Branching properties of random trees in the boundary of Outer space

Ilya Kapovich

Hunter College of CUNY

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Based on joint work with Joseph Maher, Catherine Pfaff and Samuel Taylor:
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Summary of the main results:

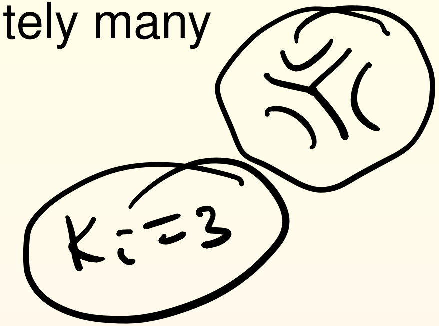
Theorem A. Let F_r be free of rank $r \geq 3$. Let μ be a “reasonable" nonelementary discrete probability measure on $Out(F_r)$ with finite support.
Let $G_0 \in CV_r$ be an arbitrary base-point in the Outer space CV_r. Let $(\phi_n)_n$ be the random walk on $Out(F_r)$ defined by μ.
Then the following hold:

1. With probability tending to 1 as $n \to \infty$, the outer automorphism $\phi_n \in Out(F_n)$ is fully irreducible with the attracting tree $T_+(\phi) \in \partial CV_r$ being trivalent and non-geometric.
2. Let $G_0 \in CV_r$ be an arbitrary basepoint, and let ν be the exit measure on ∂CV_r for the projected walk $(\phi_n G_0)_n$. Then for a ν-a.e. point $T \in \partial CV_r$, the $\mathbb{R}$-tree T is trivalent and non-geometric.

Motivation: mapping class groups

Let Σ be a closed oriented surface of genus $g \geq 2$, and let $MCG(\Sigma)$.
If $(\mathcal{F}, \lambda)$ is a measured foliation on Σ, then $\mathcal{F}$ has finitely many singularities with valencies $k_1, \dots, k_m \geq 3$.
Then, by the Poincare-Hopf theorem, we have

$$\sum_{i=1}^{m} [k_i - 2] = 2|\chi(\Sigma)| = 4g - 4.$$



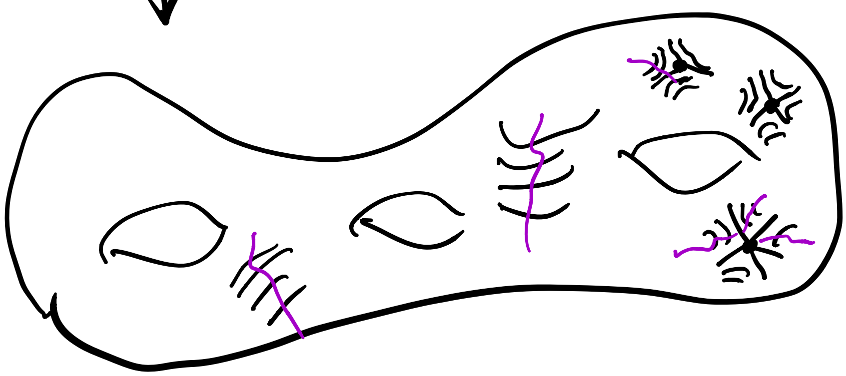
Thus the above index sum is independent of $\mathcal{F}$, but the "index list" $k_1, \dots, k_m$ depends on $\mathcal{F}$ and admits finitely many possibilities.

If $\mathcal{F}$ is arational with no saddle connections, then lifting $\mathcal{F}$ to $\widetilde{\Sigma} = \mathbb{H}^2$, collapsing the leaves to points and using the lifted transverse measure $\tilde{\lambda}$ to define distances, produces a "dual" $\mathbb{R}$-tree T_λ with a free isometric action of $\pi_1(\Sigma)$.

This tree T_λ has m orbits of branch points, of valencies $k_1, \dots, k_m$.

