Laplacian comparison theorem on Riemannian manifolds with modified m-Bakry-Emery Ricci lower bounds for $m \leq 1$

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1 Framework and Result

(M,g): n-dim. complete smooth Riemannian mfd, $\partial M = \emptyset$

$\mathfrak{m} := \mathrm{vol}_g$: volume element, V: C^1-vector field

$\Delta_V = \Delta - \langle V, \nabla\cdot\rangle$: V-Laplacian

$\mathcal{L}_V g(X,Y) := \langle \nabla_X V, Y\rangle + \langle \nabla_Y V, X\rangle$: Lie derivative of g

$\mathrm{Ric}_V^m := \mathrm{Ric}_g + \frac{1}{2}\mathcal{L}_V g - \frac{V^*\otimes V^*}{m-n}$ $(m \in]-\infty, 1[\cup[n,+\infty[)$

:modified m-Bakry-Émery Ricci tensor

$\Delta_f := \Delta - \langle \nabla f, \nabla\rangle$: Witten Laplacian $(f \in C^2(M))$

$\mathrm{Ric}_f^m := \mathrm{Ric}_g + \nabla^2 f - \frac{df\otimes df}{m-n}$ $(m \in]-\infty, 1[\cup[n,+\infty[)$

:m-Bakry-Émery Ricci tensor.

Hereafter, we fix a point $p \in M$. The following might be new.

$$V_\gamma(r) := \int_0^r \langle V_{\gamma_s}, \dot\gamma_s\rangle \mathrm{d}s, \quad \gamma : \text{ unit speed geo.}$$

$$f_V(x) := \inf\left\{ \int_0^{r_p(x)} \langle V_{\gamma_s}, \dot\gamma_s\rangle \mathrm{d}s \;\middle|\; \begin{array}{l} \gamma : \text{unit speed geodesic} \\ \gamma_0 = p, \gamma(r_p(x)) = x \end{array} \right\},$$

f_V depends on p and $f_V(p) = 0$,

If $V = \nabla f$ with $f \in C^2(M)$, then $f_V(x) = f(x) - f(p)$.

Def 1.1 (Invariant measure for Δ_V) A Radon measure μ is said to be an invariant measure for Δ_V if $\Delta_V^* \mu = 0$, i.e.

$$\int_M \Delta_V v \mathrm{d}\mu = 0 \quad \text{for all} \quad v \in C_0^\infty(M).$$

It is well-known that the invariant measure exists uniquely up to a multiplicative constant provided M is compact, or

$$k(r) := \inf\left\{\mathrm{Ric}_V^{\infty}(\nabla r_p, \nabla r_p) \,\middle|\, r_p(x) \geq r, x \notin \mathrm{Cut}(p)\right\}$$

satisfies $\int_0^{\infty} k(r)\mathrm{d}r = +\infty$ by Ikeda-Watanabe (1989), Bogachev-Röckner-Wang (01).

Q: $\mu_V := e^{-f_V}\mathfrak{m}$ is an invariant measure for Δ_V?

If $V = \nabla f$, then $\mu = e^{-f}\mathfrak{m}$ is an invariant measure for Δ_f. We do not have the answer.

For the fixed point $p \in M$, we consider a free constant $C_p = C_p(m) > 0$ and assume $m \leq 1$! For $x \in M$, we define

$$s_p(x) := \inf\left\{ C_p \int_0^{r_p(x)} e^{-\frac{2V_\gamma(t)}{n-m}} \mathrm{d}t \;\middle|\; \begin{array}{l} \gamma : \text{unit speed geo.} \\ \gamma_0 = p, \gamma(r_p(x)) = x \end{array} \right\}.$$

$s(p,x) := s_p(x)$: $s(x,y) \geq 0$ & $s(x,y) = 0 \Leftrightarrow x = y$

$s(x,y) \neq s(y,x)$ in general!

If $V = \nabla f$, $C_p := \exp\left(-\frac{2f(p)}{n-m}\right)$ & $C_q := \exp\left(-\frac{2f(q)}{n-m}\right)$, then

$$s_p(x) := \inf\left\{ \int_0^{r_p(x)} e^{-\frac{2f(\gamma_t)}{n-m}} \mathrm{d}t \;\middle|\; \begin{array}{l} \gamma : \text{unit speed geo.} \\ \gamma_0 = p, \gamma(r_p(x)) = x \end{array} \right\}.$$

& $s(p,q) = s(q,p)$. $s_p(x)$ was introduced by Wylie-Yeroshkin (16) for the case $V = \nabla f$ and $m = 1$.

