High-dimensional ellipsolds converge to Gaussian spaces

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This talk is based on a joint work with
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Today's talk

Main result (Kazukawa-S.)

For any sequence of ellipsoids, E_n, $n = 1, 2, \dots$, with $\dim E_n \to \infty$, there is a subsequence of $\{E_n\}$ converging to an (infinite-dimensional) Gaussian space $(\mathbb{R}^\infty, \gamma)$ in the sense of Gromov's concentration/weak topology.

Plan of talk:

1. Gromov's theory of mm-spaces
2. Known examples of convergence
3. Convergence of ellipsoids
4. Idea of proof of Main Thm

Gromov's theory of mm-spaces

Def (mm-space)

A metric measure space (mm-space) is a triple (X, d_X, μ_X), where (X, d_X) is a complete separable metric space and μ_X a Borel probability measure on (X, d_X).

Examples

- complete Riemannian manifold of finite volume, where the volume measure is normalized to be probability.
- n-dim Gaussian space $\Gamma^n := (\mathbb{R}^n, \|\cdot\|_2, \gamma^n)$, where $\|\cdot\|_2$ is the l_2 (Euclidean) norm and γ^n the n-dimensional standard Gaussian measure on $\mathbb{R}^n$.

Throughout this talk,
let X, Y, X_n, Y_n, $n = 1, 2, \ldots$, be mm-spaces.

Def (Box distance, Gromov-Prohorov distance)

A Borel m'ble map $\varphi : I := [\,0,1\,) \to X$ is called a parameter of X if $\varphi_* \mathcal{L}^1 = \mu_X$, where $\mathcal{L}^1$ is the one-dim Lebesgue measure.

$$\square(X,Y) := \inf\{\, \varepsilon \ge 0 \mid {}^\exists\varphi,\psi\text{: params of } X,Y,\ {}^\exists I_0 \subset I\text{: Borel s.t. } \mathcal{L}^1(I_0) \ge 1-\varepsilon,\ |\varphi^* d_X(s,t) - \psi^* d_Y(s,t)| \le \varepsilon,\ {}^\forall s,t \in I_0 \,\},$$

$$d_{\mathrm{GP}}(X,Y) := \inf\{\, d_{\mathrm{P}}(\mu_X,\mu_Y) \mid X,Y \hookrightarrow Z \text{ isometrically} \,\},$$

where $\varphi^* d_X(s,t) := d_X(\varphi(s),\varphi(t))$ and d_P is the Prohorov distance.

- $\square$ and d_{GP} are distance functions on the set of isomorphism classes of mm-spaces and satisfy $d_{\mathrm{GP}} \le \square \le 2\, d_{\mathrm{GP}}$.
- $\square$ and d_{GP} are considered roughly to be the metrizations of measured Gromov-Hausdorff convergence.

Def (Observable distance, concentration topology)

$$d_{\mathrm{conc}}(X,Y) := \text{“distance between } \mathcal{L}ip_1(X) \text{ and } \mathcal{L}ip_1(Y)\text{”},$$

$$\left(:= \inf_{\varphi,\psi} d_{\mathrm{H}}(\varphi^*\mathcal{L}ip_1(X), \psi^*\mathcal{L}ip_1(Y)) \right),$$

where $\mathcal{L}ip_1(X) := \{ f : X \to \mathbb{R} \mid \text{1-Lip} \}$,

$$X_n \overset{conc}{\to} X \text{ (concentration)} \overset{def}{\iff} d_{\mathrm{conc}}(X_n, X) \to 0,$$

The topology induced from d_{conc} is called the concentration topology.

- $X_n \to X$ w.r.t. measured Gromov-Hausdorff
  $\implies X_n \overset{\square}{\to} X \implies X_n \overset{conc}{\to} X$.
- The idea of the concentration topology comes from the concentration of measure phenomenon due to Lévy and Milman.
- The concentration topology is effective to capture the high-dimensional phenomena.

Def (Lévy family)

$\{X_n\}$ is called a Lévy family if $X_n \overset{conc}{\longrightarrow} *$,
where $*$ denotes one-point mm-space.