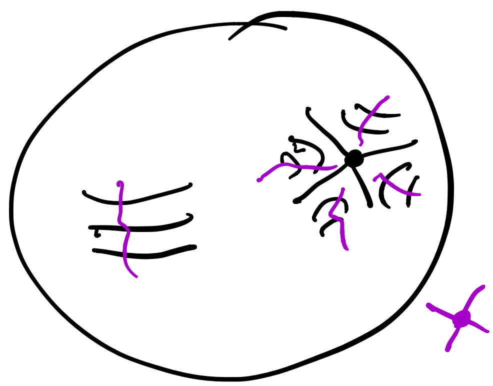
$\circlearrowright \varphi \to$ p-A

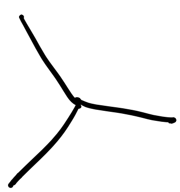
$\xi = (\mathcal{F}, \lambda)$



Σ

$\mathbb{H}^2 = \tilde{\Sigma}$





$G = \pi_1(\Sigma) \curvearrowright \underline{\underline{T_\lambda}}$ $\mathbb{R}$-tree

Motivation: mapping class group

Let μ be a "reasonable" non-elementary discrete probability measure on $G = MCG(\Sigma)$ and let $(\phi_n)_n$ be the random walk on G defined by μ. Also pick a basepoint $\rho \in \mathcal{T}(\Sigma)$ and consider the projected walk $(\phi_n\rho)_n$ on the Teichmuller space $\mathcal{T}(\Sigma)$.

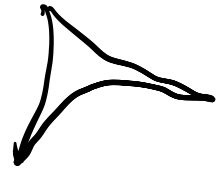
It is well-known that:

(a) [Kaimanovich-Masur] A.e. trajectory $\phi_n\rho$ of the projected walk converges to a point of $\partial\mathcal{T}(\Sigma) = PMF(\Sigma)$, and therefore the projected walk defines a μ-stationary "exit" measure ν on $PMF(\Sigma)$, supported on uniquely ergodic elements of $PMF(\Sigma)$.

(b) [Rivin, Maher, ...] With probability tending to 1 as $n \to \infty$, the element $\phi_n \in MCG(\Sigma)$ is pseudo-Anosov.

Q1: What can be said about the singularity properties of a ν-a.e. point in $PMF(\Sigma)$, and of the stable/unstable foliations of a "random" pseudo-Anosov ϕ_n?

$G \curvearrowright X$ → Gromov-hyp space (not necessarily proper)



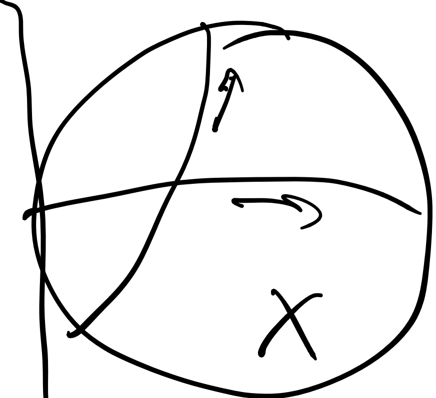
μ – "reasonable" non-elem. measure on G

MT:

