Plot of 100 experimental domains (left), union of disks (right).

## Open questions

- characterize  $\sup_{|\Omega|=1} \lambda(\Omega) T^q(\Omega)$  when  $q > 1$ ;
- prove (or disprove) the conjecture for convex sets;
- simply connected domains or star-shaped domains? The bounds may change;
- full Blaschke-Santaló diagram;
- $p$ -Laplacian instead of Laplacian?
- efficient experiments (random domains?).