- $X_n \overset{conc}{\longrightarrow} *$ (Lévy family)
  $\iff \mathcal{L}ip_1(X_n) \to \mathcal{L}ip_1(*) = \{\text{constants}\}$
  $\iff$ any 1-Lip func on X_n is almost constant for large n
  (Concentration of measure phenomenon)

Lévy's lemma

$$S^n(1) \overset{conc}{\longrightarrow} * \text{ as } n \to \infty.$$

Very different from (measured) Gromov-Hausdorff convergence !

- Any subsequence of $\{S^n(1)\}$ is $\square$-divergent (Funano).

Def (Lipschitz order)

$$X \prec Y \overset{def}{\Longleftrightarrow} {}^\exists f : Y \to X \ \text{ 1-Lip with } f_*\mu_Y = \mu_X$$

$\mathcal{X}$: the set of isomorphism classes of mm-spaces.

Π : Gromov's compactification of $(\mathcal{X}, d_{\mathrm{conc}})$ with the weak topology.

$\prec$ extends to Π and we have the largest element $\infty \in \Pi$,

$$\underset{\text{(smallest)}}{*} \prec X \prec \underset{\text{(largest)}}{\infty} \qquad \text{for any } X \in \Pi.$$

Def (Dissipation)

A convergence $X_n \xrightarrow{w} \infty$ is called a dissipation.

Prop (Characterization of dissipation)

$X_n \xrightarrow{w} \infty \Longleftrightarrow {}^\exists A_{n,1}, \dots, A_{n,N_n} \subset X_n$ with $N_n \to \infty$ s.t.

$$\mu_{X_n}\Big(\bigcup_{i=1}^{N_n} A_{n,i}\Big) \to 1, \ \max_{i=1}^{N_n} \mu_{X_n}(A_{n,i}) \to 0, \ \min_{i \neq j} d_{X_n}(A_{n,i}, A_{n,j}) \to \infty.$$

Known examples of convergence

Def (∞-dim Gaussian space)

Let $\gamma^\infty_{\{a_i^2\}} := \bigotimes_{i=1}^\infty \gamma^1_{a_i^2}$ be the ∞-dim centered Gaussian measure on $\mathbb{R}^\infty$ with variance $\{a_i^2\}$. $\Gamma^\infty_{\{a_i^2\}} := (\mathbb{R}^\infty, \|\cdot\|_2, \gamma^\infty_{\{a_i^2\}})$ is called an ∞-dim Gaussian space, actually defined as an element of Π.

- $\sum_{i=1}^\infty a_i^2 < \infty \iff \Gamma^\infty_{\{a_i^2\}}$ is an mm-space.

Thm (Convergence of spheres)

1. $r_n/\sqrt{n} \to 0 \iff S^n(r_n) \xrightarrow{conc} *$ (Gromov-Milman).
2. $r_n/\sqrt{n} \to \infty \iff S^n(r_n) \xrightarrow{w} \infty$ (S.).
3. If $r_n/\sqrt{n} \to a \in (0, +\infty)$, then
   - (i) $S^n(r_n) \xrightarrow{w} \Gamma^\infty_{\{a^2\}}$ (S.);
   - (ii) any subsequence of $\{S^n(r_n)\}$ is $\square$-divergent and is not d_{conc}-Cauchy (S.).

Other results

- Weak convergence of projective spaces (S.).
- Weak convergence of Stiefel and flag manifolds (S.-Takatsu).

Easy processes to generate a new concentration sequence $X_n \overset{conc}{\to} X$:

1. $X_n \overset{conc}{\to} X$, $Y_n \overset{conc}{\to} Y \implies X_n \sqcup Y_n \overset{conc}{\to} X \sqcup Y$.
2. $X_n \overset{\square}{\to} X$, $Y_n \overset{conc}{\to} * \implies X_n \times Y_n \overset{conc}{\to} X$ (Gromov).
   (more generally Y_n-fibrations over X_n)
3. $X_n \overset{conc}{\to} X$, X'_n is a little modification of $X_n \implies X'_n \overset{conc}{\to} X$.

A sequence not generated by these three processes is called an irreducible sequence.