① $P_\mu(g_n \curvearrowright X$ loxodr$) \xrightarrow[n\to\infty]{} 1$

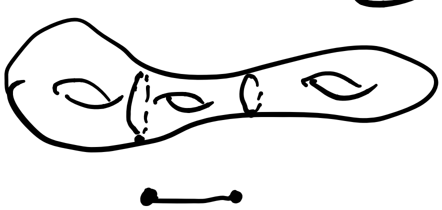
$g_n = s_1 s_2 \dots s_n$

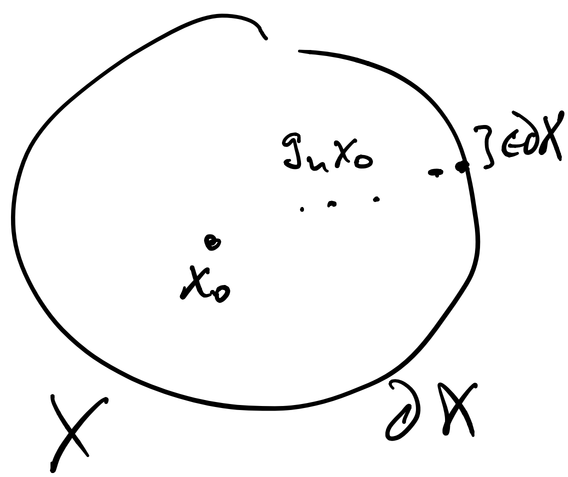
② $g_n x_0 \longrightarrow \xi \in \partial X$ a.s.



$G = MCG(\Sigma)$

$X = \mathcal{C}(\Sigma)$





Motivation: mapping class groups

Why do (a), (b) above hold?
Let G be a group acting by isometries on a separable Gromov-hyperbolic separable geodesic metric space X. Let μ be a “reasonable" non-elementary discrete probability measure on G and let $(w_n)_n$ be the random walk on G defined by μ. Let $x_0 \in X$ be a basepoint.
[Maher-Tiozzo] Then we have:

- A.e. trajectory $(w_n x_0)_n$ of the projected walk converges, as $n \to \infty$, to a point of ∂X, thus giving rise to a μ-stationary exit measure ν on ∂X.
- With probability tending to 1 as $n \to \infty$, the element $w_n \in G$ acts loxodromically on X.

Now apply to $G = MCG(\Sigma)$ and $X = \mathcal{C}(\Sigma)$, the curve complex of Σ. An element $\phi \in G$ is pseudo-Anosov iff ϕ cats loxodromically on $\mathcal{C}(\Sigma)$.

Hence we get (b), and, with a bit more work, (a).

Motivation: mapping class groups

The above argument gives no information about the answers to $Q1$. Recent results of Gadre and Maher imply that, for a reasonable non-elementary probability measure μ on $G = MCG(\Sigma)$, we have:

• For a.e. trajectory of the bi-infinite projected walk $(\phi_n \rho)_n$ on $\mathcal{T}(\Sigma)$ the limiting Teichmuller geodesic is uniquely ergodic and is defined by a tangent direction on the *principal stratum* of $\mathcal{Q}(\Sigma)$; the vertical folation for this geodesic has no saddle connections and has all of its singularities trivalent.

• With probability tending to 1 as $n \to \infty$, periodic Teichmuller geodesic axis of ϕ_n in $\mathcal{T}(\Sigma)$ is defined by a vector in the principal stratum of $\mathcal{Q}(\Sigma)$. The stable foliation of ϕ_n has no saddle connections and has all of its singularities trivalent.

F_r

$\varphi \in Out(F_r)$ is **fully irreducible** (f.i.)

if there do not exist
$n \geq 1$ and $F_r = A * B$, $A \neq 1$, $B \neq 1$

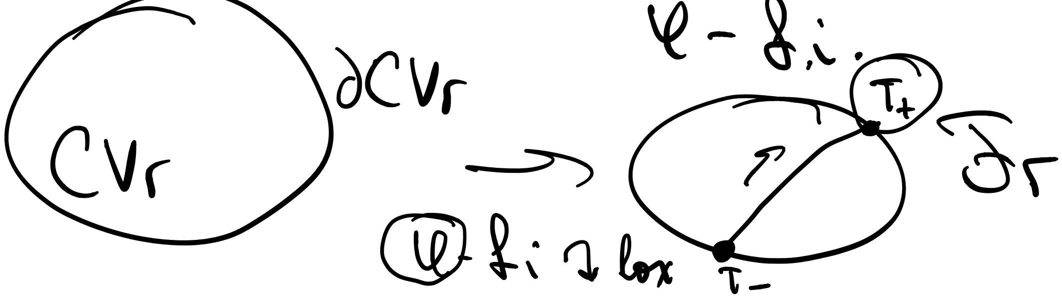
s.t. $\varphi^n([A]) = [A]$

$\mathcal{T}(\Sigma) \longleftrightarrow CV_r$

$\mathcal{C}(\Sigma) \longleftrightarrow \mathcal{F}_r$ free factor complex

$G = Out(F_r) \curvearrowright \mathcal{F}_r$

$\varphi \curvearrowright \mathcal{F}_r$ lox $\Leftrightarrow$ φ - f.i.