(M, d_g, V): (V, m)-complete at p

$$\overset{\text{def}}{\Longleftrightarrow} \quad \overline{\lim_{r\to+\infty}} \inf\left\{ \int_0^r e^{-\frac{2V_{\gamma}(t)}{n-m}} \mathrm{d}t \,\middle|\, \begin{array}{l} \gamma \text{ unit speed geo} \\ \gamma_0 = p \end{array} \right\} = +\infty \qquad (1)$$

(M, d_g, V): (V, m)-complete

$\overset{\text{def}}{\Longleftrightarrow}$ (M, d_g, V): (V, m)-complete at any $p \in M$.

$\kappa : [0, +\infty[\to \mathbb{R}$: continuous function on $[0, +\infty[$.

$\mathfrak{s}_\kappa$ is the unique sol. to Jacobi eq:

$$\psi''(s) + \kappa(s)\psi(s) = 0, \quad \psi(0) = 0, \quad \psi'(s) = 1.$$

$\delta_\kappa := \inf\{s > 0 \mid \mathfrak{s}_\kappa(s) = 0\}$. If κ is constant, then $\mathfrak{s}_\kappa(s) = \frac{\sin\sqrt{\kappa}s}{\sqrt{\kappa}}$ $(\kappa > 0)$, $= s$ $(\kappa = 0)$, $= \frac{\sinh\sqrt{-\kappa}s}{\sqrt{-\kappa}}$ $(\kappa < 0)$, & $\delta_\kappa = \frac{\pi}{\sqrt{\kappa_+}}$.

Thm 1.1 (Laplacian Comparison Thm, K-Shukuri (20))

Fix $p \in M$. Take $R \in]0, +\infty]$. Let f_V be the function defined above.

Suppose that

$$\mathrm{Ric}_V^m(\nabla r_p, \nabla r_p)_x \geq (n-m)\,\kappa(s_p(x))e^{-\frac{4f_V(x)}{n-m}}C_p^2 \qquad (2)$$

holds under $r_p(x) < R$ with $x \in (\mathrm{Cut}(p) \cup \{p\})^c$. Then

$$(\Delta_V r_p)(x) \leq (n-m)\ \cot_\kappa(s_p(x))e^{-\frac{2f_V(x)}{n-m}}C_p \qquad (3)$$

holds if $x \in \{r_p < R, s_p < \delta_\kappa\} \setminus (\mathrm{Cut}(p) \cup \{p\})$.

Here $\cot_\kappa(s) := \frac{\mathfrak{s}_\kappa'(s)}{\mathfrak{s}_\kappa(s)}$.

Rem 1.1 In K-Li ('19), we prove **Thm** 1.1 under $V = \nabla f$ with $f \in C^2(M)$ and $C_p = \exp\left(-\frac{2f(p)}{n-m}\right)$.

2 Known works

Thm 2.1 (**Greene-Wu** ('79), **Kasue** ('82), **Bakry-Qian** (05), **Li** (05))

$\mathrm{Ric}_f^m \geq (m-1)\kappa \Rightarrow \Delta_f r_p(x) \leq (m-1)\cot_\kappa(r_p(x))\ {}^\forall x \notin \{p\} \cup \mathrm{Cut}(p).$

K-Shioya ('10) Laplacian comparison for Alex sp. under $\mathrm{BG}(K, N)$.

Gigli ('15) Laplacian comparison under $\mathrm{CD}(K, N)$-space, $N > 1$.

Thm 2.2 (**Wylie-Yeroshkin** ('16)) : $\mathrm{Ric}_f^1 \geq (n-1)\kappa e^{-\frac{4f}{n-1}} \Rightarrow$

$\Delta_f r_p(x) \leq (n-1)\cot_\kappa(s_p(x))e^{-\frac{2f(x)}{n-1}}$ for all $x \notin \{p\} \cup \mathrm{Cut}(p)$.