Problem

Find an example of an irreducible $X_n \overset{conc}{\to} X$ that is $\square$-divergent.

Our main theorem provide the first discovered solution to this problem !

Convergence of ellipsoids

Consider a sequence of solid ellipsoids,

$$E_j := \{ x \in \mathbb{R}^{n(j)} \mid \sum_{i=1}^{n(j)} \frac{x_i^2}{\alpha_{ij}^2} \le 1 \},$$

$$\alpha_{ij} > 0,\ i = 1, 2, \dots, n(j),\ j = 1, 2, \dots.$$

We equip each E_j the normalized Lebesgue measure.
In the case where $\sup_j n(j) < \infty$,

- if $\sup_{i,j} \alpha_{ij} < \infty$, then $^\exists$ a subsequence of $\{E_j\}$ Hausdroff-converges (and hence $\square$-converges) to a (possibly degenerate) ellipsoid in $\mathbb{R}^N$, where $N := \max_j n(j)$;
- if $\max_i \alpha_{ij} \to \infty$ as $j \to \infty$, then $E_j \xrightarrow{w} \infty$.

Assume that $n(j) \to \infty$ as $j \to \infty$ and set $a_{ij} := \alpha_{ij}/\sqrt{n(j)}$.
If $\max_i a_{ij} \to \infty$ as $j \to \infty$, then $E_j \xrightarrow{w} \infty$.

We assume the following:

(A0) $n(j) \to \infty$ and $\sup_{i,j} a_{ij} < \infty$.

Taking a subsequence we have:

(A1) $n(j)$ is monotone nondecreasing in j.

(A2) a_{ij} is monotone nonincreasing in i for each j.

(A3) $a_{ij} \to {}^\exists a_i$ as $j \to \infty$ for each i.

Main Thm (Kazukawa-S.)

Under (A0)–(A3), we have the following.

1. $E_j \xrightarrow{w} \Gamma^\infty_{\{a_i^2\}}$.
2. $E_j \xrightarrow{conc} \Gamma^\infty_{\{a_i^2\}} \iff \sum_{i=1}^\infty a_i^2 < \infty$.
3. $\{E_j\}$ is d_{conc}-Cauchy $\iff \lim_{i\to\infty} a_i = 0$.
4. $E_j \xrightarrow{\square} \Gamma^\infty_{\{a_i^2\}} \iff \sum_{i=1}^\infty a_i^2 < \infty$ and $\lim_{j\to\infty} \sum_{i=1}^{n(j)} (a_{ij} - a_i)^2 = 0$.

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(A0) $n(j) \to \infty$ and $\sup_{i,j} a_{ij} < \infty$.

Taking a subsequence we have:

(A1) $n(j)$ is monotone nondecreasing in j.

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Main Thm (Kazukawa-S.)

Under (A0)–(A3), we have the following.

1. $E_j \xrightarrow{w} \Gamma^{\infty}_{\{a_i^2\}}$. (nontrivial)
2. $E_j \xrightarrow{conc} \Gamma^{\infty}_{\{a_i^2\}} \iff \sum_{i=1}^{\infty} a_i^2 < \infty$. (follows from ①)
3. $\{E_j\}$ is d_{conc}-Cauchy $\iff \lim_{i\to\infty} a_i = 0$. (not so difficlut by ①)
4. $E_j \xrightarrow{\square} \Gamma^{\infty}_{\{a_i^2\}} \underset{\Longleftarrow}{\Longrightarrow} \displaystyle\sum_{i=1}^{\infty} a_i^2 < \infty$ and $\displaystyle\lim_{j\to\infty} \sum_{i=1}^{n(j)} (a_{ij} - a_i)^2 = 0$.

The Main Thm implies the following.

Cor (Kazukawa-S.)

Supposing

$$\sum_{i=1}^{\infty} a_i^2 < \infty \quad \text{and} \quad \liminf_{j\to\infty} \sum_{i=1}^{n(j)} (a_{ij} - a_i)^2 > 0,$$

we have

1. $E_j \overset{conc}{\to} \Gamma^{\infty}_{\{a_i^2\}}$;
2. any subsequence of $\{E_j\}$ is $\square$-divergent.