φ - f.i. $\curvearrowright$ lox

Index properties for $Out(F_r)$

Let F_r be the free group of finite rank $r \geq 2$. The free group analog of $\mathcal{T}(\Sigma)$ is given by the *Culler-Vogtmann Outer space* CV_r. A point of CV_r is finite connected volume-1 metric graph G with $b_1(G) = r$ and with a specific isomorphism or "marking" $F_r \cong \pi_1(G)$.

The boundary ∂CV_r consists of (equivalence classes) of $\mathbb{R}$-trees T with a minimal isometric “very small" (e.g. free) action of F_r on T. A tree $T \in \partial CV_r$ is *geometric* if it arises as the dual tree of a singular measured foliation on a finite 2-complex (which does not have to be a surface) with $\pi_1 \cong F_r$.

An element $\phi \in Out(F_r)$ is *fully irreducible* if there do not exist $t \geq 1$ and a proper free factor A of F_r such that $\phi^t[A] = [A]$. Being f.i. is an $Out(F_r)$ analog of being pseudo-Anosov.

Index properties for $Out(F_r)$

For an $\mathbb{R}$-tree T, for a point $p \in T$ the *valency* $\mathrm{val}_T(p)$ is the number of components (or "directions at p") of $T - \{p\}$. We say that $p \in T$ is a *branch point* if $\mathrm{val}_T(p) \geq 3$. We say that T is *trivalent* if for every branch point $p \in T$ we have $\mathrm{val}_T(p) = 3$.

For $r \geq 3$, "most" $T \in \partial CV_r$ are F_r-free, with dense F_r-orbits. In this case T/F_r is not a graph, but still has some graph-like features: There are only finitely many F_r-orbits of branch points in T, and each branch point has finite valency. We then define

$$ind_{geom}(T) := \sum_{i=1}^{m} [\mathrm{val}_T(p) - 2]$$

Then, by a result of Gaboriau-Levitt, we have

$$ind_{geom}(T) \leq 2r - 2$$

with the equality occurring if and only if T is geometric.

Index properties for $Out(F_r)$

Every f.i. $\phi \in Out(F_r)$ acts on $CV_r \cup \partial CV_r$ with the "North-South dynamics", with the attracting/repelling fixed point $T_{\pm}(\phi) \in \partial CV_r$.

If $\phi \in Out(F_r)$ is fully irreducible, then it is known that either ϕ is of “surface type" (i.e. ϕ is induced by a pseudo-Anosov homeo of a compact surface Σ with one boundary component and $\pi_1(\Sigma) \cong F_r$), or else the action of F_r on $T_+(\phi)$ is free with dense orbits.

The *free factor graph* $\mathcal{F}_r$ has as its vertices the conjugacy classes of proper free factors of F_r. Two distinct vertices $[A], [B]$ are adjacent in $\mathcal{F}_r$ if there exist representatives $A \in [A], B \in [B]$ such that $A \leq B$ or $B \leq A$.

It is known (Bestvina-Feighn) that $\mathcal{F}_r$ is Gromov-hyperbolic, and that $\phi \in Out(F_r)$ is f.i. if and only if ϕ acts loxodromically on $\mathcal{F}_r$.