Thm 2.3 (**K-Li** ('19)) : $m \leq 1, \quad \mathrm{Ric}_f^m \geq (n-m)\kappa e^{-\frac{4f}{n-m}} \Rightarrow$

$\Delta_f r_p(x) \leq (n-m)\cot_\kappa(s_p(x))e^{-\frac{2f(x)}{n-m}}$ for all $x \notin \{p\} \cup \mathrm{Cut}(p)$.

3 Geometric consequences

Thm 3.1 (Weighted Myers' Theorem)

(M, g) : n-dim. complete C^∞-Riemannian mfd

V: C^1-vector field. Fix $p \in M$. Suppose $\delta_\kappa < \infty$ &

$$\mathrm{Ric}_V^m(\nabla r_p, \nabla r_p)_x \geq (n-m)\,\kappa(s_p(x))e^{-\frac{4f_V(x)}{n-m}}C_p^2 \qquad (4)$$

holds ${}^\forall x \in (\mathrm{Cut}(p) \cup \{p\})^c$. Then $s(p,q) \leq \delta_\kappa$ for all $q \in M$.

Cor 3.1 Let (M, g), V be as above. Fix $p \in M$ and $\delta_\kappa < \infty$. Assume that (4) holds for all $x \in (\mathrm{Cut}(p) \cup \{p\})^c$ and (M, g, V) is (V, m)-complete at p. Then M is compact.

Thm **3.2 (Bishop-Gromov Volume Comparison)** Fix $p \in M$. Let (M, g), V as above. Assume that (4) holds for all $x \in (\mathrm{Cut}(p) \cup \{p\})^c$. Then

$$r \mapsto \frac{\mu_V(B_r(p))}{\nu_p(\kappa, r)} \uparrow \quad \text{as} \quad r \downarrow 0 \tag{5}$$

Here $\nu_p(\kappa, r) := \int_0^r \int_{\mathbb{S}^{n-1}} \mathfrak{s}_\kappa^{n-m}(s_p(r, \theta)) \mathrm{d}r \mathrm{d}\theta$ and $\mu_V := e^{-f_V} \mathfrak{m}$.

Thm **3.3 (Ambrose-Myers' Theorem)** Let (M, g), V as above. Fix $p \in M$. Assume that (M, g, V) is (V, m)-complete at p and $^\exists$a unit speed geodesic γ with $\gamma_0 = p$ satisfying

$$\int_0^\infty e^{\frac{2V_\gamma(t)}{n-m}} \mathrm{Ric}_V^m(\dot{\gamma}_t, \dot{\gamma}_t) \mathrm{d}t = +\infty. \tag{6}$$

Then M is compact.

Thm 3.4 (Cheng's Maximal Diameter Theorem)

Let (M, g) be as above and $V = \nabla f$ with $f \in C^2(M)$. Assume that κ is a positive constant and (4) holds for all $x \in (\mathrm{Cut}(p) \cup \{p\})^c$ and set $C_p = \exp\left(-\frac{2f(p)}{n-m}\right)$. If there is a point q such that $d^h(p, q) = \pi/\sqrt{\kappa}$, then $m = 1$, f is rotationally symmetric around p, i.e., f is a function depending only on radial r, and g is a warped product metric of the form

$$g = \mathrm{d}r^2 + e^{\frac{2f(r)+2f(0)}{n-1}} \cdot \frac{\sin^2(\sqrt{\kappa}(s(r)))}{\kappa} g_{\mathbb{S}^{n-1}}, \quad 0 \le r \le d(p, q),$$

where $s(r) = \int_0^r e^{-\frac{2f(t)}{n-1}} \mathrm{d}t$ and $s(d(p, q)) = \pi/\sqrt{\kappa}$.

4 Sketch of Proof

Lem 4.1 $\gamma(r)$: unit speed geod. $\gamma_0 = p$ and $\dot\gamma_0 = \theta \in \mathbb{S}^{n-1}$. Let $\mathrm{d}s = C_p e^{\frac{-2V_{\gamma}(r)}{n-m}}\mathrm{d}r$ and $\lambda(r,\theta) = C_p^{-1}(e^{\frac{2f_V}{n-m}}\Delta_V r_p)(r,\theta)$. Then

$$\frac{\mathrm{d}\lambda}{\mathrm{d}s} \le -\frac{\lambda^2}{n-m} - C_p^{-2} e^{\frac{4V_{\gamma}(r)}{n-m}} \mathrm{Ric}_V^m(\dot\gamma_r, \dot\gamma_r), \quad (r,\theta) \notin (\mathrm{Cut}(p) \cup \{p\}) \tag{7}$$

If "$=$"is achieved at a point, then $m = 1$ and at that point $\nabla_{\nabla r_p}$ has at most one non-zero eigenvalue with multiplicity $n-1$.