- $\{E_j\}$ is a solution to the problem before.

Idea of proof of Main Thm

1. $E_j \xrightarrow{w} \Gamma^{\infty}_{\{a_i^2\}}$.

It suffices to prove

(a) $\Gamma^{\infty}_{\{a_i^2\}} \prec \lim_{j\to\infty} E_j$,

(b) $\lim_{j\to\infty} E_j \prec \Gamma^{\infty}_{\{a_i^2\}}$.

(a) easily follows from the Maxwell-Boltzmann distribution law.
(b) is much more difficult. To prove (b), it suffices to show $E_j \prec \Gamma^{n(j)}_{\{a_i^2\}}$, but this does not hold. Our idea is to reduce the problem to the following special case:

$$(*) \begin{cases} a_i = a_N & \text{for all } i \geq N, \\ a_{ij} = a_i & \text{for all } i, j. \end{cases}$$

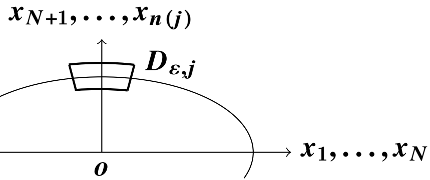
We need a delicate discussion to prove '(b) in $(*)$ $\implies$ (b)'.

Assume $(*)$. We set

$$r(x) := \left(\sum_{i=1}^{n(j)} \frac{x_i^2}{\alpha_{ij}^2} \right)^{\frac{1}{2}}, \qquad x \in \mathbb{R}^{n(j)}.$$

Note that $E_j = \{ r \le 1 \}$. Consider, for $\varepsilon > 0$,

$$D_{\varepsilon,j} := \{ x \in \mathbb{R}^{n(j)} \mid |x_j| \le \varepsilon \|x\|_2,\ j = 1, \dots, N, \\ (1 + N\varepsilon)^{-1/2} \le r(x) \le (1 + N\varepsilon)^{1/2} \}.$$



It holds that $\lim_{j\to\infty} \mu_{E_j}(D_{\varepsilon,j}) = 1$ and $\lim_{j\to\infty} \gamma^{n(j)}_{\{a_i\}}(D_{\varepsilon,j}) = 1$. We have a transprot map from $\mu_{E_j}|_{D_{\varepsilon,j}}$ to $\gamma^{n(j)}_{\{a_i\}}|_{D_{\varepsilon,j}}$ with Lipschitz constant converging to 1. With some careful error estimate, we obtain (b). □

4. $E_j \xrightarrow{\square} \Gamma^\infty_{\{a_i^2\}} \implies \sum_{i=1}^\infty a_i^2 < \infty$ and $\lim_{j\to\infty} \sum_{i=1}^{n(j)} (a_{ij} - a_i)^2 = 0$.

We prove the contraposition. Supposing that

$$\text{(i)} \ \sum_{i=1}^\infty a_i^2 = \infty \quad \text{or} \quad \text{(ii)} \ \liminf_{j\to\infty} \sum_{i=1}^{n(j)} (a_{ij} - a_i)^2 > 0,$$

we are going to prove that $\{E_j\}$ is $\square$-divergent. If (i) holds, then the weak limit is not an mm-space and so $\{E_j\}$ does not $\square$-converge. Assume (ii). Our idea is to shrink E_j to another ellipsoid E'_j satisfying

$$\lim_{j\to\infty} a'_{ij} = a'_i = 0 \quad \text{and} \quad \liminf_{j\to\infty} \sum_{i=1}^{n(j)} {a'_{ij}}^2 > 0.$$

Then, $E'_j \xrightarrow{conc} *$. We prove that $\{E'_j\}$ has no $\square$-convergent subseq., which together with $E'_j \prec E_j$ implies the $\square$-divergence of $\{E_j\}$. $\square$

4. $E_j \xrightarrow{\square} \Gamma^\infty_{\{a_i^2\}} \implies \sum_{i=1}^\infty a_i^2 < \infty$ and $\lim_{j\to\infty} \sum_{i=1}^{n(j)} (a_{ij} - a_i)^2 = 0$.

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Thank you for your attension.