Index properties for $Out(F_r)$

Hence, if μ is a "reasonable" non-elementary probability measure on $Out(F_r)$ (where $r \geq 3$) then for the corresponding random walk $(\phi_n)_n$:

• By Maher-Tiozzo applied to the action of $Out(F_r)$ on $\mathcal{F}_r$, with probability tending to 1 as $n \to \infty$ the element $\phi_n \in Out(F_r)$ is fully irreducible (and of non-surface type).

• By Namazi-Pettet-Reynolds, if $G_0 \in CV_r$ is an arbitrary basedpoint, then a.e. trajectory $\phi_n G_0$ of the projected walk converges to a point of ∂CV_r; hence the projected walk defines a μ-stationary exit measure ν supported on free uniquely ergodic trees in ∂F_r.

Q2. What can we say about the index properties of a ν-random $T \in \partial CV_r$ and of $T_+(\phi_n)$?

Main results

Theorem A. Let F_r be free of rank $r \geq 3$. Let μ be a "reasonable" nonelementary discrete probability measure on $Out(F_r)$ with finite support.
Let $G_0 \in CV_r$ be an arbitrary base-point in the Outer space CV_r. Let $(\phi_n)_n$ be the random walk on $Out(F_r)$ defined by μ.
Then the following hold:

1. With probability tending to 1 as $n \to \infty$, the outer automorphism $\phi_n \in Out(F_n)$ is fully irreducible with the attracting tree $T_+(\phi_n) \in \partial CV_r$ being trivalent and non-geometric.
2. Let $G_0 \in CV_r$ be an arbitrary basepoint, and let ν be the exit measure on ∂CV_r for the projected walk $(\phi_n G_0)_n$. Then for a ν-a.e. point $T \in \partial CV_r$, the $\mathbb{R}$-tree T is trivalent and non-geometric.

In particular, Theorem A applies if μ has finite support with $\langle supp(\mu)\rangle_+ = Out(F_r)$, in which case one can also conclude that both trees $T_\pm(\phi_n)$ are trivalent and non-geometric.

Ingredients of the proof

The key ingredients of the proof are:

• The notion of a *principal* fully irreducible: A fully irreducible $\phi \in Out(F_r)$ is called *principal* if ϕ is of non-surface type, the attracting tree $T_+ = T_+(\phi)$ is trivalent, has $ind_{geom}(T_+) = 2r - 3$, and every turn at a branch-point in T_+ is "taken" by the expanding lamination of ϕ_n.

• It follows from the work of Pfaff and Mosher that a principal ϕ has the "lone axis" property in CV_r: there exists a unique folding line from $T_-(\phi)$ to $T_+(\phi)$, and this line is a ϕ-periodic geodesic.

• It was proved by Algom-Kfir, Kapovich and Pfaff that if $L \subseteq CV_r$ is the axis of a principal ϕ, and if L_n are axes of fully irreducibles ψ_n that fellow travel L for longer and longer, closer and closer, then for $n \to \infty$ the tree $T_+(\psi_n)$ is trivalent and non-geometric.

• A strong version of the "Bounded Geodesic Image" property for the axis $L \subseteq CV_r$ of a principal $\phi \in Out(F_r)$.

Ingredients of the proof

Sketch of the proof of part (1) of Theorem A.
Use the natural coarsely Lipschitz projection $\pi : CV_r \to \mathcal{F}_r$.

The "reasonable" assumption on μ implies that $\langle supp(\mu)\rangle_+$ contains some principal $\phi \in Out(F_r)$.

Using Maher-Tiozzo-Sisto, probabilistic considerations then imply that for a "random" ϕ_n, some translate A_n of the axis of ϕ_n in $\mathcal{F}_r$ C'-fellow travels the axis $A \subseteq \mathcal{F}_r$ of ϕ for an arbitrary long time as $n \to \infty$.

The BGI property for ϕ then implies that in CV_r some translate L_n of the axis of ϕ_n C-fellow travels the axis $L \subseteq CV_r$ of ϕ for an arbitrary long time.