Proof. $|\nabla r_p|^2 = 1 \Rightarrow (\nabla^2 r_p)(\nabla r_p) = 0 \Rightarrow 0$ is an eigenvalue.
$|\nabla^2 r_p|^2 \ge \frac{(\Delta r_p)^2}{n-1} \ge \frac{(\Delta r_p)^2}{n-m}$ & Bochner-Weizenböck formula:
$0 = \frac{1}{2}\Delta_V |\nabla r_p|^2 \overset{\mathrm{BW}}{=} |\nabla^2 r_p|^2 + \mathrm{Ric}_V^\infty(\nabla r_p, \nabla r_p) + \langle \nabla \Delta_V r_p, \nabla r_p\rangle$.

$$\Longrightarrow \frac{\mathrm{d}}{\mathrm{d}r}(\Delta_V r_p)(r,\theta) \leq -\frac{(\Delta r_p(r,\theta))^2}{n-m} - \mathrm{Ric}_V^m(\dot{\gamma}_r, \dot{\gamma}_r) + \frac{|\langle V, \nabla r_p\rangle(r,\theta)|^2}{n-m}.$$

If "$=$" holds for (7) at some $x = (r_0, \theta) \notin \mathrm{Cut}(p) \cup \{p\}$, then

$$0 = \frac{(\Delta r_p)^2}{n-m} + \mathrm{Ric}_V^m(\nabla r_p, \nabla r_p) - \frac{|\langle V, \nabla r_p\rangle|^2}{n-m} + \langle \nabla \Delta_V r_p, \nabla r_p\rangle$$
$$\geq \frac{(\Delta r_p)^2}{n-1} + \mathrm{Ric}_V^m(\nabla r_p, \nabla r_p) - \frac{|\langle V, \nabla r_p\rangle|^2}{n-m} + \langle \nabla \Delta_V r_p, \nabla r_p\rangle$$

holds at $x \notin \mathrm{Cut}(p) \cup \{p\}$. This and $m \leq 1$ yield

$$\frac{m-1}{(n-m)(n-1)}(\Delta r_p)^2(x) = 0.$$

Since M has an upper bound $\kappa_\varepsilon > 0$ of the sectional curvature on some $B_\varepsilon(p) \subset \mathrm{Cut}(p)^c$, $\Delta r_p(x) > 0$. Therefore we obtain $m = 1$ and $|\nabla^2 r_p|^2 = \frac{(\Delta r_p(x))^2}{n-1}$ at x. This implies that $\nabla_{\nabla r_p}$ at x has at most one non-zero eigenvalue of multiplicity $n-1$. $\square$

Let $\kappa \in C([0,+\infty[)$. Suppose

$\mathrm{Ric}_V^m(\nabla r_p, \nabla r_p) \geq (n-m)\kappa(s_p(x))e^{-\frac{4f_V(x)}{n-m}}C_p^2$ for $s_p(x) < S$ with $x \notin \mathrm{Cut}(p) \cup \{p\}$. ($S := s(R) = C_p \int_0^R \exp\left(-\frac{2V_\gamma(t)}{n-m}\right) \mathrm{d}t$).

From (7) we have the usual Riccati inequality

$$-\frac{\mathrm{d}\lambda}{\mathrm{d}s}(s) \geq (n-m)\kappa(s) + \frac{\lambda(s)^2}{n-m} \quad \text{for} \quad s \in]0, S[. \tag{8}$$

Lem 4.2 Let (M, g), f be as above.

Assume (2) for $r_p(x) < R$ with $x \notin \mathrm{Cut}(p) \cup \{p\}$. Let γ, s, and λ be as in Lem 4.1. Then

$$\lambda(r,\theta) \leq (n-m)\cot_\kappa(s) \quad r < R, \quad s < \delta_\kappa \tag{9}$$

$x = (r,\theta) \notin \mathrm{Cut}(p) \cup \{p\}$. Here $s = s(r) = C_p \int_0^r \exp\left(-\frac{2V_\gamma(t)}{n-m}\right) \mathrm{d}t$.