Then the lone axis property can be used to show that L_n ϵ-fellow travels L for arbitrarily long time with arbitrarily small ϵ.

Then by AKKP, he tree $T_+(\phi_n)$ is trivalent and non-geometric, and part (1) of Thm A is proved.

Ingredients of the proof

Rmk. In the $\mathcal{T}(\Sigma)$ setting, a crucial point in the proofs of Gadre-Maher for $MCG(\Sigma)$ is that the principal stratum is open in $\mathcal{Q}(\Sigma)$.

In the CV_r setting, the notion of being "principal" is only defined for periodic geodesics, not for arbitrary geodesics (because the notion of an "expanding lamination" only makes sense in the periodic case).

Moreover, even in the space of periodic geodesics in CV_r, the property of being principal is **not** open.

It can happen (AKKP) that L is the axis of a principal fully irreducible ϕ, that L_n is an axis of a fully irreducible ψ_n with an overlap $L_n \cap L$ being arbitrarily long, but such that $ind_{geom}(T_+(\psi_n)) < 2r - 3$, so that ψ_n are not principal.

Thus a conceptually different argument is needed in the $Out(F_r)$ case.

Ingredients of the proof

About the proof of part (2) of Thm A.
medskip
Main new difficulty to overcome: Show that for a ν-random $T \in \partial CV_r$ every branch point $p \in T$ is “visible" for some $t > 0$ in the graph G_t along a geodesic folding ray $(G_t)_t$ towards T in CV_r.

Here a branch point $p \in T$ of valency $k \geq 3$ is *visible* in G_t if for the F_r-equivariant edge-isometric map $f_t : \widetilde{G}_t \to T$ there exist a vertex $x \in: \widetilde{G}_t$ and k edges $e_1, \dots, e_k$ starting at x such that $f_t(e_1), \dots, f_t(e_k)$ represent k distinct directions at T.

• We say that a geodesic folding ray $(G_t)_t$ towards T is *eventually legalizing* if for every $t_1 > 0$ and every immersed path γ in $\widetilde{G}_{t_1}$ there exists $t_2 > t_1$ such that the tightened image γ' of γ in $\widetilde{G}_{t_2}$ is *legal*, that is f_{t_2} is locally injective on γ'.

Ingredients of the proof

Ket Facts:

• We prove that if $T \in \partial CV_r$ is free and "arational" and if $(G_t)_t$ is an eventually legalizing geodesic folding ray towards T in CV_r, then every branch point p of T is visible in some G_t.

• We then prove that if $T \in \partial CV_r$ is ν-random (and hence free and aratiional by other known results) then every geodesic folding ray $(G_t)_t$ towards T is eventually legalizing.

Then, for part (2) of Thm A, let $T \in \partial CV_r$ be ν-random.
smallskip
We take a geodesic folding ray $(G_t)_t$ towards T in CV_r. We then use arguments similar to part (1) to show that this ray fellow travels for longer and longer, closer and closer, translates of the axis $L \subseteq CV_r$ of some principal $\phi \in Out(F_r)$.
smallskip
This implies that G_t is a trivalent graph for infinitely many arbitrarily large values of t. If $p \in T$ is a branch-point of valency $k \geq 3$, then by

Further questions

Suppose we are in the setting of Thm A.

1. Is it true that, with probability tending to 1, the element $\phi_n \in Out(F_n)$ is principal?
   smallskip
   We prove that $ind_{geom}(T_+(\phi_n)) \leq 3r - 3$. For being principal need $= 3r - 3$ here.
2. Is it true that for a ν-random $T \in \partial CV_r$ we have $ind_{geom}(T_+(\phi_n)) = 3r - 3$?
   smallskip
   Again, we can only prove that $ind_{geom}(T_+(\phi_n)) = 3r - 3$ right now.
   smallskip
3. Can one define a reasonable notion of a “principal" geodesic $L \subseteq CV_r$ without assuming L to be periodic?
   smallskip

Thank you for your attention!