Suppose further that the equality in (9) holds for some $r_0 < R$ with $s_0 := s(r_0) < \delta_\kappa$. We choose an orthonormal basis $\{e_i\}_{i=1}^n$ of T_pM with $e_n = \dot\gamma_0$. Let $\{Y_i\}_{i=1}^{n-1}$ be the Jacobi fields along γ with $Y_i(0) = o_p$ and $Y_i'(0) = e_i$. Then we have $m = 1$, and at $x = (r, \theta)$ with $r \le r_0$, $\nabla_{\nabla r_p}$ has at most one non-zero eigenvalue which is of multiplicity $n-1$.

For all i we have $Y_i(r) = C_p^{-1} F_\kappa(r) E_i(r)$ for $r \in [0, r_0]$, where

$$F_\kappa(r) := \exp\left(\frac{V_\gamma(r)}{n-1}\right)\mathfrak{s}_\kappa(s_p(\gamma_r)), \tag{10}$$

and $\{E_i(r)\}_{i=1}^{n-1}$ are the parallel vector fields with $E_i(0) = e_i$. Then,

$$g_{\gamma_r} = dr^2 + C_p^{-2} e^{\frac{2V_\gamma(r)}{n-1}} \mathfrak{s}_\kappa^2(s(r)) g_{\mathbb{S}^{n-1}}. \tag{11}$$

Here $g_{\mathbb{S}^{n-1}}$ is the standard metric on the sphere $\mathbb{S}^{n-1}$.

Proof. Note that $m_\kappa(s) := (n-m)\cot_\kappa(s)$ satisfies

$$-\frac{\mathrm{d}m_\kappa}{\mathrm{d}s}(s) = (n-m)\kappa(s) + \frac{m_\kappa(s)^2}{n-m} \tag{12}$$

Set $\beta(s) := \mathfrak{s}_\kappa^2(s)(\lambda - m_\kappa(s))$. Then, by (12) and (8), for $s < S$

$$\beta'(s) = 2\mathfrak{s}_\kappa'(s)\mathfrak{s}_\kappa(s)(\lambda - m_\kappa(s)) + \mathfrak{s}_\kappa^2(s)\left(\frac{\mathrm{d}\lambda}{\mathrm{d}s} - m_\kappa'(s)\right)$$

$$= 2\mathfrak{s}_\kappa^2(s)\cot_\kappa(s)(\lambda - m_\kappa(s)) + \mathfrak{s}_\kappa^2(s)\left(\frac{\mathrm{d}\lambda}{\mathrm{d}s} + (n-m)\kappa(s) + \frac{m_\kappa^2(s)}{n-m}\right)$$

$$\leq \frac{\mathfrak{s}_\kappa^2(s)}{n-m}\left(2m_\kappa(s)\lambda - 2m_\kappa^2(s)\right) + \frac{\mathfrak{s}_\kappa^2(s)}{n-m}\left(m_\kappa^2(s) - \lambda^2\right)$$

$$= -\frac{\mathfrak{s}_\kappa^2(s)}{n-m}(\lambda - m_\kappa(s))^2 \leq 0.$$

If $\beta(0) = 0$, then $\beta(s) \leq \beta(0) = 0$, i.e. the conclusion holds.

For proving $\beta(0) = 0$, it suffices to prove that

$$\lim_{s \to 0} s(\lambda - m_\kappa(s)) = n - 1 - (n - m) = m - 1$$

We already know that $\lim_{s \to 0} s\, m_\kappa(s) = n - m$ and the ratio $s/r = s(r)/r$ converges to C_p as $r \to 0$. So it suffices to prove $\lim_{r \to 0} s(r)\lambda(r, \theta) = n - 1$, equivalently $\lim_{r \to 0} r\Delta r_p(r, \theta) = n - 1$, because $\lim_{r \to 0} r\langle V, \nabla r_p\rangle(r, \theta) = 0$.

$$(n - 1)\cot_{K_\varepsilon}(r) \leq \Delta r_p(r, \theta) \leq (n - 1)\cot_{\kappa_\varepsilon}(r) \text{ for } r < \varepsilon.$$

This implies the desired assertion. $\square$

Спасибо за Ваше внимание!

Thank you very much for your attention!